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On the measurement of the (multidimensional) inequality of health distributions

by

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On the measurement of the (multidimensional) inequality of health distributions*

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Abstract

Health outcomes are often described according to two dimensions: quality of life and quantity of life. We analyze the measurement of inequality of health distributions referring to these two dimensions. Our analysis relies on a novel treatment of the quality-of-life dimension, which might not have a standard mathematical structure. We single out two families of (absolute and relative) multidimensional health inequality indices, inspired by the classical normative approach to income inequality measurement. We also discuss how to extend the analysis to deal with the related problem of health deprivation measurement in this setting.

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1 Introduction

Equity is typically recognized as a relevant policy objective in the health care field, as witnessed by the chapters devoted to this topic in the two volumes of the *Handbook of Health Economics* (e.g., Wagstaff and van Doorslaer, 2000; Williams and Cookson, 2000; Fleurbaey and Shokkaert, 2012). Most of the literature on health inequality is inspired by the standard economic approach to the measurement of income inequality (e.g., Bleichrodt and van Doorslaer, 2006).[1] However, such standard framework is unidimensional and, as such, not sufficiently rich to analyze relevant aspects in the health care field. More precisely, it is frequently argued that the benefit a patient derives from a particular health care intervention is defined according to two dimensions: quality of life and quantity of life (e.g., Pliskin et al., 1980). Therefore, it seems that an appropriate approach to the measurement of health inequality should take into account the multidimensionality of the problem.

Research on multidimensional economic inequality has played an important role since the seminal articles by Kolm (1977) and Atkinson and Bourguignon (1982).[2] The recent awake of interest in multidimensional inequality is driven by the recognition that univariate indices of income inequality provide an inadequate basis for comparing the inequality of well-being within and between populations. Multivariate generalizations of the procedures used to construct univariate inequality indices from social evaluation abound in the literature.[3] Nevertheless, all these procedures rely on a common assumption stating that all dimensions have an Euclidean structure (i.e., they lie within the space of real numbers), which eases their formulation. This may, however, not be totally justified in the context of health; in particular, if we endorse the view that the benefit a patient derives from a particular health care intervention is defined according to the two dimensions mentioned above. To wit, whereas the "quantity of life" is naturally represented by a real number (years), objections might be raised concerning the "quality of life", which needs a richer description than what can be represented by a single number (or even by vectors of numbers).

The model we analyze here builds onto the models of Østerdal (2005) and Hougaard, Moreno-Ternero and Østerdal (2013), in which we assume that each individual in the pop-

[1]The literature has also paid considerable attention to socioeconomic health inequalities focussing on the association between income and health (e.g., Wagstaff, Paci and van Doorslaer, 1991). We do not deal with this issue here.

[2]See Weymark (2006) for an excellent survey.

[3]Again, see Weymark (2006) for further details.

ulation is described by a duplet indicating the level achieved in quality of life and quantity of life. The set of possible "quality of life" states is defined generally enough so that no specific mathematical structure is imposed on it. We only assume that such a set contains a superior element, referred to as *perfect health*. In doing so, our model is able to accommodate, not only standard modeling assumptions endorsing an Euclidean structure in each dimension, but also recent approaches assuming that health data is reported categorically (e.g., Allison and Foster, 2004; Abu-Naga and Yalcin, 2008).[4]

The aim of this paper is then to develop a normative foundation to the measurement of (multidimensional) health inequality in such a general context. In particular, following the analysis in Hougaard, Moreno-Ternero and Østerdal (2013), we impose a number of basic assumptions on social preferences over health profiles. Thanks to these assumptions, we derive a family of population health evaluation functions (PHEFs), which rank health profiles according to social desirability, that depend on the distribution of healthy years equivalents (HYEs).

Adding an invariance property with respect to re-scaling of life years, the above family of PHEFs is restricted to impose a power concave function of HYEs. Based on the resulting family, we derive the so-called *Atkinson family of multidimensional indices of health inequalities*, which inherit the property of being invariant to arbitrary re-scaling of life years (i.e., they all are relative indices of inequality).

Instead of the previous invariance property, we can add an axiom stating that if all agents have perfect health then the evaluation of the profile will be invariant to positive translation of the distribution of life spans. If so, we would obtain a PHEF which is exponentially concave in HYEs. Based on this, we derive the so-called *Kolm-Pollak family of multidimensional indices of health inequalities*, which inherit the property of being invariant to adding a common constant to all life years (i.e., they all are absolute indices of inequality).

