On Core Stability, Vital Coalitions, and Extendability

by

Evan Shellshear

and

Peter Sudhölter

Discussion Papers on Business and Economics
No. 3/2008

FURTHER INFORMATION
Department of Business and Economics
Faculty of Social Sciences
University of Southern Denmark
Campusvej 55
DK-5230 Odense M
Denmark

Tel.: +45 6550 3271
Fax: +45 6550 3237
E-mail: lho@sam.sdu.dk
http://www.sdu.dk/osbec

ISBN 978-87-91657-18-4

On Core Stability, Vital Coalitions, and Extendability

Evan Shellshear*†

Institute of Mathematical Economics (IMW), University of Bielefeld, Germany

Peter Sudhölter‡§

Department of Business and Economics, University of Southern Denmark at Odense

Abstract

If a TU game is extendable, then its core is a stable set. However, there are many TU games with a stable core that are not extendable. A coalition is vital if there exists some core element x such that none of the proper subcoalitions is effective for x. It is exact if it is effective for some core element. If all coalitions that are vital and exact are extendable, then the game has a stable core. It is shown that the contrary is also valid for assignment games, for simple flow games, and for minimum coloring games.

Keywords: TU game, core, stable set, extendability, vital coalition

Journal of Economic Literature Classification: C71

1 Introduction

The core of a cooperative game is called stable if it is a stable set in the sense of von Neumann and Morgenstern (1953). In this paper we restrict out attention to TU games. Several sufficient conditions for core stability may be found in the literature. For details see, e.g., van Gellekom, Potters, and Reijnierse (1999). A weak and simple sufficient condition, introduced by Kikuta and Shapley (1986) is called *extendability*. A TU game is extendable if each core element of any subgame may be extended to a core element of the entire game. The main part of the present paper is devoted to relaxing extendability in such a way that the modified extendability properties (1) are still sufficient conditions and (2) become necessary conditions for core stability

*Email: evan.shellshear@wiwi.uni-bielefeld.de

†Financial support from the German Science Foundation (DFG) is gratefully acknowledged.

‡Email: psu@sam.sdu.dk

§Financial support from the Spanish Ministry of Education and Science, Project No. SEJ2006-05455, is acknowledged.

when restricting the attention to some nontrivial important classes of games. We show that the game has a stable core if certain coalitions are extendable, namely those that are *vital* in the sense of Gillies (1959) and exact in the sense of Shapley (1971). For some classes of games, e.g., for the class of symmetric games (see Biswas, Parthasarathy, Potters, and Voorneveld (1999)), necessary and sufficient conditions for core stability have been found. We show that vital-exact extendability is also a necessary condition for core stability for three important classes of games: Assignment games, simple flow games, and minimum coloring games. Moreover, our approach enables us to reprove two characterization results of Solymosi and Raghavan (2001) and Bietenhader and Okamoto (2006) in a simple way.

The paper is organized as follows. In Section 2 the basic notation and the relevant definitions are presented and some relevant well-known results are recalled. Section 3 is devoted to three new extendability concepts. Theorem 3.3 states that the new variants of extendability are still sufficient for core stability. By means of examples it is shown that the modified conditions are weaker than extendability but still not necessary for core stability. Some properties of the new conditions are also discussed. Subsection 3.1 is devoted to the proof of Theorem 3.3 and in Subsection 3.2 it turns out that, if the vital and exact coalitions exhibit any of two additional properties (see Theorem 3.8 and Corollary 3.12), then the relaxed extendability condition is necessary for core stability. Section 4 is devoted to three classes of games that have an easy characterization of core stability. It is shown that the relaxed extendability condition is a necessary condition for core stability in these three cases. Also, in the case of minimum coloring games, the well-known characterization result is generalized.

2 Preliminaries

In this section we recall definitions of some relevant concepts and well-known results that may be found in von Neumann and Morgenstern (1953) or Gillies (1959) unless otherwise specified.

A (cooperative TU) *game* is a pair (N, v) such that $\emptyset \neq N$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Let (N, v) be a game. For $S \subseteq N$ we denote by $\mathbb{R}^S$ the set of all real functions on S. So $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D.) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write $x \gg y$ if $x_i > y_i$ for all $i \in S$.

Let $X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ denote the set of *Pareto optimal allocations* (*preimputations*) of (N, v). We use $x(S) = \sum_{i \in S} x_i$ $(x(\emptyset) = 0)$ for every $S \in 2^N$ and every $x \in \mathbb{R}^N$. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$.

The *core* of (N,v), $\mathcal{C}(N,v)$, is given by

$$\mathcal{C}(N,v) = \{x \in X(N,v) \mid x(S) \geq v(S)\ \forall S \subseteq N\}.$$

The set of *imputations* of (N,v), $I(N,v)$, is $I(N,v) = \{x \in X(N,v) \mid x_i \geq v(\{i\})\ \forall i \in N\}$.

A *coalition* (in N) is a nonempty subset of N. A *subgame* of (N,v) is a game (T, v^T) where T is a coalition and $v^T(S) = v(S)$ for all $S \subseteq T$. The subgame (T, v^T) will also be denoted by (T,v).

Let $x, y \in \mathbb{R}^N$ and $S \in 2^N \setminus \{\emptyset\}$. We say that x *dominates* y via S (at (N,v)), written $x \operatorname{dom}_S y$, if $x(S) \leq v(S)$ and $x_S \gg y_S$. Also, we define $x \operatorname{dom} y$, that is, x *dominates* y (at (N,v)), if there exists a coalition S in N such that $x \operatorname{dom}_S y$. Let $X \subseteq \mathbb{R}^N$. We say that X is *internally stable* (with respect to (w.r.t.) (N,v)) if for any $x \in X$ and $y \in \mathbb{R}^N$, $x \operatorname{dom} y$ implies that $y \notin X$. Moreover, X is *externally stable* (w.r.t. (N,v)) if for any $y \in I(N,v) \setminus X$ there exists $x \in X$ such that $x \operatorname{dom} y$. The set X is *stable* if it is internally and externally stable.

Note that $\mathcal{C}(N,v)$ is internally stable and that any externally stable set contains $\mathcal{C}(N,v)$. We say that (N,v) *has a stable core* if $\mathcal{C}(N,v)$ is *stable*, that is, externally stable, w.r.t. (N,v). We also remark that, if $I(N,v) = \emptyset$, then $\emptyset = \mathcal{C}(N,v)$ is stable. Hence, we shall not further consider the case that $\sum_{i \in N} v(\{i\}) > v(N)$.

We now recall some relevant results. The proof of the well-known Proposition 2.1 is presented, because its statement will be used several times.

