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by

Ryan Tierney

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Department of Economics
Faculty of Business and Social Sciences
University of Southern Denmark
Campusvej 55, DK – 5230 Odense M
Denmark
Email: astni@sam.sdu.dk
https://www.sdu.dk/en/om_sdu/institutter_centre/oekonomiskinstitut

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INCENTIVES AND EFFICIENCY IN MATCHING WITH TRANSFERS: TOWARDS NONQUASILINEAR PACKAGE AUCTIONS

RYAN TIERNEY

University of Southern Denmark, Department of Economics, Campusvej 55, Odense M, Denmark 5230

Abstract. We study the package assignment model and its consequences for the model of matching with transfers. We show that on rich domains, strategy-proofness, joint monotonicity (of Barberà, Berga, and Moreno [*American Economic Review, 106 (2016)*]), anonymity in welfare, and continuity in welfare together imply *conditional efficiency*: the allocation cannot be improved by re-allocation of packages, *keeping packages intact.* Thus, rules are restricted to choosing, for each problem, a set of objects to distribute and a partitioning of these.

Labor markets are auctions with unit demand, once anonymity is modified to account for productivity differences. In this case, conditional efficiency is *no blocking* (by matched pairs), the core component of the standard solution concept of *stability.* Thus, while it is known that stable outcomes can be strategy-proof, we show that a component of stability is necessary for incentives.

These results are derived from the following result, also discovered here, on the restricted quasilinear domain: weak pairwise strategy-proofness, anonymity in welfare, and continuity in welfare imply no-envy.

JEL codes: C78, D44, D47

Key words: Assignment game, Package auctions, Strategy-proofness

In mechanism design and social choice theory, we frequently encounter tension between incentives and efficiency. In this manuscript, we find that a restricted form of efficiency is *implied* by dominant-strategy incentives. We study the package assignment model, wherein each agent is to be given a set of discrete objects and a monetary

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E-mail address: `RyanETierney@gmail.com`.

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transfer. We restrict attention to deterministic rules satisfying welfare-anonymity—ruling out dictatorial rules—and welfare-continuity. We then find several theorems in which an incentive constraint implies *conditional efficiency*: taking as given the set of packages allocated, they are distributed optimally.[1] Thus, by dropping efficiency and considering only incentives, we gain less freedom to design allocation rules than one might expect; the rule can decide only *which* sets of packages it will distribute at a given problem, it cannot decide to *whom* these are distributed.

We study both general and quasilinear domains of preferences. For the full domain, the incentive conditions we impose are strategy-proofness and a welfare monotonicity condition based on, but weaker than, Maskin monotonicity. For the quasilinear domain, the incentive condition is weak pairwise strategy-proofness. Most of the work is in showing the theorem for this restricted domain, as there are fewer manipulations available to the agents and thus fewer tools available for a uniqueness proof. The correspondence between the two theorems is then achieved via a strengthening of the theorem of Barberà, Berga, and Moreno (2016).

Two important applications of these findings are found in package sales in the presence of wealth effects, and in labor market matching. In the first application, the literature has seen a burst of interest in wealth effects in recent years. The results are sometimes surprising. Baisa and Burkett (2019) find that, if in addition to wealth effects there are also interdependent preferences, then the existence of *ex post* implementable and efficient English auctions depends on whether the setting is a sales auction or a procurement auction.[2]

Morimoto and Serizawa (2015) provided the first characterization of rules for selling heterogeneous items to non-quasilinear buyers. They find that the *minimal-price Walrasian* rules are the unique efficient and strategy-proof (deterministic) mechanisms that induce voluntary participation and do not provide subsidies. Showing the difficulty that wealth effects introduce, they restricted attention to the case when each buyer could obtain at most one item. Without this restriction, it is not guaranteed that minimal-price Walrasian rules are defined (Gul and Stacchetti (1999); Bikhchandani and Ostroy (2002)), but with it, these rules generalize the Vickrey-Clarke-Groves

[1] In the standard auction sense for quasilinear preferences or the Morimoto and Serizawa (2015) sense for general preferences.

[2] In non-quasilinear preferences, *valuations* are not well-defined.

schemes, and so make Morimoto and Serizawa's work the closest analogue to Holmström's (1979) characterization on the general domain.[3] Unfortunately, Kazumura and Serizawa (2016a) then showed that if packages are to be sold, there *is no* rule that satisfies Morimoto and Serizawa's desiderata.

Thus, there is still much to be uncovered in this important problem. Our results yield a structure theorem, without any *a priori* restriction on the transfers to the agents or the packages they may buy, that can contribute to the theory of the second-best. In the meantime, work has continued on related questions such as finding notions of monotonicity that imply incentive compatibility (Kazumura et al., 2020a) and seeking desirable rules (Malik and Mishra, 2021).

In classical models of job matching with salaries, we discover a near equivalence between *ex ante* dominant strategy incentives and *ex post* coalitional incentives. In these models, conditional efficiency is equivalent to the *ex post* notion of *no blocking* (by matched pairs; see footnote), which when combined with non-wastefulness and individual rationality becomes the familiar solution concept of *stability* (see Section 3.2).[4] Stability and strategy-proofness have a well studied relationship in matching, both with and without continuous transfers (Shapley and Shubik, 1971; Dubins and Freedman, 1981; Kelso Jr and Crawford, 1982; Leonard, 1983; Demange and Gale, 1985; Hatfield and Kojima, 2009; Svensson, 2009; Hirata and Kasuya, 2017; Tierney, 2019). To oversimplify: Researchers begin by studying stability, for its inherent appeal, in their problem. The set of stable allocations is then found to have a nice structure (that of a welfare lattice, or semi-lattice). This structure allows us to identify a unique best stable allocation for some side of the market, which when chosen as the solution, yields a strategy-proof rule. Our results reverse this course and thereby show an intuitive "if and only if," though not a formal one. In particular, we require the ancillary conditions mentioned above. However, we only deduce a *part* of stability, and it is easy to see there are many *unstable* rules that satisfy our desiderata.

It is worth noting here that we do not impose even weak notions of efficiency, such as efficiency when constrained to the range, total distribution of the objects, or even non-wastefulness. All of these requirements are natural and have elicited interest in

[3]For the case of unit-demand for *homogenous* items, Sakai (2007) and Saitoh and Serizawa (2008) provide earlier characterizations.

[4]Depending on the context, non-wastefulness may be part of the definition of no-blocking. We instead distinguish the cases when an agent is not matched at all versus when it is "wrongly" matched.

the literature (Andersson et al. (2022); Alva and Manjunath (2019); Adachi (2014); Mukherjee (2013); Sakai (2012); Ashlagi and Serizawa (2012); Sakai (2007); Kazumura et al. (2020b)). Thus, we isolate, to a greater extent, the effects of the incentive constraints.

The foregoing gives an overview of our relative contribution, but since we interact with several literatures, a full analysis would be cumbersome. We thus make further connections to existing work in the natural course of the manuscript and via our corollaries. Therefore, we proceed with the model and solution concepts in Section 1 and the main results in Sections 2.1 (quasilinear preferencs) and 2.2 (general preferences). In Section 3.2, we formalize the application to labor market matching and in Section 3.3, the application to the package assignment model. As usual, our results require some richness of the preference domain, and since our richness requirement is completely standard, we postpone its formalization to Section 4. We then, finally, take up an extension in Section 5, wherein we consider the case when items may come in multiple copies.