As the above PHEFs are obtained in a fixed-population setting, we also extend the framework in Hougaard, Moreno-Ternero and Østerdal (2013) to deal with a variable-population setting. The resulting families of inequality indices in this setting coincide with those derived in the fixed-population setting mentioned above.

Finally, we also explore how to extend the previous normative analysis of inequality measurement to deal with the related problem of health deprivation measurement.

The rest of the paper is organized as follows. In Section 2, we set up the preliminaries, mostly

[4]They deal with non-Euclidean spaces but with more mathematical structure than ours, as they assume the existence of an ordering of health states.

summarizing the model and basic notions in Hougaard, Moreno-Ternero and Østerdal (2013). In Section 3, we present our contribution to deal with the measurement of multidimensional health inequality, and further extensions to the model are elaborated in Section 4. We conclude in Section 5. For a smooth passage we defer the proofs to an appendix.

2 The preliminaries

Let us identify the population (society) with a fixed finite set of individuals $N = \{1, ..., n\}$, $n \geq 3$. The health of each individual in the population is described by a duplet indicating the level achieved in two parameters: quality of life and quantity of life.[5] Assume that there exists a set of possible health states, A, defined generally enough to encompass all possible health states for everybody in the population. Quantity of life is simply described by a set of nonnegative real numbers, $T \subset \mathbb{R}$. In what follows, we assume that $T = [0, +\infty)$. Formally, let $h_i = (a_i, t_i) \in A \times T$ denote the health duplet of individual i.[6] A population health distribution (or, simply, a health profile) $h = [h_1, \ldots, h_n] = [(a_1, t_1), ..., (a_n, t_n)]$ specifies the health duplet of each individual in society. Denote the set of all possible health profiles by H, i.e., H is the n-Cartesian product of the set $A \times T$. Even though we do not impose a specific mathematical structure on the set A, we assume that it contains a specific element, a_*, referred to as *perfect health* (typically understood as absence of abnormal conditions) and which is identified by the policy maker, as a "superior" state.

The policy maker's preferences (or social preferences) over health profiles are expressed by a preference relation $\succsim$, to be read as "at least as preferred as". As usual, $\succ$ denotes strict preference and $\sim$ denotes indifference. Assume the relation $\succsim$ is a weak order, i.e., it is complete (for each pair of health profiles h, h', either $h \succsim h'$, or $h' \succsim h$, or both) and transitive (if $h \succsim h'$ and $h' \succsim h''$ then $h \succsim h''$).

A *population health evaluation function* (PHEF) is a real-valued function $P : H \to \mathbb{R}$. We say that P represents $\succsim$ if

$$P(h) \geq P(h') \Leftrightarrow h \succsim h',$$

for each $h, h' \in H$. Note that if P represents $\succsim$ then any strictly increasing transformation of

[5]Here, as in much of the literature on inequality measurement, it is assumed that the population is homogeneous in the sense that individuals do not differ in welfare-relevant characteristics other than the attributes that are the focus of the analysis.

[6]For ease of exposition, we establish the notational convention that $h_S \equiv (h_i)_{i \in S}$, for each $S \subset N$.

P will also do so.

We now list a set of basic axioms for social preferences that we shall endorse in this paper.[7] *Anonymity* (in short, ANON) says that the evaluation of the population health should depend only on the list of quality-quantity duplets, not on who holds them. *Separability* (in short, SEP) says that if the distribution of health in a population changes only for a subgroup of agents in the population, the relative evaluation of the two distributions should only depend on that subgroup. *Continuity* (in short, CONT) says that, for fixed distributions of health states, small changes in lifetimes should not lead to large changes in the evaluation of the population health distribution. *Perfect health superiority* (in short, PHS) says that replacing the health status of an agent by that of perfect health, ceteris paribus, cannot worsen the evaluation of the population health. *Time monotonicity at perfect health* (in short, TMPH) says that if each agent is at perfect health, increasing the time dimension is strictly better for society. *Positive lifetime desirability* (in short, PLD) says that society improves if any agent moves from zero lifetime to positive lifetime (for a given health state). Finally, the *social zero condition* (in short, ZERO) says that if an agent gets zero lifetime, then her health state does not influence the social desirability of the health distribution. Formally,

ANON: $h \sim h_\pi$ for each $h \in H$, and each $\pi \in \Pi^N$.