Proposition 2.1 (Gillies (1959)) *Let (N,v) be a game such that $I(N,v) \neq \emptyset$. If (N,v) has a stable core, then, for each $i \in N$, there exists $x \in \mathcal{C}(N,v)$ such that $x_i = v(\{i\})$.*

Proof: As (N,v) has a stable core and $I(N,v) \neq \emptyset$, $\mathcal{C}(N,v) \neq \emptyset$. Assume, on the contrary, that there exists $k \in N$ such that $x_k > v(\{k\})$ for all $x \in \mathcal{C}(N,v)$. As $\mathcal{C}(N,v)$ is a compact set, $t = \min\{x_k \mid x \in \mathcal{C}(N,v)\}$ exists so that $t > v(\{k\})$. Choose $x \in \mathcal{C}(N,v)$ with $x_k = t$, let $\varepsilon > 0$ satisfy $t - (|N|-1)\varepsilon \geq v(\{k\})$, and define $y \in \mathbb{R}^N$ by $y_i = x_i + \varepsilon$ for all $i \in N \setminus \{k\}$ and $y_k = x_k - (|N|-1)\varepsilon$. Then $y \in I(N,v) \setminus \mathcal{C}(N,v)$. Hence, there exist $z \in \mathcal{C}(N,v)$ and $\emptyset \neq T \subseteq N$ with $z \operatorname{dom}_T y$. For any $S \subseteq N \setminus \{k\}$, $y(S) = \varepsilon|S| + x(S) \geq x(S) \geq v(S)$. Hence, $k \in T$ so that $z_k \geq t = x_k$. As $y_{N\setminus\{k\}} > x_{N\setminus\{k\}}$, we conclude that $z(T) > x(T) \geq v(T)$ and the desired contradiction has been obtained. **q.e.d.**

The foregoing proposition has the following interesting consequence[1].

Corollary 2.2 *If the game (N,v) has a stable core, then any preimputation of (N,v) is dominated by some element of $\mathcal{C}(N,v)$, provided that $I(N,v) \neq \emptyset$.*

[1] Corollary 2.2 may not hold for an arbitrary stable set. Indeed, if (N,v) is the three-person majority game, defined by $N = \{1,2,3\}$, $v(N) = v(S) = 1$, if $|S| = 2$, and $v(T) = 0$, if $|T| \leq 1$, then $X = \left\{\left(c, \frac{1}{2} - c, \frac{1}{2}\right) \middle| \, 0 \leq c \leq \frac{1}{2}\right\}$ is a well-known stable set, but the preimputation $(1,1,-1)$ is not dominated by an element of X.

In order to recall the Bondareva-Shapley theorem (see Bondareva (1963) and Shapley (1967)) which gives necessary and sufficient conditions for the non-emptiness of the core, the following notation is useful. For $T \subseteq N$, denote by $\chi^T \in \mathbb{R}^N$ the *characteristic vector* of T, defined by

$$\chi_i^T = \begin{cases} 1 & , \text{if } i \in T, \\ 0 & , \text{if } i \in N \setminus T. \end{cases}$$

A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called *balanced* (over N) if positive numbers $\delta^S, S \in \mathcal{B}$, exist such that $\sum_{S \in \mathcal{B}} \delta^S \chi^S = \chi^N$. The collection $(\delta^S)_{S \in \mathcal{B}}$ is called a system of *balancing weights* for $\mathcal{B}$.

Theorem 2.3 (The Bondareva-Shapley Theorem) *Let (N, v) be a game. Then $\mathcal{C}(N, v) \neq \emptyset$ if and only if for each balanced collection $\mathcal{B}$ over N and any system $(\delta^S)_{S \in \mathcal{B}}$ of balancing weights of $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \delta^S v(S) \leq v(N)$.*

The foregoing theorem motivates calling a game (N, v) a *balanced game* if $\mathcal{C}(N, v) \neq \emptyset$. Note that (N, v) is *totally balanced* if, for any $\emptyset \neq S \subseteq N$, (S, v) is balanced. The *totally balanced cover* of (N, v), (N, v^{tb}), is given by

$$v^{tb}(S) = \max \left\{ \sum_{T \in \mathcal{B}} \delta^T v(T) \;\middle|\; \begin{array}{l} \mathcal{B} \text{ is a balanced collection over } S \text{ and} \\ (\delta^T)_{T \in \mathcal{B}} \text{ is system of balancing weights for } \mathcal{B} \end{array} \right\} \; \forall S \subseteq N. \quad (2.1)$$

The formulation of a weak sufficient condition for core stability requires some notation. Let (N, v) be a game and let $\emptyset \neq S \subseteq N$. The coalition S is called *extendable* (w.r.t. (N, v)) if, for any $x \in \mathcal{C}(S, v)$, there exists $y \in \mathcal{C}(N, v)$ such that $x = y_S$. Moreover, (N, v) is *extendable* if all coalitions are extendable. The proof of the following well-known result is straightforward.

Theorem 2.4 (Kikuta and Shapley (1986)) *Any extendable game (N, v) has a nonempty stable core.*

3 Relaxing Extendability

This section is organized as follows. The present part introduces conditions that are weaker than extendability. The main result of this section, Theorem 3.3, states that these new variants of extendability are sufficient conditions for core stability. Moreover, properties and relations of the new variants of extendability are presented. Subsection 3.1 is devoted to the proof of the main result and in Subsection 3.2 we show that certain assumptions on the structure of a game guarantee that the new conditions are necessary for core stability.

We now recall two possible properties of a coalition w.r.t. a game. Let (N, v) be a game and $\emptyset \neq S \subseteq N$. The coalition S is called *exact* (w.r.t. (N, v)) if there exists $x \in \mathcal{C}(N, v)$ such that

$x(S) = v(S)$. In this case S is *effective* for x. If all coalitions are exact, then (N, v) is called *exact* (see Shapley (1971) or Schmeidler (1972)). We say that a balanced game (N, v) is *exact extendable* if all exact coalitions are extendable. Moreover, S is called *vital* (w.r.t. (S, v)) if there exists $x \in \mathcal{C}(S, v)$ such that $x(T) > v(T)$ for[2] all $T \in 2^S \setminus \{\emptyset, S\}$. We say that a balanced game (N, v) is *vital extendable* if all vital coalitions w.r.t. (N, v) are extendable.

Remark 3.1 There is a simple characterization of a vital coalition (see Gillies (1959)). Indeed, S is vital if and only if for any balanced collection $\mathcal{B}$ over S, $S \notin \mathcal{B}$, and any system $(\delta^T)_{T \in \mathcal{B}}$ of balancing weights for $\mathcal{B}$, $\sum_{T \in \mathcal{B}} \delta^T v(T) < v(S)$.

Denote by $\mathcal{E}(N, v)$ the set of all coalitions S that are effective for x for all $x \in \mathcal{C}(N, v)$ or $S = \emptyset$, that is,

$$\mathcal{E}(N, v) = \{S \subseteq N \mid x(S) = v(S) \ \forall x \in \mathcal{C}(N, v)\}. \tag{3.1}$$

Definition 3.2 *Let (N, v) be a balanced game. A coalition $S \subseteq N$ is called* **strongly vital-exact** *(w.r.t. (N, v)) if S is vital and if there exists $x \in \mathcal{C}(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus (\{S\} \cup \mathcal{E}(N, v))$. The game (N, v) is* **vital-exact extendable** *if all strongly vital-exact coalitions are extendable.*

Theorem 3.3 *Any balanced, vital-exact extendable game (N, v) has a stable core.*

Thus, Theorem 3.3 shows relations that may be summarized in the following diagram:

$$\begin{array}{ccccc} & \nearrow & \text{exact extendability} & \searrow & \\ \text{extendability} & & & & \text{vital-exact extendability} \Rightarrow \text{core stability}. \\ & \searrow & \text{vital extendability} & \nearrow & \end{array} \tag{3.2}$$

By means of examples we will show that none of the opposite implications of (3.2) is valid and that exact extendability may not imply vital extendability and vice versa. Moreover, there are balanced games that are vital-exact extendable and have non-extendable coalitions that are vital and exact.

Example 3.4 Let $N = \{1, \ldots, 7\}$ and let (N, v_1) be defined as follows. Let $T = \{1, 2\}, T^i = \{2, i\}$ for $i = 3, 4, 5$, and $T^j = \{1, j\}$ for $j = 6, 7$, and let $v_1(N) = 16, v_1(T^k) = 4$ for all

[2]Gillies (1959) introduced vital coalitions of at least two elements, whereas according to our definition singletons are always vital.