1. Primitives

1.1. **Model.** There is a finite set Ω of indivisible objects and there is money. Agents are permitted to consume a single item from a set $X \subseteq 2^{\Omega} \setminus \{\emptyset\}$; that is, agents may not have access to every package of objects. They may be permitted to consume a null item, denoted 0, representing the state of consuming none of the other items. Let $\overline{X} := X \cup \{0\}$. Our first salient restriction arises: if $x \cap y \neq \emptyset$, then it is infeasible for one agent to consume x while another consumes y.

The set of agents is N, each of whom has a continuous and monotone preference relation over the consumption space $\overline{X} \times \mathbb{R}$. A typical preference is denoted R with asymmetric part P and symmetric part I. Preferences are strictly monotone in money: for each $x \in \overline{X}$ and $m', m \in \mathbb{R}$ with $m' > m$, $(x, m')\ P\ (x, m)$. We also make the standard *possibility of compensation* assumption: for each (x, m) and each $y \in \overline{X}$, there is $t \in \mathbb{R}$ such that $(y, t)\ I\ (x, m)$. The set of preferences is denoted $\overline{\mathcal{R}}$. For part of the paper we focus on the subset of quasilinear preferences, denoted $\overline{\mathcal{V}}$. For each $R \in \overline{\mathcal{V}}$, there is a vector $v \in \mathbb{R}^{\overline{X}}$ such that the function

$$v(x) + m$$

represents R. Note that we use functional notation $v(x)$ to identify the x coordinate of vector $v \in \mathbb{R}^{\overline{X}}$ and that $v(x)$ may be negative. Given a bundle (x, m), denote by $\boldsymbol{U}[R, (x, m)]$ the upper-contour set of R at (x, m). The lower-contour set is $\boldsymbol{L}[R, (x, m)]$.

For $v \in \mathbb{R}^{\overline{X}}$, denote by $R[\![v]\!]$ the weak preference relation induced by v. As above, $P[\![v]\!]$ and $I[\![v]\!]$ denote the corresponding strict and indifference relations. For notational simplicity, we often identify the relation $R[\![v]\!]$ by the vector v, but we must be careful with this. We shall not, as is customary, assume that $v(0) = 0$, as it will be useful at times to have different representations at our disposal. Let $\boldsymbol{e}$ denote the constant vector of ones. Given $v \in \mathbb{R}^{\overline{X}}$ and $\lambda \in \mathbb{R}$, $R[\![v + \lambda \boldsymbol{e}]\!]{=}R[\![v]\!]$; it is easy to see that this actually characterizes the set of equivalent, additively separable representations of $R[\![v]\!]$. When two vectors u and v represent the same preference, we write $u \simeq y$.

Given a set $A \subseteq \overline{X} \times \mathbb{R}$, denote by $\mathfrak{C}(R, A) \subseteq \overline{X} \times \mathbb{R}$ the maximal ("chosen") elements of preference relation R in A. Given $\boldsymbol{a} \in \mathbb{R}^{\overline{X}}$, we abuse notation and denote by $\mathfrak{C}(R, \boldsymbol{a})$ the set of items chosen by R from a choice set where a^x is the amount of money to be consumed together with item x. Formally, $\mathfrak{C}(R, \boldsymbol{a})$ is the projection onto $\overline{X}$ of $\mathfrak{C}(R, \{(y, t) : t = a^y\})$.

Denote by Z the set of feasible allocations $\varphi = (\xi, \mu) \in \overline{X}^N \times \mathbb{R}^N$. As noted above, for each $\varphi \in Z$, and each pair $i, j \in N$, $\xi_i \cap \xi_j = \emptyset$; however, the set Z might be further restricted. With abuse of notation, a rule is a function $\varphi = (\xi, \mu) : \overline{\mathcal{R}}^N \to Z$.

1.2. **Desiderata.** As is clear from the model, the profile $\boldsymbol{R} \in \overline{\mathcal{R}}^N$ fully determines the state of the economy. We regard each component R_i, as private information. Thus, to elicit the state of the economy from the agents, we require a notion of incentive compatibility. In the interest of robustness, we study

Strategy-proofness (StP): For each $\boldsymbol{R} \in \overline{\mathcal{R}}^N$, each $i \in N$, and each $R'_i \in \overline{\mathcal{R}}$, $\varphi_i(\boldsymbol{R})\ R_i\ \varphi_i(R'_i, \boldsymbol{R}_{-i})$.[5]

A rule is *strategy-proof* if and only if its induced manipulation game has a dominant-strategy equilibrium. Such rules require minimal rationality of participants.

We shall also study pairwise incentives that are similarly robust. In particular, we shall want to rule out the possibility that a pair of agents should collude and *both*

[5]Here, $(R'_i, \boldsymbol{R}_{-i})$ is, as usual, the profile derived from $\boldsymbol{R}$ by replacing component R_i with R'_i and changing nothing else.

improve their outcome, independent of any side payments or other compensation they might give each other. Such considerations are important in practice. The majority of people display some level of others-oriented preference and the desire for cooperation (Andreoni and Miller (2002); Van Lange (1999)). Moreover, even a small fraction of others-oriented players can induce fully selfish players to show others-orientation if they know it will reinforce a cooperative outcome in which they benefit (McKelvey and Palfrey (1992)). Thus, if a pair of players can benefit from a joint misreport to the rule, without generating any side payments that could incriminate them, then the only barrier to doing so is coordination. While coordination in auctions is necessarily hidden, for legal concerns, it was open at least in the old Boston mechanism for school choice (Abdulkadiroglu et al. (2006)). This is an environment in which no money is involved, and in which parents could defend their collusion on the ground of ensuring a better outcome for their children. Finally, it is worth noting that Li's (2017) empirically successful notion of *obvious strategy-proofness* implies our notion of pairwise incentive compatibility.

Weak Pairwise Strategy-proofness (2StP): For each $\boldsymbol{R} \in \overline{\mathcal{R}}^N$, each subset $N' \subseteq N$ with $|N'| \leq 2$, and each partial profile $\boldsymbol{R}'_{N'} \in \overline{\mathcal{R}}^{N'}$, there is at least one agent $k \in N'$ for whom $\varphi_k(\boldsymbol{R})\ R_k\ \varphi_k(\boldsymbol{R}'_{N'}, \boldsymbol{R}_{-N'})$.

Weak pairwise strategy-proofness nests *StP* by allowing N' to be a singleton as well. Obviously, **weak group strategy-proofness**, which is simply the above definition absent the restriction that $|N'| \leq 2$, implies *2StP*.

The sets of *strategy-proof* and *weak pairwise strategy-proof* rules are immense and diverse. In particular, they contain priority-based rules that maximally favor one agent over another based only on their index. To eliminate such rules from consideration, both because they are unfair and impractical (such discrimination might invite a lawsuit), we impose

Anonymity in Welfare (WAnon): Let $\boldsymbol{R}$ and $\boldsymbol{R}' \in \overline{\mathcal{R}}^N$ be such that there is a bijection $\sigma : N \to N$ with $R'_j = R_i$ when $j = \sigma(i)$. Then $\varphi_j(\boldsymbol{R}')\ I_i\ \varphi_i(\boldsymbol{R})$.