SEP: $\left[h_S, h_{N\setminus S}\right] \succsim \left[h'_S, h_{N\setminus S}\right] \Leftrightarrow \left[h_S, h'_{N\setminus S}\right] \succsim \left[h'_S, h'_{N\setminus S}\right]$, for each $S \subseteq N$, and $h, h' \in H$.

CONT: Let $h, h' \in H$, and $h^{(k)}$ be a sequence in H such that, for each $i \in N$, $h_i^{(k)} = (a_i, t_i^{(k)}) \to (a_i, t_i) = h_i$. If $h^{(k)} \succsim h'$ for each k then $h \succsim h'$, and if $h' \succsim h^{(k)}$ for each k then $h' \succsim h$.

PHS: $[(a_*, t_i), h_{N\setminus\{i\}}] \succsim h$, for each $h = [h_1, \ldots, h_n] \in H$ and $i \in N$.

TMPH: If $t_i \geq t'_i$, for each $i \in N$, with at least one strict inequality, then $[(a_*, t_1), \ldots, (a_*, t_n)] \succ [(a_*, t'_1), \ldots, (a_*, t'_n)]$.

PLD: $h \succsim [h_{N\setminus\{i\}}, (a_i, 0)]$, for each $h = [h_1, \ldots, h_n] \in H$ and $i \in N$.

ZERO: For each $h \in H$ and $i \in N$ such that $t_i = 0$, and $a'_i \in A$, $h \sim [h_{N\setminus\{i\}}, (a'_i, 0)]$.

In what follows, we refer to the set of axioms introduced above as our *basic structural axioms*; in short, **BASIC**.

[7] The reader is referred to Hougaard, Moreno-Ternero and Østerdal (2013) for further discussion of the axioms.

In Hougaard, Moreno-Ternero and Østerdal (2013) it is demonstrated that BASIC implies the existence of a PHEF that depends on the healthy years equivalents (in short, HYEs) only. In what follows, we show how this fact can be used to establish indices of multidimensional health inequality.

3 Indices of multidimensional health inequality

3.1 The Atkinson-Kolm-Sen approach

We first explore the application to our setting of the so-called "Atkinson-Kolm-Sen approach". This approach, originated in the seminal articles of Atkinson (1970) and Kolm (1969) on univariate inequality measurement, and later popularized by Sen (1973), basically amounts to constructing an inequality index from a social evaluation ordering. Now, in contrast with those seminal contributions in the unidimensional case, we shall derive first such a social welfare function (or social evaluation ordering) axiomatically, rather than proposing it exogenously. In order to do that, let us add two more axioms to the basic structural axioms from the previous section.

The first one, *time scale independence at perfect health* (in short, TSIPH), is a specific form of homotheticity, a notion with a long tradition in the literature on income inequality measurement, and particularly embedded in the Atkinson-Kolm-Sen approach. It says that if all agents in society are enjoying the perfect health status, then the evaluation of this society will be invariant to a proportional scaling of the distribution of lifespans. Formally,

TSIPH: For each $c > 0$, and $h = [(a_*, t_i)_{i \in N}]$, $h' = [(a_*, t'_i)_{i \in N}]$,

$$h \succsim h' \Rightarrow [(a_*, ct_i)_{i \in N}] \succsim [(a_*, ct'_i)_{i \in N}].$$

The second one, *Pigou-Dalton transfer at perfect health* (in short, PDTPH), is a specific form of the Pigou (1912)-Dalton (1920) transfer principle, which is also deeply rooted in the literature on income inequality measurement, that hence refers to the distributional sensitivity of the social evaluation. It states that a hypothetical progressive transfer of lifespans between two agents at perfect health would be welcomed.

Formally,

PDTPH: For each $h = [(a_*, t_k)_{k\in N}]$, and $i, j \in N$, such that $t_i \neq t_j$,

$$\left[\left(a_*, \frac{t_i + t_j}{2}\right), \left(a_*, \frac{t_i + t_j}{2}\right), h_{N\setminus\{i,j\}}\right] \succ h.$$