$k = 3, \ldots, 7$, $v_1(T) = 1$, and for all other $S \subseteq N$, let $v_1(S) = 0$. Then $(3, 3, 2, 2, 2, 2, 2) \in \mathcal{C}(N, v_1)$ so that $\mathcal{E}(N, v_1) = \{\emptyset, N\}$. With

$$y^1 = (12, 4, 0, 0, 0, 0, 0), y^2 = (0, 2, 2, 2, 2, 4, 4), y^3 = (4, 0, 4, 4, 4, 0, 0)$$

note that $y^i \in \mathcal{C}(N, v_1)$ for $i = 1, 2, 3$. The coalition T is vital, but not exact. Indeed, let $y \in \mathcal{C}(N, v_1)$. As $y(T^k) \geq 4$, $k = 3, \ldots, 7$,

$$y_i \geq 4 - y_2 \ \forall i \in \{3, 4, 5\} \text{ and } y_j \geq 4 - y_1 \ \forall j \in \{6, 7\} \tag{3.3}$$

so that $16 = y(N) \geq 20 - y(T) - y_2$, that is, $y(T) \geq 2$. We conclude that a coalition $S \subsetneqq N$ satisfying $v_1(S) > 0$ is exact if and only if it is one of the coalitions T^j, $j = 3, \ldots, 7$, and that these coalitions are extendable. An exact coalition S with $v_1(S) = 0$ is also extendable, because $\mathcal{C}(S, v_1)$ is a singleton. Hence, (N, v_1) is exact extendable, but not vital extendable. Let (N, v_1') be the game that differs from (N, v_1) only inasmuch as $v_1'(T) = 0$. Then (N, v_1') is vital extendable (because T is not vital w.r.t. (N, v_1')) and exact extendable, but T is still not extendable.

Example 3.5 Now, let (N, v_2) be the game that differs from (N, v_1) defined in Example 3.4 only inasmuch as $v_2(N) = 18$. Any singleton and any of the coalitions $T^j, j = 3, \ldots, 7$, are still extendable which follows from the fact that $y^k + 2\chi^{\{i\}} \in \mathcal{C}(N, v_2)$ for any $k = 1, 2, 3$, and $i \in N$. Moreover, $z = (0, 1, 3, 3, 3, 4, 4)$ is the unique element in $\mathcal{C}(N, v_2)$ that satisfies $z(T) = v_2(T)$. Hence, T is vital and exact, but not strongly vital-exact. We conclude that (N, v_2) is vital-exact extendable, but neither exact extendable nor vital extendable. Now, if the the worth of N is further increased, that is, let $0 < \varepsilon < 1$ and (N, v_3) differ from (N, v_2) only inasmuch as $v_3(N) = v_2(N) + \varepsilon$, then $(\varepsilon, 1 - \varepsilon, 3 + \varepsilon, 3 + \varepsilon, 3 + \varepsilon, 4 - \varepsilon, 4 - \varepsilon) \in \mathcal{C}(N, v)$ so that T is strongly vital-exact. Now, T is not extendable, because if $y \in \mathcal{C}(N, v_3)$ satisfies $y_2 = 0$, then $y_1 \geq 2 - \varepsilon > 1$, that is, $y(T) > v_3(T)$. Nevertheless, (N, v_3) has a stable core. Indeed, if $x \in I(N, v_3) \setminus \mathcal{C}(N, v_3)$, then two cases may occur. If $x(T^j) \geq 4$ for all $j = 3, \ldots, 7$, then, by (3.3) applied to x, $x_2 + x(T) \geq 2 - \varepsilon$. As $x(T) < 1$, $x_2 > 1 - \varepsilon$ and $x_1 < \varepsilon$ so that x is dominated by some core element via T. In the other case there exists $\ell \in \{3, \ldots, 7\}$ such that $x(T^\ell) < v(T^\ell)$ and extendability of T^ℓ guarantees that x is dominated by some core element.

Together with Example 4.2 (the game (N, v_4) discussed in Section 4) the foregoing examples show that the relations summarized in (3.2) are strict even if balancedness is assumed:

$$\text{core stability} \overset{v_3}{\not\Rightarrow} \text{vital-exact extendability} \begin{array}{c} \not\nearrow \\ v_2 \\ \not\searrow \end{array} \begin{array}{c} \text{vital extendability} \\ v_4 \not\Downarrow \qquad \not\Uparrow v_1 \\ \text{exact extendability} \end{array} \begin{array}{c} \not\searrow \\ v_1' \\ \not\nearrow \end{array} \text{extendability.}$$

The properties of the games (N, v_1) and (N, v_3) of Example 3.4 also show that neither "exact extendability" nor "vital-exact extendability" are strong *prosperity properties* in the sense of van Gellekom, Potters, and Reijnierse (1999, Definition 2.1) who showed that "extendability" is a strong prosperity property. Note that in a similar way (Indeed, a nonempty proper coalition in N is or is not vital regardless of the "prosperity" of N.) it may be shown that "vital extendability" is a strong prosperity property.

An interesting invariance property shared by two of the new variants of "extendability" and by "core stability" is contained in the following statements. Let (N, v) be a balanced game and (N, v^{tb}) its totally balanced cover (see (2.1)):

(1) (N, v) has a stable core $\Longleftrightarrow$ (N, v^{tb}) has a stable core.

(2) (N, v) is vital extendable $\Longleftrightarrow$ (N, v^{tb}) is vital extendable.

(3) (N, v) is vital-exact extendable $\Longleftrightarrow$ (N, v^{tb}) is vital-exact extendable.

For a proof of (1) see van Gellekom, Potters, and Reijnierse (1999, p. 220) who also show by means of Example 2 that there exists an extendable game whose balanced cover is not extendable. By (2.1), $\mathcal{C}(N, v) = \mathcal{C}(N, v^{tb})$. We conclude that a coalition is vital w.r.t. (N, v) iff it is vital w.r.t. (N, v^{tb}). Again by (2.1), $\mathcal{E}(N, v) = \mathcal{E}(N, v^{tb})$ and we may conclude that a coalition is strongly vital-exact w.r.t. (N, v) iff it is strongly vital-exact w.r.t. (N, v^{tb}). Hence, (2) and (3) are valid. The totally balanced cover of (N, v_1), (N, v_1^{tb}), is not exact extendable. Indeed, it is straightforward to verify that $v_1^{tb}(\{1, 2, 3, 6\}) = 8$ and that $(1, 0, 4, 3) \in \mathcal{C}(\{1, 2, 3, 6\}, v_1^{tb})$. However, this vector is not the restriction of any element of $\mathcal{C}(N, v_1^{tb})$.

3.1 The Proof of Theorem 3.3

We now prove two useful lemmata. Let (N, v) be a balanced game.