We limit our attention to rules that satisfy continuity in welfare space. We view continuity as a regularity condition, but there are several reasons why it is independently appealing. The first is ease of calculation. This is relevant even if the rule in question admits a closed form, as most practical procedures for implementing a social choice rule involve some dynamic convergence of messages (think of the ascending auction). The second reason is the avoidance of disputes: while a discontinuity may be easily justified to experts, who understand the problem deeply, to the layman it may appear capricious.

To define continuity of a rule, we first require a topology on the space of problems. Given $R \in \overline{\mathcal{R}}$, with corresponding indifference relation I, and $x \in \overline{X}$, for any $(y, m) \in \overline{X} \times \mathbb{R}$, let $U^x\left(y, m \middle| R\right)$ be the quantity of money satisfying[6]

$$\left(x, U^x\left(y, m \middle| R\right)\right) \; I_i \; (y, m).$$

This is well-defined given the possibility of compensation axiom. Since preferences are continuous, then for any x, the function $(y, m) \mapsto U^x\left(y, m \middle| R\right)$ is a continuous utility function representation for R. We say that a sequence R^n converges to R if and only if, for each $x \in \overline{X}$, $U^x\left(\cdot \middle| R^n\right)$ compact-converges to $U^x\left(\cdot \middle| R\right)$.[7]

Welfare Continuity: For each $i \in N$, and each $x \in \overline{X}$, the function $U^x\left(\varphi_i(\cdot) \middle| \cdot\right)$ is continuous.

1.3. **Solution Concepts.** The main contribution of the paper is to show that rules satisfying our desiderata must be price-based. If an allocation is supported by prices, we call it a quasi-equilibrium. As there is no feasibility constraint on the consumption of money, our notion of quasi-equilibrium has nothing to say about this. Finally, package prices need not be linear.

Vector $\boldsymbol{p} \in \mathbb{R}^{\overline{X}}$ is a **quasi-equilibrium price vector** for economy $\boldsymbol{R} \in \overline{\mathcal{R}}^N$ if there is an allocation $\varphi = (\xi, \mu) \in Z$ such that, for each agent $i \in N$,

$$\xi_i \in \mathfrak{C}\left(R_i, -\boldsymbol{p}\right) \text{ and } \mu_i = -p^{\xi_i}.$$

Allocation φ is called a **quasi-equilibrium**. The price of the null item may be non-zero; the notion of quasi-equilibrium does not inherently respect the outside option (getting the null and paying zero) or forbid subsidy. There may be several quasi-equilibria for each quasi-equilibrium price vector, but these will all be welfare equivalent, as each agent's maximal welfare is uniquely determined by the price vector.

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[6]The technique here is the same as used by Demange and Gale (1985).

[7]A sequence of functions f^n compact-converge to f if the converge uniformly on each compact set. This induces the topology of closed convergence on $\overline{\mathcal{R}}$.

For quasilinear preferences, additive separability gives us the classical notion of an *optimal* object assignment as one that maximizes $\sum_{i\in N} v_i(\xi_i)$ when each v_i has $v_i(0) = 0$. Quasi-equilibrium allocations need not be optimal, but are better, in this sense, than any other allocation with the same set of distributed items. This is because, fixing the set of distributed items, we are left with a classical assignment game as in Shapley and Shubik (1971). Thus, we conclude that each quasi-equilibrium yields an assignment that is

Conditionally optimal: A feasible item assignment $\xi \in \overline{X}^N$ is conditionally optimal for a profile of quasilinear preferences $\boldsymbol{v} \in \overline{\mathcal{V}}^N$ if, letting $A = \{x \in \overline{X} : \exists i \in N, \xi_i = x\}$, it maximizes the sum $\sum_{i\in N} v_i(\xi'_i) - v_i(0)$ among feasible allocations $\xi' \in A^N$.

For general preferences, we cannot separate the analysis of the item and money distributions, so we consider a kind of Pareto efficiency. The set of Pareto-efficient allocations, in the standard sense, is empty as we have not put an upper bound on the consumption of money. Rather than imposing any such bounds, we follow Morimoto and Serizawa (2015) in imagining that the money distributed by the rule comes from an administrator with simple preferences: distributing less money is better. Thus, an allocation is deemed inefficient if there is another feasible allocation that all agents, including the administrator, would unanimously vote to adopt instead; i.e., if a better distribution of objects and money can be found without increasing the administrator's contribution. It is then easy to see that each quasi-equilibrium is

Conditionally efficient: An allocation $(\xi, \mu) \in Z$ is conditionally efficient for $\boldsymbol{R} \in \overline{\mathcal{R}}^N$ if, letting $A = \{x \in \overline{X} : \exists i \in N, \xi_i = x\}$, it is Pareto efficient relative to feasible allocations $(\xi', \mu') \in A^N \times \mathbb{R}^N$ with $\sum_{i\in N} \mu'_i \leq \sum_{i\in N} \mu_i$.

In addition to the foregoing optimality properties, quasi-equilibrium is in fact equivalent to the following well-studied fairness condition. Since each agent has access to the same choice set, each agent finds their own bundle at least as good as everyone else's. Suppose, on the other hand that, at φ, we have

No Envy: For each $\{i, j\} \subseteq N$, $\varphi_i \; R_i \; \varphi_j$.

Then we can construct a price vector to support φ as follows: for each $x \in \overline{X}$ that is consumed by some agent i, set $p_x = -\mu_i$. If $x \in \overline{X}$ is not consumed by any agent, then set p_x sufficiently high. In sum, an allocation is a quasi-equilibrium if and only if it satisfies *no-envy*.

2. Main Results

Our theorems hold true only on domains that contain enough preference relations. For our purposes, such a domain is convex and contains enough Maskin-monotonic transforms of its members. As our requirements are simultaneously tedious to formalize and completely standard, we postpone discussion of them to Section 4. However, we summarize them now.

We study quasilinear and general domains. For a quasilinear domain to be rich, it must first be convex. Then, given two preferences on the interior of the domain and any bundle in the consumption space, there must be a *strict* Maskin monotonic transform of *both* preferences through that bundle. Additive preferences are rich. We demonstrate, in Section 3.3.2, a rich domain of preferences satisfying the gross substitutes condition and that contains the additive domain. For a general domain to be rich, it must contain a rich quasilinear domain, and satisfy the conditions of Barberà, Berga, and Moreno (2016). We formalize these notions in Section 4 and discuss, informally, how a rich general domain may be constructed from a rich quasilinear domain via convex interpolation operations.

2.1. Quasilinear Domains.

Our main theorem states that a rule satisfying our conditions can be, essentially, reduced to a pricing function taking values in $\mathbb{R}^{\overline{X}}$; after that there remains only the selection of the precise allocation, which is irrelevant for welfare.

Theorem 1. *Let $\mathcal{V} \subseteq \overline{\mathcal{V}}$ be rich. If a rule $\varphi : \mathcal{V}^N \to Z$ satisfies weak pairwise strategy-proofness, anonymity in welfare, and welfare continuity, then for each $v \in \mathcal{V}^N$, $\varphi(v)$ is a quasi-equilibrium.*

We cannot say, at present, if *welfare continuity* is essential for this result, however, the other two conditions are. For *anonymity in welfare*, consider the virtual valuation second-price auctions of Myerson (1981). For *weak pairwise strategy-proofness*, consider the first-price auction.