It turns out, as the next result shows, that the so-called *power (concave) HYE* PHEFs, formally defined next, are characterized by the axioms described above. To wit,

$$P^{ph}[h_1, \ldots, h_n] = P^{ph}[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} f(a_i, t_i)^\gamma, \tag{1}$$

where $\gamma \in (0, 1)$, and $f : A \times T \to T$ is a (continuous with respect to its second variable) function indicating the HYEs for each individual, i.e., for each $h = [h_1, \ldots, h_n] = [(a_1, t_1), \ldots, (a_n, t_n)] \in H$, and each $i \in N$,

$$h \sim [(a_*, f(a_i, t_i))_{i\in N}].$$

Theorem 1 *(Hougaard, Moreno-Ternero and Østerdal, 2013) The following statements are equivalent:*

1. $\succsim$ *is represented by a PHEF satisfying* (1)*.*
2. $\succsim$ *satisfies BASIC, TSIPH and PDTPH.*

We then assume that the social welfare associated to a given health profile is evaluated by (1). It is worth mentioning that the representation for the social ordering we obtain is of the separable form. As such, it could be interpreted as being the value assigned to a health profile by a utilitarian social welfare function that uses the HYE function (which might look like a utility function) to convert individual health duplets into an interpersonally-comparable measure of health. That is indeed the starting motivation in the Atkinson-Kolm-Sen approach. Pursuing such approach from this point, we define the *equally-distributed-equivalent HYE*, $E(h)$, associated with a given health profile h, as the per capita HYE which, if distributed equally, would be indifferent to the actual health profile according to the social preference relation $\succsim$. Formally, $E^{ph}(h)$ is defined implicitly by

$$[(a_*, E^{ph}(h)), \ldots, (a_*, E^{ph}(h))] \sim h \sim [(a_*, f(a_i, t_i))_{i\in N}], \tag{2}$$

for each $h \in H$. By (1), it follows that

$$E^{ph}(h) = \left(\frac{1}{n}\sum_{i=1}^{n} f(a_i, t_i)^\gamma\right)^{1/\gamma},$$

for each $h \in H$.

Therefore, letting $\mu^f(h)$ denote the mean of the distribution of HYEs associated with h, the *Atkinson family of multidimensional health inequality indices*, associated to any social ordering $\succsim$ satisfying BASIC, TSIPH and PDTPH, would be given by

$$I^A(h) = 1 - \frac{E^{ph}(h)}{\mu^f(h)} = 1 - \frac{\left(\frac{1}{n}\sum_{i=1}^n f(a_i,t_i)^\gamma\right)^{1/\gamma}}{\frac{1}{n}\sum_{i=1}^n f(a_i,t_i)} = 1 - \left(\frac{1}{n}\sum_{i=1}^n \left(\frac{n \cdot f(a_i,t_i)}{\sum_{i=1}^n f(a_i,t_i)}\right)^\gamma\right)^{1/\gamma}. \quad (3)$$

The previous family is defined by means of the parameter $\gamma \in (0,1)$, which can be interpreted as the degree of inequality aversion, or the relative sensitivity to transfers at different HYE levels, reflected by the index. As γ rises, more weight is given to transfers at the lower end of the distribution (of HYEs), and less weight to transfers at the top. Note that the index (3) is a relative inequality index in the sense that it is invariant to proportional changes in life years when all agents enjoy perfect health.

It is worth mentioning that a more specific expression for the Atkinson family could be given in the so-called *Quality-Adjusted-Life-Years* (in short, QALYs) case, i.e., the case in which $f(a_i,t_i) = q(a_i)t_i$, for each $(a_i,t_i) \in A \times T$, where $q : A \to [0,1]$ is a function satisfying $0 \le q(a_i) \le q(a_*) = 1$, for each $a_i \in A$.[8]

3.2 The Kolm-Pollak approach

As mentioned above, the Atkinson family of multidimensional health inequality indices, just presented, is a family of relative indices of inequality. In other words, each index within the family is invariant to a scaling of HYEs. This is a consequence of the TSIPH axiom we have imposed on the underlying social ordering. One might wonder if an alternative family could be constructed so that each index within the family would be invariant instead to an increase or decrease of all of its variables by a common amount. We pursue such an aim in the next lines, by following the seminal route proposed by Kolm (1969) and Pollak (1971). To do so, we need to formalize first the following counterpart to the TSIPH axiom. In words, *time translatability at perfect health* (in short, TTRPH), says that if all agents in society are enjoying the perfect health status, then the evaluation of this society will be invariant to a positive translation of the distribution of lifespans. Formally,

[8]It turns out that the corresponding social ordering could be characterized by replacing TSIPH in Theorem 1 with a stronger axiom requiring time scale invariance at any health state. QALYs constitute the standard currency in the methodology of cost-utility analyses, probably the most widely accepted methodology in the economic evaluation of health care nowadays (e.g., Drummond et al., 2005).