Lemma 3.6 *For any $x \in X(N, v) \setminus \mathcal{C}(N, v)$ there exists a strongly vital-exact coalition P such that $x(P) < v(P)$.*

Proof: By the definition of $\mathcal{E}(N, v)$ and the convexity of the core, there exists $x^0 \in \mathcal{C}(N, v)$ such that $x^0(S) > v(S)$ for all $S \in 2^N \setminus \mathcal{E}(N, v)$. For $\lambda \in \mathbb{R}$ denote $x^\lambda = \lambda x + (1 - \lambda)x^0$. As $\mathcal{C}(N, v)$ is convex and closed, there exists $\widehat{\lambda}, 0 \leq \widehat{\lambda} < 1$, such that

$$\lambda \geq 0 \text{ and } x^\lambda \in \mathcal{C}(N, v) \Longleftrightarrow 0 \leq \lambda \leq \widehat{\lambda}.$$

Then there exists $P \subseteq N$ such that $x(P) < v(P)$ and $x^{\widehat{\lambda}}(P) = v(P)$. Hence, P is exact. Now, let P be minimal (w.r.t. inclusion) such that $x(P) < v(P)$ and $x^{\widehat{\lambda}}(P) = v(P)$. By minimality of P,

$$Q \subsetneqq P \text{ and } x(Q) < v(Q) \Longrightarrow x^{\widehat{\lambda}}(Q) > v(Q) \tag{3.4}$$

By (3.4), for all $Q \subsetneqq P$ and all $\lambda, 0 < \lambda \leq 1$,

$$\left(Q \in \mathcal{E}(N, v) \Longrightarrow x^{\lambda}(Q) \geq v(Q)\right) \text{ and } \left(x(Q) > v(Q) \Longrightarrow x^{\lambda}(Q) > v(Q)\right).$$

Hence, $x_P^{\widehat{\lambda}} \in \mathcal{C}(P, v)$, $x^{\widehat{\lambda}}(Q) > v(Q)$ for all $Q \in 2^P \setminus \mathcal{E}(N, v)$, $Q \neq P$, and there exists $\varepsilon > 0$ such that $x^{\widehat{\lambda}+\varepsilon}(Q) \geq v(Q)$ for all $Q \subsetneqq P$. Then $d = v(P) - x^{\widehat{\lambda}+\varepsilon}(P) > 0$. Now, with $y = x^{\widehat{\lambda}+\varepsilon} + \frac{d}{|P|}\chi^P$ observe that $y(P) = v(P)$ and $y(Q) > v(Q)$ for all $Q \in 2^P \setminus \{\emptyset, P\}$. Hence, P is strongly vital-exact. **q.e.d.**

Lemma 3.7 *If (N, v) is vital-exact extendable and $x \in X(N, v) \setminus \mathcal{C}(N, v)$, then there exists a strongly vital-exact coalition S such that $x(S) < v(S)$ and $x(T) \geq v(T)$ for all $T \subsetneqq S$.*

Proof: By Lemma 3.6 there exists a strongly vital-exact coalition P such that $x(P) < v(P)$. Let P be a minimal coalition that satisfies the foregoing conditions. Assume, on the contrary, that there exists $Q \subsetneqq P$ such that $x(Q) < v(Q)$. Define

$$y = x + \frac{v(P) - x(P)}{|P \setminus Q|}\chi^{P \setminus Q}$$

and observe that $x \leq y$, $x(Q) = y(Q)$, and $y(P) = v(P)$. Hence $y_P \in X(P, v) \setminus \mathcal{C}(P, v)$. By Lemma 3.6 applied to (P, v) and y_P, there exists a strongly vital-exact coalition T w.r.t. (P, v) such that $y(T) < v(T)$ and, hence, $x(T) < v(T)$. As P is extendable, T is strongly vital-exact w.r.t. (N, v) so that the desired contradiction has been obtained. **q.e.d.**

Proof of Theorem 3.3: Let $z \in X(N, v) \setminus \mathcal{C}(N, v)$. By Lemma 3.7 there exists a strongly vital-exact $\emptyset \neq S \subseteq N$ such that $z(S) < v(S)$ and $z(T) \geq v(T)$ for all $T \subsetneqq S$. Let $y \in \mathbb{R}^S$ be given by $y_i = z_i + \frac{v(S)-z(S)}{|S|}$. Then $y(S) = v(S)$ and $y \gg z$, hence $y(T) > v(T)$ for all $\emptyset \neq T \subsetneqq S$. We conclude that $y \in \mathcal{C}(S, v)$. As S is extendable, there exists $x \in \mathcal{C}(N, v)$ such that $x_S = y$. Thus $x \operatorname{dom}_S z$. **q.e.d.**

3.2 Two Consequences of Theorem 3.3

This subsection serves to show that all strongly vital-exact coalition are extendable, if the set of strongly vital-exact coalitions exhibits a certain structure. We say that (N, v) has *disjoint antichains of strongly vital-exact coalitions* if, for all strongly vital-exact coalitions S and T, $S \subseteq T$ or $T \subseteq S$ or $S \cap T = \emptyset$ (that is, the elements of any antichain of the partially ordered set of strongly vital-exact coalitions, ordered by inclusion, are pairwise disjoint).

Theorem 3.8 *If (N, v) is a balanced game that has disjoint antichains of strongly vital-exact coalitions, then (N, v) has a stable core.*

Proof: Let S be a strongly vital-exact coalition. By Theorem 3.3 it suffices to show that S is extendable. To this extent let $x \in \mathcal{C}(S, v)$. As S is exact, there exists $y \in \mathcal{C}(N, v)$, $y(S) = v(S)$. Let $z \in \mathbb{R}^N$ be given by $z_S = x$ and $z_{N \setminus S} = y_{N \setminus S}$. We conclude that $z(N) = v(N)$, $z(T) = y(T) \geq v(T)$ for all $T \subseteq N \setminus S$ and all $S \subseteq T \subseteq N$, and $z(P) = x(P) \geq v(P)$ for all $P \subseteq S$. Hence, $z(Q) \geq v(Q)$ for all strongly vital-exact coalitions Q. By Lemma 3.6, $z \in \mathcal{C}(N, v)$ and the proof is complete. **q.e.d.**

Balanced games that have disjoint antichains of strongly vital-exact coalitions may be constructed as follows. Let N be a finite nonempty set, let $x \in \mathbb{R}^N$, and let (N, v) satisfy $v(S) = x(S)$ for all $S \subsetneqq N$ and $v(N) \geq x(N)$. Then the strongly vital-exact coalitions are the singletons and N provided that $v(N) > x(N)$. Hence (N, v) has the desired property. Now let $(N^1, v^1), \ldots, (N^k, v^k)$ be k balanced games that have disjoint antichains of strongly vital-exact coalitions such that the N^ℓ are pairwise disjoint. With $N = \bigcup_{\ell=1,\ldots,k} N^\ell$ let (N, v) be a game that satisfies $v(S) = \sum_{\ell=1}^k v^\ell(S \cap N^\ell)$ for all $S \subsetneqq N$ and $v(N) \geq \sum_{\ell=1}^k v^\ell(N^\ell)$. Then (N, v) has the desired property.

The following theorem reveals some structure of the set of strongly vital-exact coalitions and will be used to show that vital-exact extendability is a necessary condition for core stability for the second class of games.