Given the properties of quasi-equilibrium, it is worth re-stating the theorem in different terminology:

Theorem 1a. *On any rich quasilinear domain, weak pairwise strategy-proofness, anonymity in welfare, and welfare continuity together imply conditional optimality.*

Theorem 1b. *On any rich quasilinear domain, weak pairwise strategy-proofness, anonymity in welfare, and welfare continuity together imply no-envy.*

2.2. **General Domains.** Thanks to the results of Barberà, Berga, and Moreno (2016) (henceforth BBM), our theorem can take a different form when preferences exhibit wealth effects. In particular, pairwise incentives can be replaced by individual incentives and the following monotonicity condition: Suppose an allocation has been determined for an economy, and then the economy changes such that all agents appraise the determined allocation better than they did before. The rule must pick something that all agents find at least as good as the original allocation, as judged by their new preferences.

Preference R' is a Maskin monotonic transform of R at (x, m) if $(y, t)\ R'\ (x, m)$ implies $(y, t)\ R\ (x, m)$, in which case we write $R' \in \mathscr{T}\,(R, (x, m))$.

Pairwise Weak Monotonicity: If for each $i \in N' \subseteq N$ with $|N'| = 2$, $R'_i \in \mathscr{T}\,(R_i, \varphi_i(\boldsymbol{R}))$, and otherwise $R'_i = R_i$, then for each $i \in N'$, $\varphi_i\,(\boldsymbol{R}')\ R'_i\ \varphi_i\,(\boldsymbol{R})$.

Theorem 2. *Let $\mathcal{R} \subseteq \overline{\mathcal{R}}$ be rich. If a rule $\varphi : \mathcal{R}^N \to Z$ satisfies strategy-proofness, pairwise weak monotonicity, anonymity in welfare, and welfare continuity, then for each $\boldsymbol{R} \in \mathcal{R}^N$, $\varphi(\boldsymbol{R})$ is a quasi-equilibrium.*

As with the quasilinear domain, we restate the theorem as an implication between normative desiderata.

Theorem 2a. *On a rich general domain, strategy-proofness, pairwise weak monotonicity, anonymity in welfare and welfare continuity together imply conditional efficiency.*

Theorem 2b. *On a rich general domain, strategy-proofness, pairwise weak monotonicity, anonymity in welfare, and welfare continuity together imply no-envy.*

This last result is reminiscent of that of Fleurbaey and Maniquet (1997), namely, that in the classical problem of consumption in $\mathbb{R}^k$, Maskin monotonicity and the following *equal treatment of equals* condition imply no-envy.

Equal Treatment of Equals (ETE): If $R_i = R_j$ then $\varphi_i(\boldsymbol{R})\ I_j\ \varphi_j(\boldsymbol{R})$.

Recall that Maskin monotonicity requires that, if the hypotheses of *pairwise weak monotonicity* hold, then $\varphi(\boldsymbol{R}') = \varphi(\boldsymbol{R})$. Thus, *pairwise weak monotonicity* is a strictly weaker requirement. We can, in addition, demonstrate a rule in our problem that fails *no-envy* and yet satisfies *strategy-proofness*, *pairwise weak monotonicity*, *equal treatment of equals*, and *welfare continuity*.

Example. Suppose there is a single real item and two agents. Agent 1 always gets $(0,0)$, the null item and no money. Define function τ so that, given R_1, $(1,\tau(R_1))\ I_1\ (0,0)$, where $(1,m)$ is the bundle including the single real item and m units of money. This is well-defined because of the possibility of compensation assumption. Agent 2 then gets $(1,\tau(R_1))$. Neither agent can influence their bundle, hence *strategy-proofness*. If $R_1' \in \mathscr{T}(R_1,(0,0))$, then $\tau(R_1') \geq \tau(R_1)$, so 2's welfare is isotone with 1's Maskin monotonic transforms, and 2's welfare is clearly continuous in 1's preferences more generally. Finally, if $R_2 = R_1$, then by construction, $\varphi_1(\boldsymbol{R})\ I_i\ \varphi_2(\boldsymbol{R})$ for each $i \in \{1,2\}$, so we have *equal treatment of equals*.

This rule demonstrates simultaneously the relative weakness of our monotonicity condition as well as the importance of the permutation condition in *welfare anonymity*. That said, many of the steps in our proof can be achieved with *equal treatment of equals* alone, and so we invoke this weaker requirement when it suffices. Because of this, we can consider an alternative set of conditions that imply quasi-equilibrium. In particular, when the rule is bounded relative to the preference domain, and when all the agents who consume the null get the same bundle of money, then *ETE* suffices.

Theorem 3. *Let $\mathcal{V} \subseteq \overline{\mathcal{V}}$ be rich and suppose $\varphi = (\xi,\mu) : \mathcal{V}^N \to Z$ satisfies weak pairwise strategy-proofness, equal treatment of equals, and welfare continuity. Assume further that 1) there is $\overline{m} \in \mathbb{R}$ such that, for each $\boldsymbol{v} \in \mathcal{V}^N$, and each $i \in N$, $\mu_i(\boldsymbol{v}) \leq \overline{m}$, and that 2) $\xi_i(\boldsymbol{v}) = \xi_j(\boldsymbol{v}) = 0$ implies $\mu_i(\boldsymbol{v}) = \mu_j(\boldsymbol{v})$. Finally, assume that 3) for each $\boldsymbol{v} \in int\left(\mathcal{V}^N\right)$ and each $i \in N$, there is $u_i \in \mathscr{T}(v_i,\varphi_i(\boldsymbol{v}))$ with $\varphi_i(\boldsymbol{v})\ P\,[\![u_i]\!]\ (y,\overline{m})$ for all $y \neq \xi_i(\boldsymbol{v})$. Then for each $\boldsymbol{v} \in \mathcal{V}^N$, $\varphi(\boldsymbol{v})$ is a quasi-equilibrium.*

The first and third conditions jointly say that agents can distinguish different items with higher intensity than the rule can compensate for. It is worth emphasizing here that we did *not* have this assumption before.

3. Applications

In this section we discuss applications of our results to labor markets and package auctions, the main distinction between these two being that labor markets are a case of one-to-one matching while package auctions match one agent to many objects. First, we review a few standard solution concepts related to the notion of quasi-equilibrium.

3.1. **Familiar Solution Concepts.** A *Vickrey-Clarke-Groves (VCG)* scheme is a rule $\varphi = (\xi, \mu) : \mathcal{V}^N \subseteq \overline{\mathcal{V}}^N \to Z$ such that ξ is optimal and, for each $i \in N$, there is $h_i : \mathcal{V}^{N\setminus\{i\}} \to \mathbb{R}$ such that

$$\mu_i(\boldsymbol{v}) = \sum_{j \in N\setminus\{i\}} v_j\left(\xi_j(\boldsymbol{v})\right) - \max_{\xi \in Z, \xi_i = 0} \sum_{j \in N\setminus\{i\}} v_j\left(\xi_j\right) + h_i(\boldsymbol{v}_{-i}).$$

It is well-known that every VCG scheme is *strategy-proof*. A *VCG pivot rule* is a VCG scheme with $h_i \equiv 0$ for all agents. Note that these rules award money consumption of 0 to those who consume item 0. By the way the VCG schemes are calculated, it is easy to see that any pivot rule satisfies *anonymity in welfare* (as the maximum value function is symmetric) and *welfare continuity* (by Berge's maximum theorem).