TTRPH: For each $c > 0$, and $h = [(a_*, t_i)_{i \in N}]$, $h' = [(a_*, t'_i)_{i \in N}]$,

$$h \succsim h' \Rightarrow [(a_*, t_i + c)_{i \in N}] \succsim [(a_*, t'_i + c)_{i \in N}].$$

It turns out, as the next result shows, that the so-called *exponential (concave) HYE* PHEFs, formally defined next, are characterized when we replace the TSIPH axiom by this one in the statement of Theorem 1. To wit,

$$P^{eh}[h_1, \ldots, h_n] = P^{eh}[(a_1, t_1), \ldots, (a_n, t_n)] = -\sum_{i=1}^{n} e^{\gamma f(a_i, t_i)}, \tag{4}$$

where $\gamma \in \mathbb{R}_{--}$, and $f : A \times T \rightarrow T$ is a function indicating the HYEs for each individual, as described above.

Theorem 2 *The following statements are equivalent:*

1. $\succsim$ *is represented by a PHEF satisfying* (4)*.*

2. $\succsim$ *satisfies BASIC, TTRPH and PDTPH.*

Consequently, we now assume that the social welfare associated to a given health profile is evaluated by (4). Thus, it follows that the equally-distributed-equivalent HYE function is now given by

$$E^{eh}(h) = \frac{1}{\gamma} \ln \left[\frac{1}{n} \sum_{i=1}^{n} e^{\gamma f(a_i, t_i)} \right].$$

Therefore, the *Kolm-Pollak family of multidimensional health inequality indices*, associated to any social ordering $\succsim$ satisfying BASIC, TTRPH and PDTPH, would be given by[9]

$$I^{KP} = \mu^f(h) - E^{eh}(h) = -\frac{1}{\gamma} \ln \left[\frac{1}{n} \sum_{i=1}^{n} e^{\gamma \left(f(a_i, t_i) - \frac{\sum_{k=1}^{n} f(a_k, t_k)}{n} \right)} \right]. \tag{5}$$

As in the case of the Atkinson family, the Kolm-Pollak family of multidimensional health inequality indices is defined by means of the parameter $\gamma \in \mathbb{R}_{--}$, which can also be interpreted as the degree of inequality aversion. Now, in contrast with the Atkinson family, each index (5) is an absolute inequality index in the sense that it is invariant to adding the same number of life years to all agents, in situations where they all enjoy perfect health.

[9]This class of inequality indices was introduced by Kolm (1969) in the univariate case. In consumer theory, the same functional form was shown by Pollak (1971) to characterize the additive utility functions that have linear Engel curves. Hence the name of the family.

Finally, as in the case of the Atkinson family, it is worth mentioning that a more specific expression for the Kolm-Pollak family could be given in the QALY case, i.e., the case in which $f(a_i, t_i) = q(a_i)t_i$, for each $(a_i, t_i) \in A \times T$, where $q : A \to [0, 1]$ is a function satisfying $0 \leq q(a_i) \leq q(a_*) = 1$, for each $a_i \in A$.

4 Further insights

4.1 Variable population

The analysis in the previous section has been made in a fixed-population setting. There are many instances in which a variable-population setting is more suitable. An obvious point in case is that in which one is interested in comparisons across countries or regions. Because population size and individual identities may be different, this is not a straightforward issue. Nevertheless, the next lines convey a plausible way to move from the fixed-population setting described in the previous sections to a variable-population setting.[10]

Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, the set of positive integer numbers, with generic elements N and M. A population health distribution is now described by a population $N \in \mathcal{N}$, and the corresponding health profile $h = [h_i]_{i \in N} = [(a_i, t_i)]_{i \in N}$ specifying the health duplet of each individual in society. Let $\mathcal{H}^N$ be the set of (health) distributions with population N and $\mathcal{H} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{H}^N$. With a slight abuse of notation, we still denote by $\succsim$ the policy maker's preferences over health distributions of different populations (that might have different sizes).