Theorem 3.9 *If (N, v) is a balanced game, then there exist a balanced collection $\mathcal{P}$ of strongly vital-exact coalitions w.r.t. (N, v) and a system $(\delta^P)_{P \in \mathcal{P}}$ of balancing weights for $\mathcal{P}$ such that*

$$\sum_{P \in \mathcal{P}} \delta^P v(P) = v(N).$$

Proof: Let (N, v) be balanced. By convexity of the core and the definition of $\mathcal{E}(N, v)$, there exists $x \in \mathcal{C}(N, v)$ such that $x(T) > v(T)$ for all $T \in 2^N \setminus \mathcal{E}(N, v)$. Therefore the following statement is true:

$$R, T \in \mathcal{E}(N, v), \emptyset \neq R \subseteq T, R \text{ is vital} \Longrightarrow R \text{ is strongly vital-exact w.r.t. } (N, v). \tag{3.5}$$

We proceed by induction on $n = |N|$. If $n = 1$, then N is vital, hence strongly vital-exact, so that the proof is finished in this case. Let the theorem be true for $n \leq t$ and some $t \in \mathbb{N}$ and assume now that $n = t + 1$. If N is vital, then the theorem is true. Hence, we may assume that N is not vital. By Remark 3.1 and Theorem 2.3, there exist a balanced collection $\widehat{\mathcal{R}}$ on N and a system $(\widehat{\delta}^R)_{R \in \widehat{\mathcal{R}}}$ of balancing weights for $\widehat{\mathcal{R}}$ such that $N \notin \widehat{\mathcal{R}}$ and $\sum_{R \in \widehat{\mathcal{R}}} \widehat{\delta}^R v(R) = v(N)$. Moreover, for

$x \in \mathcal{C}(N,v)$, $v(N) = x(N) = \sum_{R \in \widehat{\mathcal{R}}} \widehat{\delta}^R x(R) = \sum_{R \in \widehat{\mathcal{R}}} \widehat{\delta}^R v(R)$ so that $R \in \mathcal{E}(N,v)$ for all $R \in \widehat{\mathcal{R}}$. As (R,v) is balanced, the inductive hypothesis implies that there exist a balanced collection $\mathcal{P}_R$ on R of strongly vital-exact coalitions w.r.t. (R,v) and a system $(\delta_R^P)_{P \in \mathcal{P}_R}$ of balancing weights for $\mathcal{P}_R$ such that $v(R) = \sum_{P \in \mathcal{P}_R} \delta_R^P v(R)$. Define, for any $P \in \mathcal{P} = \bigcup_{R \in \widehat{\mathcal{R}}} \mathcal{P}_R$,

$$\delta^P = \sum_{R \in \{R \in \widehat{\mathcal{R}} \mid P \in \mathcal{P}_R\}} \widehat{\delta}^R \delta_R^P.$$

We conclude that $\sum_{P \in \mathcal{P}} \delta^P \chi^P = \chi^N$ and $\sum_{P \in \mathcal{P}} \delta^P v(P) = v(N)$. Thus, $\mathcal{P}$ is a balanced collection on N and $\mathcal{P} \subseteq \mathcal{E}(N,v)$ so that the proof is finished by (3.5). **q.e.d.**

Now, the second class of games is constructed as follows. Let (N,v) be a game that satisfies the following property:

$$S \text{ is strongly vital-exact} \Longrightarrow |S| \leq 2. \tag{3.6}$$

For all $x, y \in X(N,v)$ and all $\alpha \geq 0$ define $z^{\alpha,x,y} \in \mathbb{R}^N$ by

$$z_i^{\alpha,x,y} = \begin{cases} x_i + \min\{y_i - x_i, \alpha\} & \text{, if } y_i \geq x_i, \\ x_i - \min\{x_i - y_i, \alpha\} & \text{, if } x_i \geq y_i \end{cases} \tag{3.7}$$

and note that z is well-defined.

Lemma 3.10 *If (N,v) satisfies (3.6), if $x, y \in \mathcal{C}(N,v)$, and if $\alpha \geq 0$, then $z^{\alpha,x,y} \in \mathcal{C}(N,v)$.*

Proof: If $\mathcal{C}(N,v) = \emptyset$, then the statement of the lemma is vacuously true. Hence, we assume now that (N,v) is balanced. By Theorem 3.9 there exist a balanced collection $\mathcal{P}$ of strongly vital-exact coalitions on N and a system $(\delta^P)_{P \in \mathcal{P}}$ of balancing weights for $\mathcal{P}$ such that $\sum_{P \in \mathcal{P}} \delta^P v(P) = v(N)$. Let $z = z^{\alpha,x,y}$ and let $i \in N$. If $y_i \geq x_i$, then $z_i \geq x_i \geq v(\{i\})$. If $y_i < x_i$, then $z_i \geq y_i \geq v(\{i\})$. Hence, z is individually rational. Let $P \in \mathcal{P}$. If $|P| = 1$, then $x(P) = y(P) = v(P)$ so that $z(P) = v(P)$. If $|P| = 2$, then $x(P) = y(P) = v(P)$ also implies $z(P) = v(P)$. By (3.6), $z(P) = v(P)$ for all $P \in \mathcal{P}$. We conclude that $z(N) = v(N)$. Now, let $S = \{i,j\}$, $i \neq j$, $i,j \in N$. By (3.6) and Lemma 3.6 it suffices to show that $z(S) \geq v(S)$. If $y_i \geq x_i$ and $y_j \geq x_j$, then $z(S) \geq x(S) \geq v(S)$. If $y_i \geq x_i$ and $y_j < x_j$, then the case $z(S) < y(S)$ may just occur, if $y_i - x_i > \alpha$. However, in this case $z(S) \geq x(S)$. The case $y_i < x_i$ and $y_j \geq x_j$ may be treated similarly. Finally, if $y_i < x_i$ and $y_j < x_j$, then $z(S) \geq y(S)$. Thus, $z \in \mathcal{C}(N,v)$. **q.e.d.**

Proposition 3.11 *If (N,v) satisfies (3.6) and if each $\{i\}$, $i \in N$, is exact, then (N,v) is vital-exact extendable.*

Proof: Let S be a strongly vital-exact coalition and $x \in \mathcal{C}(N,v)$ such that $x(S) = v(S)$. If $|S| = 1$, then the proof is already finished. Hence, we may assume that $S = \{k, \ell\}$ for some

$k, \ell \in N$, $k \neq \ell$. Let $y \in \mathcal{C}(N,v)$ such that $y_k = v(\{k\})$ and let $\alpha = x_k - v(\{k\})$. By Lemma 3.10, $z = z^{\alpha,x,y} \in \mathcal{C}(N,v)$. Now, $z_k = y_k = v(\{k\})$ and $z_\ell = \alpha + x_\ell = v(\{k,\ell\})$. By convexity of $\mathcal{C}(N,v)$, S is extendable. **q.e.d.**

Proposition 2.1 implies the following result.

Corollary 3.12 *If (N,v) is a balanced game that satisfies (3.6), then the following conditions are equivalent:*

(1) (N,v) has a stable core.

(2) (N,v) is vital-exact extendable.

(3) For each $i \in N$, the singleton $\{i\}$ is an exact coalition.

4 Three Remarkable Classes of Games

Example 3.4 shows that the reverse of Theorem 3.3 does not hold. However, as Corollary 3.12 suggests, there are remarkable classes of games such that the extendability of all strongly vital-exact coalitions is also necessary for core stability.

4.1 Assignment Games

Shapley and Shubik (1972) introduced assignment games. For finite sets S and T an *assignment* of (S,T) is a bijection $b : S' \to T'$ such that $S' \subseteq S$, $T' \subseteq T$, and $|S'| = |T'| = \min\{|S|,|T|\}$. We shall identify b with $\{(i,b(i)) \mid i \in S'\}$. Let $\mathcal{B}(S,T)$ denote the set of assignments. A game (N,v) is an *assignment game* if there exist a partition $\{P,Q\}$ of N and a nonnegative real matrix $A = (a_{ij})_{i \in P, j \in Q}$ such that

$$v(S) = \max_{b \in \mathcal{B}(S\cap P, S\cap Q)} \sum_{(i,j)\in b} a_{ij}. \tag{4.1}$$

Let (N,v) be an assignment game defined by the matrix A on $P \times Q$.