The calculation of VCG schemes clearly depends on quasilinearity. Strategy-proofness, however, does not. Demange and Gale (1985) showed that, at least for the unit demand case, which is when an agent can consume at most one object, the *minimal price Walrasian rules* (defined below) are *strategy-proof* even on non-quasilinear domains. Morimoto and Serizawa (2015) show that, when there are more agents than items, such rules are the *only efficient* and *strategy-proof* rules that, like the pivot rules, give money consumption 0 to the agents who consume item 0. Unfortunately, Kazumura and Serizawa (2016b) find that no rule can satisfy those conditions when packages of objects are available. We hope the results herein can provide some helpful structure to this space, so that even if an ideal rule cannot be found, we may nonetheless optimize within what is possible.

Given an economy $\boldsymbol{R} \in \overline{\mathcal{R}}^N$, let $\mathbb{P}(\boldsymbol{R})$ denote its quasi-equilibrium price vectors. In the unit-demand case, $\mathbb{P}(\boldsymbol{R})$ is a lattice under the *supremum* and *infimum* operations induced by the usual vector order $\leq$ (Shapley and Shubik (1971); Demange and Gale (1985)). This implies that for each pair $\{p, p'\} \subseteq \mathbb{P}(\boldsymbol{R})$, the point $p \wedge p' := (\min\{p_x, p'_x\})_{x \in X}$ also belongs to $\mathbb{P}(\boldsymbol{R})$. It further implies that for each point $\boldsymbol{b} \in \mathbb{R}^{\overline{X}}$, there is a unique minimal $p \in \mathbb{P}(\boldsymbol{R})$ satisfying $p \geq \boldsymbol{b}$. Denote this element $p^*(\boldsymbol{R}; \boldsymbol{b})$. Recall that prices may be negative, so in fact the agents may be offered payment to take an object. The vector $p^*(\boldsymbol{R}; \boldsymbol{b})$ is the buyer-optimal quasi-equilibrium price that respects the bounds given by $\boldsymbol{b}$.

Given a vector $\boldsymbol{b} \in \overline{\mathbb{R}}^{\overline{X}}$ as a parameter, we may construct a rule $w^{\boldsymbol{b}}$ as follows: For each $\boldsymbol{R} \in \overline{\mathcal{R}}^N$, calculate $p^*(\boldsymbol{R}; \boldsymbol{b})$. Now let $w^{\boldsymbol{b}}(\boldsymbol{R})$ be an equilibrium generated by prices $p^*(\boldsymbol{R}; \boldsymbol{b})$. We have thus defined an $|\overline{X}|$-dimensional family of rules, which are called the **minimal price Walrasian rules**.

3.2. **Application: Job Matching.** For this section, we imagine that X is a set of tasks. Each agent is to be given at most one task and agents cannot share a task. Thus, X is the set of singleton subsets of Ω. For notational continuity, we use X nonetheless. When $i \in N$ is given task $x \in X$, a publicly observable quantity $\pi_{ix} \in \mathbb{R}$ of money is generated. The firm overseeing task x then pays a salary to i. We shall consider this the primitive model for job matching and call it the *worker-firm* model, but we will map it into our model above, which we call the *bidder-seller* model. In the worker-firm model, workers have *worker preferences*, which are marked with a star, as in $R_i^\star$.

To map the worker-firm model into the bidder-seller model, imagine that π_{ix} accrues first to i who then returns some profit to the firm. This induces an auction of heterogeneous objects with buyers having unit-demand preferences. Formally, suppose i consumes (x, w_i) in the worker-firm model; she does task x and gets paid wage w_i. If we view π_{ix} as first accruing to i, then this means she pays $\pi_{ix} - w_i$ to the firm. Note that i's welfare is *decreasing* in this quantity. Since we assume welfare is increasing in money, we say she consumes $\mu_i = w_i - \pi_{ix}$ money in the bidder-seller model, which is negative for normal applications (and fits the bidder-seller model, as payments are usually negative consumption). Thus, (x, w_i) maps to $(x, w_i - \pi_{ix})$ and money consumption is simply computed net of productivity. It follows that we map worker preference $R_i^\star$ of agent i to *bidder preference* R_i of agent i via

$$(x, w_i)\ R_i^\star (y, w_i') \iff (x, w_i - \pi_{ix})\ R_i\ (y, w_i' - \pi_{iy}).$$

We shall now explore the consequences of our prior results, found in the bidder-seller model, in the worker-firm model.

We first examine what *anonymity in welfare* means in this context. In the bidder-seller model, if $\boldsymbol{R}$ is permuted to $\boldsymbol{R}'$, with $R_j' = R_i$, then we have $(\xi_j', \mu_j')\ I_i\ (\xi_i, \mu_i)$, where $(\xi_j', \mu_j') = \varphi_j(\boldsymbol{R}')$ and $(\xi_i, \mu_i) = \varphi_i(\boldsymbol{R})$. If we substitute $\mu_i = w_i - \pi_{i\xi_i}$ and

$\mu'_j = w_j - \pi_{j\xi'_j}$, and then map the result to worker preferences of i, we get

$$(\xi'_j, w'_j - \pi_{j\xi'_j} + \pi_{i\xi'_j})\ I^\star_i\ (\xi_i, w_i).$$

So if $\pi_{j\xi'_j} > \pi_{i\xi'_j}$, i might prefer j's job and wage bundle after permutation. This may be true even if $R_j = R_i$. However, this is reasonable for the application as it reflects the differential productivity of the two workers. *Anonymity in welfare* thus becomes

No Discrimination: Let $\varphi^\star$ be a rule on the worker-firm model. Let $\boldsymbol{R}^\star$ and $\boldsymbol{R}'^\star$ be preference profiles such that there is a permutation $\sigma : N \to N$ with $R'^\star_j = R^\star_i$ when $j = \sigma(i)$. Then, letting $(\xi, \boldsymbol{w}) = \varphi^\star(\boldsymbol{R}^\star)$ and $(\xi', \boldsymbol{w}') = \varphi^\star(\boldsymbol{R}'^\star)$,

$$(\xi'_j, w'_j - (\pi_{j\xi_j} - \pi_{i\xi_j}))\ I^\star_i\ (\xi_i, w_i).$$

Consider an allocation $(\xi, \mu) \in \overline{X} \times \mathbb{R}$ that is *not* a quasi-equilibrium in the bidder-seller model. There are $i, j \in N$ such that $(\xi_j, \mu_j)\ P_i\ (\xi_i, \mu_i)$. In worker preferences,

$$(\xi_j, \mu_j + \pi_{i\xi_j})\ P^\star_i\ (\xi_i, w_i).$$

For $\varepsilon > 0$, sufficiently small, i could propose wage

$$w'_i = \mu_j + \pi_{i\xi_j} - \varepsilon$$

to task ξ_j and would still prefer (ξ_j, w'_i) to her current allotment. The firm managing the task would then be receiving an offer to earn profit

$$\pi_{i\xi_j} - w'_i = -\mu_j + \varepsilon > -\mu_j = \pi_{j\xi_j} - w_j.$$

We assume the firm would take such an offer, and thus i and ξ_j would form a blocking pair. It follows, therefore, that quasi-equilibrium, when applied to the job matching model, is equivalent to

No-blocking: An allocation $(\xi, \boldsymbol{w})$ is *blocked* if there are $i, j \in N$ and $w' \in \mathbb{R}$ such that $\pi_{i\xi_j} - w' > \pi_{j\xi_j} - w_j$ and $(\xi_j, w')\ P^\star_i\ (\xi_i, w_i)$.