We now consider the following standard axiom in the literature, which involves a replication operation that is often considered in economic modelling. Let $N \in \mathcal{N}$ and k be a positive integer. By a k-replica of a health distribution with population set N we mean a distribution in which each member of N has $k - 1$ clones, each of whom is endowed with a health duplet equal to his. Formally, given $h \in \mathcal{H}^N$, and if M designates the set of agents in the replica problem and $h' \in \mathcal{H}^M$ the replica problem, we have $N \subset M$, $|M| = k|N|$, and there is a partition of M into $|N|$ groups of k agents indexed by $i \in N$, $(N^i)_{i \in N}$, such that for each $i \in N$ and each $j \in N^i$, $h'_j = h_i$. The requirement of *replication invariance* (in short, RI) says that a k-replica of a distribution is indifferent to such distribution.

RI: For each $N \in \mathcal{N}$, each $h = [(a_i, t_i)_{i \in N}] \in \mathcal{H}^N$, each $M \supset N$, and each $h' \in \mathcal{H}^M$, if h' is a

[10] The reader is referred to Blackorby, Bossert and Donaldson (2005) for a thorough account of population issues in welfare economics and related areas.

k-replica of h then $h \sim h'$.

It turns out, as the next results show, that the *average* versions of the (power and exponential) concave HYE PHEFs considered in Section 3, and formally defined next, can be characterized thanks to this axiom, building onto the analysis of Section 3. To wit,

$$P^{aph}[h] = P^{aph}[(a_i, t_i)_{i \in N}] = \frac{1}{|N|} \sum_{i \in N} f(a_i, t_i)^{\gamma}, \tag{6}$$

where $\gamma \in (0, 1)$, and f is constructed as in (1).

$$P^{aeh}[h] = P^{aeh}[(a_i, t_i)_{i \in N}] = -\frac{1}{|N|} \sum_{i \in N} e^{\gamma f(a_i, t_i)}, \tag{7}$$

where $\gamma \in \mathbb{R}_{--}$, and f is constructed as in (1).

The following results are obtained:

Theorem 3 *In the variable-population setting, the following statements are equivalent:*

1. $\succsim$ *is represented by a PHEF satisfying* (6), *for each* $h \in \mathcal{H}$.

2. $\succsim$ *satisfies BASIC, TSIPH and PDTPH, for each population* N, *and RI.*

Theorem 4 *In the variable-population setting, the following statements are equivalent:*

1. $\succsim$ *is represented by a PHEF satisfying* (7), *for each* $h \in \mathcal{H}$.

2. $\succsim$ *satisfies BASIC, TTRPH and PDTPH, for each population* N, *and RI.*

If we now assume that the social welfare associated to a given health profile is evaluated by (6), then the equally-distributed-equivalent HYE would be exactly defined as in (2). Therefore the corresponding Atkinson family of multidimensional health inequality indices, in this variable population setting, would be exactly defined as in (3). Likewise, if we assume that the social welfare associated to a given health profile is evaluated by (7), then the corresponding Kolm-Pollak family of multidimensional health inequality indices, in this variable population setting, would be exactly defined as in (5). It is worth mentioning that each index within both families would inherit the property of replication invariance from the underlying (variable-population) PHEFs.

4.2 Health deprivation measurement

The concept of deprivation can be traced back to Runciman (1966), who formulated the idea that a person's feeling of deprivation in a society arises out of comparing its situation with those who are better off. This intuition was early used by Sen (1976) and Yitzhaki (1979), among others, in order to obtain a measure of deprivation in the unidimensional space of income. Now, as pointed out by philosophers (e.g., Rawls, 1971) and economists (e.g., Ravallion, 1996) alike, income is often not a perfect indicator of deprivation. Consequently, attention has recently shifted to study poverty (deprivation) in a multidimensional framework (e.g., Decancq, Fleurbaey and Maniquet, 2013). The concern for cumulative deprivation, with the poorest being at the same time less healthy, might constitute one of the main motivations for the sizable scientific literature and the rapidly increasing policy interest in socio-economic inequities in health (e.g., Fleurbaey and Shokkaert, 2012). Our main aim in this section is, precisely, to extend our analysis from the previous sections to deal with health deprivation measurement in the multidimensional framework we consider in this work.

It is well known that the normative approach to inequality measurement can be extended to poverty measurement, as pioneered by Sen (1976) and Blackorby and Donaldson (1980). Therefore, we can build onto the analysis of Section 3 to derive ethical indices for the measurement of health deprivation (poverty) in our setting.

More precisely, suppose that the deprivation threshold is a pre-specified level of healthy year equivalents z, and that the set of the *deprived* (those people for whom the healthy years equivalents are at or below the deprivation threshold) is $Z(z)$.[11] Let $n(z)$ denote the cardinality of $Z(z)$.