Lemma 4.1 *If $S \subseteq N$, $|S| \geq 2$, is a vital coalition, then $|S \cap P| = |S \cap Q| = 1$.*

Proof: Let $S \subseteq N$, $|S| \geq 2$. If $S \subseteq P$ or $S \subseteq Q$, then $0 = v(S) = \sum_{i \in S} v(\{i\})$ and, hence, S is not vital. Assume now, that $|S \cap P| \geq 2$ or $|S \cap Q| \geq 2$, let us say $|S \cap P| \leq |S \cap Q|$, and let $b \in \mathcal{B}(S \cap P, S \cap Q)$ satisfy $v(S) = \sum_{i \in S \cap P} a_{ib(i)}$. Thus, for any $i \in S \cap P$, $v(S) = v(\{i,b(i)\}) + v(S \setminus \{i,b(i)\})$. We conclude that S is not vital. **q.e.d.**

By Lemma 4.1 and Corollary 3.12, (N, v) has a stable core if and only if (N, v) is strongly vital-exact extendable. The theory developed so far enables us to reprove Theorem 1 of Solymosi and Raghavan (2001): Assume without loss of generality that $|P| \leq |Q|$ and let b denote an optimal assignment. The assignment game (N, v) has a stable core if and only if

$$a_{ib(i)} = \max_{r \in Q} a_{ir} = \max_{r \in P} a_{rb(i)} \text{ and } a_{ij} = 0 \; \forall i \in P, j \in Q \setminus \{b(r) \mid r \in P\}. \tag{4.2}$$

In order to verify the if direction, assume that (4.2) is valid and define $x, y \in \mathbb{R}^N$ by $x_i = a_{ib(i)} = y_{b(i)}$ and $x_{b(i)} = y_i = x_j = y_j = 0$ for all $i \in P$ and all $j \in Q \setminus b(P)$. Then $x, y \in \mathcal{C}(N, v)$ and, hence, by Proposition 3.11 and Theorem 3.3, (N, v) has a stable core.

In order to verify the only if direction, assume that (N, v) has a stable core and let $i, j \in P$ and $r \in Q$. By Proposition 2.1 there exists $x \in \mathcal{C}(N, v)$ such that $x_r = 0$. Then $x \geq 0$, $x_i + x_{b(i)} = a_{ib(i)}$, and $x_i + x_r \geq a_{ir}$ so that $a_{ib(i)} \geq x_i \geq a_{ir}$. Similarly we may deduce that $a_{ib(i)} \geq a_{jb(i)}$ so that (4.2) is shown.

The following example shows that exact extendability is not necessary for core stability for assignment games.

Example 4.2 Let

$$A = \begin{pmatrix} 6 & 4 & 0 \\ 0 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix},$$

and let (N, v_4) be the assignment game defined by A, where $1, 2$, and 3 are the "row" players and $4, 5$, and 6 are the "column" players. The unique optimal assignment b is given by $b(i) = 3 + i$ for $i = 1, 2, 3$. Hence, (4.2) is satisfied so that (N, v_4) has a stable core. Moreover, $x = (3, 5, 1, 3, 1, 5) \in \mathcal{C}(N, v_4)$ and $x(S) = v_4(S)$, where $S = \{1, 3, 4, 5\}$. Now, S is not extendable, because $(4, 0, 4, 0) \in \mathcal{C}(S, v_4)$ and any $y \in \mathcal{C}(N, v_4)$ must assign $a_{ib(i)}$ to any coalition $\{i, b(i)\}$ of optimally matched players, e.g., satisfies $y_1 + y_4 = a_{14} = 6$. We conclude that (N, v_4) is not exact extendable. In order to show that (N, v_4) is vital extendable, it suffices to show that $\{1, 5\}$ and $\{3, 4\}$ are extendable. A careful inspection of the core elements $(0, 2, 0, 6, 4, 6), (4, 6, 6, 2, 0, 0), (6, 6, 4, 0, 0, 2), (2, 0, 0, 4, 6, 6)$ shows that they are extendable. It should be noted that there are also assignment games with a stable core that are not vital extendable. Indeed, let (N, v_5) be the assignment game defined by B. As each pair (i, j), $i \in P, j \in Q$, belongs to an optimal matching except the pair $(3, 4)$, we conclude that $\mathcal{C}(N, v_5) = \{(\alpha, \alpha, \alpha, 2 - \alpha, 2 - \alpha, 2 - \alpha) \mid 0 \leq \alpha \leq 2\}$. Consequently, the vital coalition $\{3, 4\}$ is not exact and, hence not extendable.

4.2 Simple Flow Games

Kalai and Zemel (1982) present two equivalent representations of totally balanced games: A game is totally balanced game if and only if (a) it is a *flow* game or (b) it is the minimum of finitely many additive games. The following example shows that even for the minimum of two additive games, the simplest nontrivial case in (b), vital-exact extendability may not be necessary for core stability. Moreover, for "simple" flow games we shall derive that vital extendability is necessary and sufficient for core stability.

Example 4.3 Let $N = \{1, \ldots, 6\}$, let $\lambda = (2, 1, 1, 2, 1, 1)$, let $N^1 = \{1, 2, 3\}$, let $N^2 = \{4, 5, 6\}$ and let (N, v) be the game given by $v(S) = \min_{i=1,2} \lambda(S \cap N^i)$. The game (N, v) is exact (see, e.g., Raghavan and Sudhölter (2005)) and it has a stable core. Indeed, $\mathcal{C}(N, v)$ is the convex hull of $(2, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 2, 1, 1)$ (see, e.g., Rosenmüller (2000)). By considering an arbitrary element of the relative interior of the core, it follows that $S = \{1, 5, 6\}$ is a strongly vital-exact coalition. This coalition cannot be extended, because for any $x \in \mathcal{C}(N, v)$, $x_5 = x_6$, but the core of (S, v) contains some y with $y_5 \neq y_6$ (e.g., given by $y_1 = y_6 = 1/2, y_5 = 1$).

Adopting the notation of Sun and Fang (2007), who characterized the simple flow games that have a stable core, $D = (V, E, s, t)$ is a *simple (directed) network* with source s and sink t, if V is the vertex set, $E \neq \emptyset$ is the arc set, and s and t are distinct vertices in V. The term "simple" refers to the fact that all arcs have the same capacity, let us say 1. The *flow game* (E, v^D) associated with $D = (V, E, s, t)$ is the TU game defined by the requirement that, for any $\emptyset \neq S \subseteq E$, $v^D(S)$ is the maximal flow from s to t in the network (V, S, s, t). A game (N, v) is a *simple flow game* if it is the game associated with some simple directed network with a source and a sink.

A *(simple) path* in a network $D = (V, E, s, t)$ is a sequence of arcs from s to t that visits each vertex at most once. It is well-known that

$$v^D(S) \text{ is the maximal number of arc-disjoint paths in } (V, S, s, t) \text{ for all } S \in 2^E \setminus \{\emptyset\}. \qquad (4.3)$$

Let $D = (V, E, s, t)$ be a simple network with source and sink and denote $v = v^D$.

Remark 4.4 If a coalition S is vital and $v(S) > 0$, then $v(S) = 1$ and $v(T) = 0$ for all $T \subsetneq S$. Indeed, by (4.3), the elements of S, suitably ordered, must form a path.