No blocking is the heart of the solution concept of *stability*, the other two ingredients being *non-wastefulness* and *voluntary participation*. The intimate relationship between stability and incentives is well-known, with several characterizations in the literature (Leonard, 1983; Morimoto and Serizawa, 2015; Svensson, 2009), but, to the author's knowledge, the corollary below is the first to show the necessity of an ingredient of *stability* without assuming either of the other two ingredients.

Corollary 1. *If a rule for the job matching model, defined on a rich quasilinear domain $\mathcal{V} \subseteq \overline{\mathcal{V}}$, satisfies weak pairwise strategy-proofness, no discrimination and welfare continuity, then it satisfies no-blocking.*

Corollary 2. *If a rule for the job matching model, defined on a rich general domain $\mathcal{R} \subseteq \overline{\mathcal{R}}$, satisfies strategy-proofness, pairwise weak monotonicity, no discrimination and welfare continuity, then it satisfies no-blocking.*

3.2.1. *Novel Characterizations of Min-price Rules.* The previous section demonstrated how the job matching model and the auction model with unit-demand map between each other: one only need to reinterpret *anonymity in welfare* and quasi-equilibrium. As the auction model is more parsimonious—doing away with productivities—we proceed with this approach. We show here some corollaries of our main results, arriving at three novel characterizations via synthesis with existing results.

Completing the synthesis requires some additional concepts. Non-wastefulness is the idea that any unconsumed item should be available for free. We have not given the quantity 0 of money any particular meaning thus far, so getting something for free has no significance. We therefore first introduce the concept of

No Subsidies: In allocation $\varphi = (\xi, \mu)$, each $i \in N$ with $\xi_i \in X$ has $\mu_i \leq 0$.

With this, it is meaningful to also require

Non-Wastefulness: In allocation $\varphi = (\xi, \mu)$, suppose there are $i \in N$ and $x \in X$ such that $(x, 0)\ P_i\ \varphi_i$. Then there is j with $\xi_j = x$.

Lemma 1. *A non-wasteful quasi-equilibrium that gives no subsidies is efficient.*

Proof. Let $\boldsymbol{p} \in \mathbb{R}^{\overline{X}}$ be a price vector supporting $\varphi = (\xi, \mu)$. Let $\varphi' = (\xi', \mu')$ be an allocation that might welfare dominate φ. Note that since quasi-equilibria are conditionally efficient, φ' can only welfare dominate φ by allocating a different set of items or distributing more money. Thus, consider a sequence $\{1, 2, \ldots, n\} \subseteq N$, relabeled for convenience, such that for $1 \leq i < n$, $\xi'_i = \xi_{i+1}$, and such that ξ_1 is not consumed under ξ' and ξ'_n is not consumed under ξ. The sequence of agents forms a trading path. Define $\boldsymbol{q} \in \mathbb{R}^n$ so that for each $i \in \{1, \ldots, n\}$, $(\xi'_i, -q_{\xi'_i})\ I_i\ \varphi_i$. Since φ is a quasi-equilibrium, for $1 \leq i < n$, $q_{\xi'_i} \leq p_{\xi'_i}$. By *non-wastefulness*, $q_{\xi'_n} \leq 0$ and by *no-subsidies*, $p_{\xi_1} \geq 0$. Altogether, we have

$$\sum_{i=1}^{n} q_{\xi'_n} \leq \sum_{i=1}^{n-1} q_{\xi'_i} \leq \sum_{i=2}^{n} p_{\xi_i} \leq \sum_{i=1}^{n} p_{\xi_i}.$$

If φ' indeed welfare dominates φ, then for each i, $\mu'_i \geq -q_{\xi'_i}$ and for some i this is strict. Then $\sum_{i=1}^{n} \mu'_i > -\sum_{i=1}^{n} p_{\xi_i}$, so any such sequence of agents receives strictly more money from the rule. It is easy to verify that the same holds for cycles. In sum, the only way for φ' to welfare dominate φ is via the distribution of more money. □

Now we can leverage Holmström's Theorem (1979) for our first corollary:

Corollary 3. *Assume $\mathcal{V} \subseteq \overline{\mathcal{V}}$ is rich. If a non-wasteful rule $\varphi : \mathcal{V}^N \to Z$ satisfies no-subsidies, weak pairwise strategy-proofness, anonymity in welfare, and welfare continuity, then it is a Vickrey-Clarke-Groves scheme with h_i independent of index.*

To extend this to the general domain we leverage the theorem of Morimoto and Serizawa (2015). This, however, requires two more restrictions. First, they show their theorem on the domain

$$\mathcal{R}^C = \left\{ R \in \overline{\mathcal{R}} : \forall x \in X, \forall m \in \mathbb{R}, (x, m)\ P\ (0, m) \right\}.$$

Proposition 1 below shows that quasilinear preferences with positive valuations are rich. We leave it to the reader to verify that $\mathcal{R}^C$ is rich, using this fact. Second, they further impose that the rule be

Voluntary: For each $\boldsymbol{R} \in \mathcal{R}$ and each $i \in N$, $\varphi_i(\boldsymbol{R})\ R_i\ (0, 0)$.

We can now state

Corollary 4. *Assume $|N| \geq \left|\overline{X}\right|$. Let $\varphi : \left(\mathcal{R}^C\right)^N \to Z$ be a non-wasteful, voluntary rule that gives no subsidies. Assume further that φ satisfies strategy-proofness, pairwise weak monotonicity, anonymity in welfare and welfare continuity. Then φ is a minimum price Walrasian rule with $\boldsymbol{b} = \boldsymbol{0}$.*

Our Theorem 3, combined with the result of Svensson (2009), yields a corollary for the case when matching must be one-to-one. That is, assume now that the null is not available and $|X| = |N|$. To apply Svensson's result, we need

Regularity: A rule $\varphi = (\xi, \mu)$ is regular if there is some $\boldsymbol{v} \in \mathcal{V}^N$ such that

$$\sum_{i \in N} \mu_i(\boldsymbol{v}) = \sup_{\boldsymbol{v}' \in \mathcal{V}^N} \sum_{i \in N} \mu_i(\boldsymbol{v}') \in \mathbb{R}.$$

Corollary 5. *Assume the null item is not available and that $|N| = |X|$. If a regular rule $\varphi : \overline{\mathcal{V}}^N \to Z$ satisfies weak group strategy-proofness, equal treatment of equals, and welfare continuity, then there is a vector $\overline{\boldsymbol{m}} \in \mathbb{R}^X$ such that, for each $\boldsymbol{v} \in \mathcal{V}^N$, $\varphi(\boldsymbol{v})$ is an equilibrium with respect to price $p^* (\boldsymbol{v}; \overline{\boldsymbol{m}})$.*

3.3. **Application: Package Assignment.** Theorem 2 contributes to the nascent study of package auctions in the presence of wealth effects (Kazumura and Serizawa (2016b); Baisa (2020); Kazumura et al. (2020a)). Our guidance in this direction (or any direction, for that matter) is of course conditional on the usefulness of rich domains, a topic we address in Section 3.3.2. We begin, however, with another corollary.