A general relative health deprivation index may be defined as

$$Q = \frac{n(z)}{n}\frac{z - E(h_p)}{z} \tag{8}$$

where $E(h_p)$ is the representative HYE of the deprived, as measured by an arbitrary (homothetic) social evaluation function. Similarly, a general absolute deprivation index may be defined as

$$Q = n(z)(z - E(h_p)). \tag{9}$$

[11] Note that z might arise from different combinations of quality and quantity of life. In other words, agents can also be considered deprived for a deficit in only one of the two dimensions of health, and not necessarily both.

The previous deprivation indices are defined with respect to social evaluation functions defined over the deprived. Nevertheless, as shown by Blackorby and Donaldson (1980), they become ethically significant (i.e., they reflect ethical judgements for the whole society) when $E(\cdot)$ is completely (additively) separable. That is precisely the case of the functions $E^{ph}(\cdot)$ and $E^{eh}(\cdot)$ considered in Section 3. Consequently, the *Atkinson family of multidimensional health deprivation indices*, associated to any social ordering $\succsim$ satisfying BASIC, TSIPH and PDTPH, given by

$$Q^A = \frac{n(z)}{n}\frac{z - E^{ph}(h_p)}{z} = \frac{n(z)z - \left(\sum_{i\in Z(z)} f(a_i,t_i)^{\gamma}\right)^{1/\gamma}}{nz}, \tag{10}$$

and the *Kolm-Pollak family of multidimensional health deprivation indices*, associated to any social ordering $\succsim$ satisfying BASIC, TTRPH and PDTPH, given by

$$Q^{KP} = n(z)(z - E^{eh}(h_p)) = n(z)\left(z - \frac{1}{\gamma}\ln\left[\frac{1}{n(z)}\sum_{i\in Z(z)} e^{\gamma f(a_i,t_i)}\right]\right), \tag{11}$$

are (relative and absolute, respectively) deprivation indices with ethical content.

5 Discussion

We have explored in this paper the measurement of inequality of health distributions referring to two dimensions: quality of life and quantity of life. Our analysis relies on a novel treatment of the quality-of-life dimension, which might not have a standard mathematical structure. We have mostly concentrated on the normative approach to the problem, adapting to our setting the classical formulation in the literature on income inequality measurement credited to Atkinson, Kolm and Sen. As a result, we have singled out several proposals for inequality indices to evaluate health distributions in our setting. They all share the feature of relying on the so-called HYE functions to move from our multidimensional setting of health to a unique dimension over which we could use standard indices that are so popular in the literature on income inequality measurement.[12] As such, our contribution is reminiscent of the two-stage aggregation procedure proposed by Maasoumi (1986) to deal with multi-attribute social evaluation and, in particular, to construct multi-attribute inequality indices. More precisely, in the first stage of such procedure, and for each individual, a utility function is used to aggregate the individual's allocation of the several attributes into a summary measure of well-being. This

[12] This was not an arbitrary choice, but rather a consequence of imposing some basic structural assumptions on the social welfare ordering over health distributions.

initial aggregation results in a unidimensional distribution of utilities. In the second stage, a univariate inequality index is applied to this distribution to obtain a measure of the inequality in the distribution matrix.[13]

Our contribution is also reminiscent to Tsui's (1995) multi-attribute (Euclidean) generalization of the Atkinson class of (univariate) indices. As in our case, Tsui's (1995) generalization is also identified axiomatically. In contrast, his axioms are formulated in terms of a social evaluation function, rather than in terms of the underlying binary relation. In any case, our differentiating aspect is to generalize the Atkinson (and the Kolm-Pollak) family to a multidimensional, but not necessarily Euclidean, space.

To conclude, it is worth mentioning that our analysis throughout this paper has been framed without making assumptions about individual preferences over length and quality of life. This is in contrast with the more standard approach in the health economics literature, where a given relationship is assumed between quality and quantity of life at the individual level, entailing the existence of possibly asymmetric individual utility functions (e.g., Østerdal, 2005). Nevertheless, information over such individual preferences is sometimes not available, either for practical or ethical reasons. Furthermore, even though there is a vast literature on assessing individual preferences over health profiles (see, for instance, Dolan (2000) and literature cited therein) recurrent criticisms are made to each of the approaches in that literature. Therefore, in situations where solid information concerning individual preference is absent, but social preferences are more easily available, our approach offers a viable alternative for inequality measurement.