An arc $e \in E$ is called a *dummy arc*[3] if there exists a path containing e and if $v(E \setminus \{e\}) = v(E)$. We recall that a *cut* of D is a coalition $C \subseteq E$ such that each path contains an arc of C. For a

[3]Sun and Fang (2007) use this term although a dummy arc is not a *dummy player*. Indeed, an arc is a dummy player if and only if it either connects s and t or it is not contained in any path.

proof of the following “max-flow min-cut” theorem see, e.g., Ford and Fulkerson (1962):

$$v^D(E) = \min\{|C| \mid C \text{ is a cut of } D\}. \tag{4.4}$$

We are now able to recall Theorem 3 of Sun and Fang (2007).

Theorem 4.5 *Let $D = (V, E, s, t)$ be a simple network with source and sink. Then (E, v^D) has a stable core if and only if E does not contain any dummy arc.*

We use the preceding theorem and the following lemma and remark to show that vital extendability is necessary for core stability in the case of simple flow games. Let $D = (V, E, s, t)$ be a simple network with source and sink.

Lemma 4.6 *If E does not contain any dummy arc and if $e \in E$ satisfies $v^D(E \setminus \{e\}) < v^D(E)$, then there exists a minimum cut C with $e \in C$.*

Proof: By (4.3) there are $v^D(E)$ arc-disjoint paths. We may assume that $v^D(E) > 1$. As $v^D(E) > v^D(E \setminus \{e\})$, the arc e must be contained in one of the paths and $v^D(E \setminus \{e\}) = v^D(E) - 1$. Hence, if C' is a minimum cut of $(V, E \setminus \{e\}, s, t)$, then $C' \cup \{e\}$ is a minimum cut of D by (4.4). **q.e.d.**

Remark 4.7 In a constructive way Kalai and Zemel (1982, p. 478) show that the core of an arbitrary flow game is nonempty. Applied to a simple flow game (N, v) associated with the simple network $D = (V, E, s, t)$ they prove that, for any minimum cut C of D, $\chi^C \in \mathcal{C}(E, v^D)$.

Proposition 4.8 *A simple flow game (N, v) has a stable core if and only if it is vital extendable.*

Proof: Let $D = (V, E, s, t)$ be a simple network with source and sink and let (E, v) be the associated simple flow game. As the if direction is valid by Theorem 3.3, we assume now that (E, v) has a stable core. Let S be a vital coalition. If $v(S) = 0$, then $|S| = 1$ and, by Proposition 2.1, S is extendable. If $v(S) > 0$, then, by Remark 4.4, $v(S) = 1$ and $v(T) = 0$ for all $T \subsetneqq S$ and the elements of S form a path. By Lemma 4.6 and Remark 4.7, for any $e \in S$, there exists $x \in \mathcal{C}(E, v)$ such that $x_e = 1$ and $x_{e'} = 0$ for all $e' \in S \setminus \{e\}$. However, $\mathcal{C}(S, v)$ is the convex hull of those core elements when restricted to S. **q.e.d.**

It should be remarked that Fang, Fleischer, Li, and Sun (2007, p. 444) present an example of a simple flow game (associated with G_3) that has a stable core and that is not extendable. (Indeed, the 4-person coalition corresponding to the arcs that are marked by + is not exact, but the core of the corresponding subgame is nonempty.)

4.3 Minimum Coloring Games

Deng, Ibaraki, and Nagamochi (1999) introduced minimum coloring games and we basically adopt the notation of Bietenhader and Okamoto (2006). A *graph* is a pair $G = (V, E)$, where V is a finite nonempty set, called the set of *vertices*, and E is a set of 2-element subsets of V, called the set of *edges*. For any $U \subseteq V, U \neq \emptyset$, let G^U denote the subgraph of G whose vertex set is U and whose edges are those edges in E that are subsets of U.

The graph G is *complete* if E is the set of all 2-element subsets of V. A nonempty set $U \subseteq V$ is a *clique* if G^U is complete. Let $\omega(G)$ denote the size of a maximum clique. A *coloring* of G is a mapping $c : V \to \mathbb{R}$ satisfying $c(i) \neq c(j)$ for all $\{i, j\} \in E$. A *minimal* coloring is a coloring c such that $|c(V)|$ is minimal. Let $\gamma(G)$ denote the *chromatic number* of G, i.e., $\gamma(G) = |c(V)|$ for any minimal coloring of G. A set $U \subseteq V$, $U \neq \emptyset$, is *independent* if $\gamma(G^U) = 1$. The graph G is *perfect* if $\omega(G^U) = \gamma(G^U)$ for all $U \in 2^V \setminus \{\emptyset\}$.

Let $G = (V, E)$ be a graph. The *minimum coloring game* on G is the TU game (N, v^G) defined by the following requirements: (1) $N = V$; (2) $v^G(S) = |S| - \gamma(G^S)$ for[4] all $S \in 2^V \setminus \{\emptyset\}$.

Theorem 4.9 *Let (N, v) be a balanced minimum coloring game. Then the following conditions are equivalent:*

(1) (N, v) has a stable core.

(2) (N, v) is vital extendable.

(3) Every singleton is exact w.r.t. (N, v).

We postpone the proof of Theorem 4.9 and first prove the following lemma.

Lemma 4.10 *Let (N, v) be a minimum coloring game on the graph $G = (V, E)$. Then $\emptyset \neq S \subseteq N$ is vital if and only if S is independent.*

Proof: If S is independent, then $v(T) = |T| - 1$ for all $\emptyset \neq T \subseteq S$. Let $x \in \mathbb{R}^S$ be defined by $x_i = \frac{|S|-1}{|S|}$ for all $i \in S$. Then $x(S) = v(S)$ and $x(T) > v(T)$ for all $\emptyset \neq T \subsetneqq S$ so that S is vital. Conversely, assume now that S is a coalition with $v(S) < |S| - 1$. It remains to show that S is not vital. Let $c : S \to \mathbb{R}$ be a minimal coloring of G^S and let $i \in S$. Then $T = \{j \in S \mid c(j) \neq c(i)\} \neq \emptyset$ and c_T (the restriction of c to T) is a minimal coloring of G^T. We conclude that $v(S) = v(T) + v(S \setminus T)$ and, hence, that S is not vital. **q.e.d.**

[4] Bietenhader and Okamoto (2006) consider the "cost" game whose coalition function simply assigns $\gamma(G^S)$ to any coalition S. We consider the "cost sharing" game instead so that, e.g., the definition of the core remains unchanged.

Proof Theorem 4.9: By Proposition 2.1, Theorem 3.3 and (3.2) it remains to show that (3) implies (2). Let S be a vital coalition and $y \in \mathcal{C}(N,v)$. For any $j \in N$, $v(N) - v(N \setminus \{j\}) + y(N \setminus \{j\}) \geq v(N) = y(N) = y_j + y(N \setminus \{j\})$. We conclude that $y_j \leq v(N) - v(N \setminus \{j\})$. As $v(N) - v(N \setminus \{j\}) \leq 1$ for any minimum coloring game, we conclude that $y_j \leq 1$. Now, let $i \in S$. By (3), there exists $x \in \mathcal{C}(N,v)$ with $x_i = v(\{i\}) = 0$. By Lemma 4.10, $v(S) = |S| - 1$. Therefore, $x_j = 1$ for all $j \in S \setminus \{i\}$ and convexity of the core completes the proof. **q.e.d.**

Let (N,v) be a balanced game. Schmeidler (1972) presents a simple necessary and sufficient condition for exactness of a singleton $\{i\}$, $i \in N$: The singleton $\{i\}$ is exact if and only if

$$v(\{i\}) = \max \left\{ \sum_{S \subsetneqq N} \delta^S v(S) - \delta^N v(N) \,\middle|\, \delta^T \geq 0 \ \forall T \subseteq N, \sum_{S \subsetneqq N} \delta^S \chi^S - \delta^N \chi^N = \chi^{\{i\}} \right\}. \tag{4.5}$$

Theorem 4.9 and (4.5) imply the following corollary.