3.3.1. *A novel observation on collusion in VCG schemes.* Our result implies that if the VCG pivot rule is *weakly pairwise strategy-proof*, then it is *envy-free*. Pápai (2003) showed that, in general, VCG schemes cannot be made *envy-free*, and thus we arrive at the following result.

Corollary 6. *No rule $\varphi : \overline{\mathcal{V}}^N \to Z$ assigns items optimally, awards 0 money to agents who get the null item, and satisfies weak pairwise strategy-proofness.*

Proof. Optimal assignment and *strategy-proofness* (as a consequence of *weak pairwise strategy-proofness*) yield VCG schemes. Since $\overline{\mathcal{V}}$ contains $(0, -1, \cdots - 1)$, each $i \in N$ has a report guaranteeing them the null item, regardless of the reports of the other. Thus, to award such an agent 0 money, it must be that $h_i \equiv 0$; φ is the pivot rule. As mentioned, this is *welfare anonymous* and *welfare continuous*, so Theorem 1 implies *envy-freeness*, which is impossible given the result of Pápai (2003). □

3.3.2. *Rich Domains for Package Assignment.* Since Kelso & Crawford (1982), the domain of quasilinear preferences satisfying their *gross substitutes* condition has been of central interest. Unfortunately, Bikhchandani and Ostroy (2002) provide an example of a problem satisfying this condition for which the Vickrey pivot rule cannot be expressed as a quasi-equlibrium. Nonetheless, given the importance of the property, and the likely necessity of sacrificing optimal allocation in many contexts, we study whether it is a rich domain. Unfortunately, it is not convex. However, it has useful sub- and superdomains that are rich.

Gross-substitutability is a property of a demand function in the face of linear pricing: when the prices of some objects increase, previously demanded packages, with the newly expensive objects removed, should be contained in newly demanded packages. Fujishige and Yang (2003) showed that demand exhibits gross-substitutability if and only if the the preference can be represented by $v \in \mathbb{R}^{\overline{X}}$ that is $M^\natural$-concave.[8] This latter condition was introduced by Murota and Shioura (1999) and is somewhat complex. Of primary importance for our purposes is that the set of $M^\natural$-concave functions is *not* convex (Murota, 2003). However, we shall demonstrate here a subdomain of $M^\natural$-concave functions that is rich.

Every $M^\natural$-concave function is submodular but the converse is not true (Murota (2003), example 6.20). We leave it to the reader to verify that submodular functions also form a rich domain.

A function $f : \mathbb{Z} \to \mathbb{R}$ is *univariate discrete concave* if, for each $t \in \mathbb{Z}$, $f(t+1) + f(t-1) \leq 2f(t)$. A function $v : 2^\Omega \to \mathbb{R}$ is *concave quasi-separable* if there is a non-decreasing univariate discrete concave function f such that, for each $x \in X$,

$$v(x) = f(|x|) + \sum_{\omega \in x} v(\{\omega\}).$$

Murota (2003) showed that concave quasi-separable functions are $M^\natural$ concave.

As we have postponed the definition of richness, so we postpone the proof of the following proposition to Section 4.

Proposition 1. *The domain of quasilinear preferences induced by non-negative, concave quasi-separable functions is rich.*

4. Rich Preference Domains

We build rich general domains from rich quasilinear domains, so we discuss the latter first.

4.1. **Rich Quasilinear Domains.** A rich quasilinear domain is convex and contains enough Maskin monotonic transforms. Convexity is defined in the obvious way. If

[8] In this model, where all objects are distinct.

R and R' are quasilinear, with representations u and v respectively, then the convex hull of R and R' is

$$\left\{R'' \in \overline{\mathcal{V}} : \exists \theta \in [0,1], R'' = R\,[\![\theta u + (1-\theta)v]\!]\right\}.$$

This is well defined: if u' and v' are alternative representations, then there are $\lambda, \gamma \in \mathbb{R}$ such that

$$\theta u' + (1-\theta)v' = \theta u + (1-\theta)v + (\theta\lambda + (1-\theta)\gamma)\,\boldsymbol{e} \simeq \theta u + (1-\theta)v.$$

If R is quasilinear, then $\mathscr{T}(R,(x,t))$ is independent of t, and we may suppress notation for money in this case, writing $\mathscr{T}(R,x)$. Relation R' is a *strict* Maskin monotonic transform of R at (x,m) if $(y,t)\ R'\ (x,m)$ and $y \neq x$ then $(y,t)\ P\ (x,m)$. In this case we write $R' \in \mathscr{T}^{str.}(R,(x,m))$. A quasilinear relation v is *regular* in domain $\mathcal{V}$ if, for each $x \in \overline{X}$, and each open $U \subseteq \mathbb{R}^{\overline{X}}$ containing v, there is $u \in U$ with $u \in \mathscr{T}^{str.}(v,x)$ and if, in addition, $x = 0$, then there is $\delta > 0$ such that, for each $y \in X$, $u(y) = v(y) - \delta$.

Richness (Quasilinear Case): A domain $\mathcal{V} \subseteq \overline{\mathcal{V}}$ is rich if

(1) It is convex;
(2) The relative interior of $\mathcal{V}$ is non-empty and consists of regular preferences;
(3) For each pair u and v of regular preferences, and each $x \in \overline{X}$, $\mathscr{T}^{str.}(u,x) \cap \mathscr{T}^{str.}(v,x) \neq \emptyset$.

We can now prove Proposition 1.

Proof. Let u and v be non-negative, concave quasi-separable. For each $\omega \in \Omega$, let $u_\omega = u(\{\omega\})$ and $v_\omega = v(\{\omega\})$, and assume f and g are the discrete concave functions for u and v respectively. Assume, moreover, that each $u_\omega, v_\omega > 0$ and that f and g are both *strictly* positive and *strictly* discrete concave. Clearly, these conditions make u and v interior elements. To show that they are regular, for $x \subseteq \Omega$, and each $\omega \in x$, set $u'_\omega = u_\omega - {}^{\delta}\!/\!_{|x|}$. For $x = 0$ (that is $x = \emptyset$), set $f' = f - \delta$.