6 Appendix: Proofs of the Theorems

For the proof of Theorem 2, we focus on its non-trivial implication, i.e., $2 \rightarrow 1$. Formally, assume $\succsim$ satisfies BASIC, TTRPH and PDTPH.

Consider the following family:

$$P^s[h_1, \ldots, h_n] = P^s[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} g\left(f(a_i, t_i)\right), \tag{12}$$

where $f : A \times T \rightarrow \mathbb{R}_+$ and $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ are such that:

- f is continuous with respect to its second variable,

[13]Maasoumi (1986) also proposed functional forms for these aggregators. For the second-stage aggregator, he suggested using a member of the class of generalized entropy inequality indices. This class of indices contains the Atkinson class and all of the indices that are ordinally equivalent to some member of the Atkinson class.

- $0 \leq f(a_i, t_i) \leq t_i$, for each $(a_i, t_i) \in A \times T$,

- $h \sim [(a_*, f(a_i, t_i))_{i \in N}]$, for each $h = [(a_1, t_1), \ldots, (a_n, t_n)] \in H$, and

- g is a strictly increasing and continuous function.

By BASIC, $\succsim$ can be represented by a PHEF, P, satisfying (12) (see Theorem 1 in Hougaard, Moreno-Ternero and Østerdal, 2013). By TTRPH,

$$\sum_{i=1}^{n} g\left(f(a_i, t_i)\right) \geq \sum_{i=1}^{n} g\left(f(a'_i, t'_i)\right) \iff \sum_{i=1}^{n} g\left(f(a_i, t_i) + c\right) \geq \sum_{i=1}^{n} g\left(f(a'_i, t'_i) + c\right),$$

for each $h = [(a_1, t_1), \ldots, (a_n, t_n)] \in H$, $h' = [(a'_1, t'_1), \ldots, (a'_n, t'_n)] \in H$ and $c > 0$.

By a classical result (e.g., Moulin, 1988) whose inspiration can be traced back to Bergson and Samuelson (e.g., Burk, 1936; Samuelson, 1965), there are only three possible functional forms for P:

- $P[h_1, \ldots, h_n] = P[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} e^{\gamma f(a_i, t_i)}$, for some $\gamma > 0$,

- $P[h_1, \ldots, h_n] = P[(a_1, t_1), \ldots, (a_n, t_n)] = -\sum_{i=1}^{n} e^{\gamma f(a_i, t_i)}$, for some $\gamma < 0$,

- $P[h_1, \ldots, h_n] = P[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} f(a_i, t_i)$,

for each $i \in N$.

It is straightforward to show that the first and last functional forms cannot satisfy PDTPH. Altogether, we have that $P = P^{eh}$, as desired. $\square$

We now move to the proof of the remaining theorems. We focus on the proof of Theorem 3, where, again, we focus on its non-trivial implication, i.e., $2 \rightarrow 1$.[14] Formally, assume $\succsim$ satisfies BASIC, TSIPH and PDTPH, for each population N, and RI.[15] By Theorem 1, when restricted to $\mathcal{H}^N$, $\succsim$ is represented by a PHEF satisfying (1). Now, let $h = [(a_i, t_i)_{i \in N}]$ and $h' = [(a'_i, t'_i)_{i \in N'}]$ be two arbitrary distributions, and denote by n and n', respectively, the cardinalities of their corresponding populations. Then, by RI, transitivity, and the previous

[14] The proof of Theorem 4 is analogous to this one and, thus, we omit it.

[15] For ease of notation, we refer to h' in the definition of the RI axiom as $k \cdot h$.

statement,

$$\begin{aligned} h \succsim h' &\Leftrightarrow n' \cdot h \succsim n \cdot h' \\ &\Leftrightarrow P^{ph}[n' \cdot h] \geq P^{ph}[n \cdot h'] \\ &\Leftrightarrow n' \cdot \sum_{i \in N} f(a_i, t_i)^\gamma \geq n \cdot \sum_{i \in N'} f(a'_i, t'_i)^\gamma \\ &\Leftrightarrow \frac{\sum_{i \in N} f(a_i, t_i)^\gamma}{n} \geq \frac{\sum_{i \in N'} f(a'_i, t'_i)^\gamma}{n'}, \end{aligned}$$

as desired. □

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