Corollary 4.11 *A balanced minimum coloring game (N,v) has a stable core if and only if, for each $i \in N$, the following implication is valid:*

$$\delta^T \geq 0 \ \forall T \subseteq N, \sum_{S \subsetneqq N} \delta^S \chi^S - \delta^N \chi^N = \chi^{\{i\}} \Longrightarrow \sum_{S \subsetneqq N} \delta^S v(S) \leq \delta^N v(N).$$

Note that, by Corollary 3.12, Corollary 4.11 is also true for balanced games that satisfy (3.6).

Remark 4.12 Let (N,v) be the minimum coloring game on the graph G.

(1) If G is perfect, then it is easy to show that, for any maximum clique S, $\chi^{N \setminus S} \in \mathcal{C}(N,v)$. Hence, (N,v) is totally balanced. In fact, according to Deng, Ibaraki, Nagamochi, and Zang (2000), the converse is also true: If the minimum coloring game on G is totally balanced, then G is perfect.

(2) If G is perfect and $x \in \mathcal{C}(N,v)$, then $x_i = 1$ for each vertex $i \in N$ that does not belong to any maximum clique. In fact, Okamoto (2003) shows that $\mathcal{C}(N,v)$ is the convex hull of the $\chi^{N \setminus S}$ of all maximum cliques S.

We now use the foregoing theorem to reprove the following result of Bietenhader and Okamoto (2006, Theorem 4.1) in a short way.

Theorem 4.13 *Let $G = (V,E)$ be a perfect graph. Then the minimum coloring game (N,v) on G has a stable core if and only if every $i \in N$ belongs to a maximum clique of G.*

Proof: By (1) of Remark 4.12, (N, v) is (totally) balanced. If (N, v) has a stable core, then by Proposition 2.1, for any $i \in N$, there exists $x \in \mathcal{C}(N, v)$ such that $x_i = 0$. By (2) of Remark 4.12, i belongs to a maximum clique. For the other direction, assume now that each edge belongs to a maximum clique. Let S be a vital coalition and let $i \in S$. Choose any maximum clique T with $i \in T$. By Lemma 4.10, $T \cap S = \{i\}$. By (1) of Remark 4.12, the vector $z^i \in \mathbb{R}^S$ given by $z^i_i = 0$ and $z^i_j = 1$ for $j \in S \setminus \{i\}$ is the restriction of a core element to S. Also, $\mathcal{C}(S, v)$ is the convex hull of the $z^i, i \in S$. Hence, S is extendable and the proof is complete by Theorem 4.9. **q.e.d.**

Example 4.14 Let G_3 be the perfect graph that consists of two disjoint triangles that are connected via one edge and may be found in Bietenhader and Okamoto (2006, p. 424). (For a characterization of extendable minimum coloring games on perfect graphs see their Theorem 4.2.) So, $G_3 = (N, E)$, where

$$N = \{1, \ldots, 6\} \text{ and } E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 4\}\}.$$

Let $v = v^{G_3}$ and $T = \{1, 4\}$. Then $x(T) \geq 1$ for any $x \in \mathcal{C}(N, v)$. Note that $(0, 0, 0, 1, 1, 1) \in \mathcal{C}(N, v)$ so that $S = \{1, 2, 4\}$ is exact. As $(0, 1, 0) \in \mathcal{C}(S, v)$, S is not extendable.

References

Bietenhader, T., and Y. Okamoto (2006): "Core stability of minimum coloring games", *Mathematics of Operations Research*, 31, 418 – 431.

Biswas, A. K., T. Parthasarathy, J. A. M. Potters, and M. Voorneveld (1999): "Large cores and exactness", *Games and Economic Behavior*, 28, 1 – 12.

Bondareva, O. N. (1963): "Some applications of linear programming methods to the theory of cooperative games", *Problemi Kibernitiki*, 10, 119–139.

Deng, X., T. Ibaraki, and H. Nagamochi (1999): "Algorithmic aspects of the core of combinatorial optimization games", *Mathematics of Operations Research*, 24, 751 – 766.

Deng, X., T. Ibaraki, H. Nagamochi, and W. Zang (2000): "Totally balanced combinatorial optimization games", *Mathematical Programming*, 87, 441 – 452.

Fang, Q., R. Fleischer, J. Li, and X. Sun (2007): "Algorithms for core stability, core largeness, exactness, and extendability of flow games", in *Computing and Combinatorics*, Vol. 4598 of *Lecture Notes in Computer Science*, pp. 440 – 447, Berlin and Heidelberg, Germany. Springer.

Ford, Jr, L. R., and D. R. Fulkerson (1962): *Flows in Networks*. Princeton University Press, New Jersey.

Gillies, D. B. (1959): "Solutions to general non-zero-sum games", in *Contributions to the Theory of Games IV*, ed. by A. W. Tucker, and R. D. Luce, Vol. 40 of *Annals of Mathematical Studies*, pp. 47 – 85, Princeton, N.J. Princeton University Press.

Kalai, E., and E. Zemel (1982): "On totally balanced games and games of flow", *Mathematics of Operations Research*, 7, 476 – 478.

Kikuta, K., and L. S. Shapley (1986): "Core-stability in n-person games", Mimeo.

Okamoto, Y. (2003): "Fair cost allocations under conflicts – a game-theoretic point of view –", in *Algorithms and Computation*, Vol. 2906 of *Lecture Notes in Computer Science*, pp. 686 – 695, Berlin and Heidelberg, Germany. Springer.

Raghavan, T. E. S., and P. Sudhölter (2005): "The Modiclus and Core Stability", *International Journal of Game Theory*, 33, 467 – 478.

Rosenmüller, J. (2000): *Games Theory: Stochastics, Information, Strategies and Cooperation*. Theory and Decision Library, Series C, Kluwer Academic Publishers, Boston/Dordrecht/London.

Schmeidler, D. (1972): "Cores of exact games", *Journal of Mathematical Analysis and Applications*, 40, 214 – 225.

Shapley, L. S. (1967): "On balanced sets and cores", *Naval Research Logistics Quarterly*, 14, 453 – 460.

——— (1971): "Cores of convex games", *International Journal of Game Theory*, 1, 11 – 26.

Shapley, L. S., and M. Shubik (1972): "The assignment game I: The core", *International Journal of Game Theory*, 2, 111 – 130.

Solymosi, T., and T. E. S. Raghavan (2001): "Assignment games with stable core", *International Journal of Game Theory*, 30, 177 – 185.

Sun, X., and Q. Fang (2007): "Core stability and flow games", in *Discrete Geometry, Combinatorics and Graph Theory*, Vol. 4381 of *Lecture Notes in Computer Science*, pp. 189 – 199, Berlin and Heidelberg, Germany. Springer.

van Gellekom, J. R. G., J. A. M. Potters, and J. H. Reijnierse (1999): "Prosperity properties of TU-games", *International Journal of Game Theory*, 28, 211 – 227.

von Neumann, J., and O. Morgenstern (1953): *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ, third edn.