Fix $x \subseteq \Omega$, and set α so that for each $y \subsetneq x$,

$$|x \setminus y|\,\alpha > \max\{f(|\Omega|), g(|\Omega|)\} + \sum_{\omega \in x \setminus y} \max\{v_\omega, u_\omega\}.$$

Let $w \in \mathbb{R}^{\overline{X}}$ be given, for each $y \in \overline{X}$, by $w(y) = \alpha \, |y \cap x|$. We shall show that $w \in \mathscr{T}^{str.}(u, x) \cap \mathscr{T}^{str.}(v, x)$. Assume $(y, t) \; R \, [\![w]\!] \; (x, m)$, with $y \neq x$. Then

$$0 \leq \alpha \, |x \setminus y| = \alpha \left(|x| - |y \cap x| \right) \leq t - m.$$

If $y \supsetneq x$, then $u(y) > u(x)$ by the positivity of valuations, and so $u(x) - u(y) < 0 \leq t - m$. Otherwise, again via non-negativity,

$$u(x) - u(y) \leq f\left(|\Omega|\right) + \sum_{\omega \in x \setminus y} u_\omega < |x \setminus y| \, \alpha \leq t - m.$$

In either case, $(y, t) \; P \, [\![u]\!] \; (x, m)$, and the same calculation holds for v. □

4.2. **Rich General Domains.** The indifference sets of a quasilinear preference relation are translates of each other. Thus, each quasilinear preference relation can be identified with a single indifference set. Viewing these as geometric objects, we can treat a quasilinear domain as a source of indifference sets from which general preference relations are constructed. Given $R \in \overline{\mathcal{R}}$, find $v \in \mathbb{R}^{\overline{X}}$ such that, for each $z \in \overline{X}$, $v(z) = -U^z\left(x, m \middle| R\right)$. The quasilinear preference v has the same indifference set as R through the bundle (x, m), and so we say it is the preference *induced* by R at (x, m). Domain $\mathcal{R}$ *induces* domain $\mathcal{V}$ by repeating this operation for each $R \in \mathcal{R}$ and each $(x, m) \in \overline{X} \times \mathbb{R}$.

A general domain is rich if it induces a rich quasilinear domain and, additionally, satisfies the richness condition of Barberà, Berga, and Moreno (2016):

Richness (General Case): The domain $\mathcal{R}$ is *rich* if it (i) induces an *open* rich quasilinear domain and (ii) if for each pair R, $R' \in \mathcal{R}$, if $(x, m) \; P \; (y, t)$, then there is $R'' \in \mathcal{R}$ such that, for each $z \in \overline{X}$.

$$U^z\left(x, m \middle| R''\right) \geq \max\left\{U^z\left(x, m \middle| R\right), U^z\left(x, m \middle| R'\right)\right\}$$
$$\text{and } U^z\left(y, t \middle| R''\right) = U^z\left(y, t \middle| R\right).$$

We may wonder then if a rich quasilinear domain provides enough indifference sets to construct a rich general domain. The answer is generically "yes," and it can be done as follows: Let v be the quasilinear preference induced by R at (x, m) and v' the one induced by R', also at (x, m). If v and v' are regular elements of a rich domain, then there is $v^* \in \mathscr{T}^{str.}(v, x) \cap \mathscr{T}^{str.}(v', x)$. Similarly, let u be induced by R at (y, t). Finally, let R'' be a relation that has an indifference set given by v^* at (x, m), an indifference set given by u at (y, t), and indifferences generated from the convex hull of v^* and u in between these two. Then R'' is the preference required by richness.

5. Extension: Items in Multiple Copies

We have assumed that only the null item can be consumed by more than one agent. We can replace this assumption by an assumption on how a rule breaks ties. *Any* strategy-proof rule can be viewed as a combination of a menu correspondence and a tie-breaking function. That is, given the preferences $\boldsymbol{R}_{-i}$ of agents not i, the rule presents agent i with menu $A_i(\boldsymbol{R}_{-i})$. Agent i then reports her favorite bundles from this menu. When there is more than one such bundle, the rule must employ its tie-breaking rule. Of course, all of these reports happen simultaneously, and so the operation we describe is infeasible in practice. Nonetheless, it shows that every strategy-proof rule can be viewed as a selection from a correspondence containing all feasible tie-breaking choices. All of these choices are welfare equivalent: all agents are indifferent between them. However, not all properties hold across selections. In particular, if a rule is *weakly pairwise strategy-proof*, the other selections from the correspondence are not guaranteed to retain this property, only the individual version. This motivates the following condition:

Neutral Tie-breaking: Let $\Phi : \mathcal{R}^N \subseteq \overline{\mathcal{R}}^N \rightrightarrows Z$ be a correspondence and suppose $\{\varphi, \varphi'\} \subseteq \Phi(\boldsymbol{R})$ are distinct. Let $\psi \in Z$ be a feasible allocation such that, for each $i \in N$, $\psi_i \in \{\varphi_i, \varphi'_i\}$. Then $\psi \in \Phi(\boldsymbol{R})$.

We shall extend our previous conditions from single-valued to set-valued rules as follows: Say that correspondence Φ satisfies condition **P** if and only if each of its selections satisfy **P**. In keeping with the spirit of our main results, we also require the rule at least be decisive in welfare space: if $\{\varphi, \varphi'\} \subseteq \Phi(\boldsymbol{R})$, then for each $i \in N$, $\varphi_i(\boldsymbol{R})\ I_i\ \varphi'_i(\boldsymbol{R})$. This requirement is known as *essential single-valuedness*. Note that, in this case, our menu correspondence interpretation of *strategy-proofness* remains valid. Moreover, we show in the appendix that the menu correspondence associated with a strategy-proof and welfare continuous rule is continuous.[9] It is then easy to see, via Maskin monotonic transforms, that ties only occur on a negligible set of problems. In this sense, *neutral tie-breaking* has bite only on a small part of the domain. It is nonetheless a required condition of the following results, also shown in the appendix.

[9]At least, on the domain of problems where it could matter for welfare.

Theorem 4. *Assume real items may come in multiple copies. Let $\mathcal{V} \subseteq \overline{\mathcal{V}}$ be rich. If a rule $\Phi : \mathcal{V}^N \rightrightarrows Z$ satisfies weak pairwise strategy-proofness, anonymity in welfare, welfare continuity and neutral tie-breaking, then for each $v \in \mathcal{V}^N$, each $\varphi \in \Phi(v)$ is a quasi-equilibrium.*

Theorem 5. *Assume real items may come in multiple copies. Let $\mathcal{R} \subseteq \overline{\mathcal{R}}$ be rich. If a rule $\Phi : \mathcal{R}^N \rightrightarrows Z$ satisfies strategy-proofness, pairwise weak monotonicity, anonymity in welfare, welfare continuity, and neutral tie-breaking then for each $\boldsymbol{R} \in \overline{\mathcal{R}}^N$, each $\varphi \in \Phi(\boldsymbol{R})$ is a quasi-equilibrium.*

Appendix A. Proof of Theorem 1

Throughout this appendix, we assume that φ satisfies our conditions and our domain $\mathcal{V}$ is rich and a subset of $\overline{\mathcal{V}}$. Denote by $\mathring{\mathcal{V}}$ the relative interior of $\mathcal{V}$; by assumption, these are regular preferences. The proof is argued on $\mathring{\mathcal{V}}$ and extends to $\mathcal{V}$ via *WCon*. We begin with some preliminary results.

A.1. **Preliminaries.**

A.1.1. *Strategy-proofness and Maskin monotonic transformation. Strategy-proofness* implies that each agent's report at each problem maximizes their welfare relative to all other reports they could make. Thus, given the reports of others, if we let $A_i(\boldsymbol{v})$ be the set of all $\varphi_i(v_i, \boldsymbol{v}_{-i})$, as v_i varies in $\mathcal{V}$, then

$$\varphi_i(v_i, \boldsymbol{v}_{-i}) \in \mathfrak{C}\left(v_i, A_i(\boldsymbol{v})\right).$$

Clearly A_i is invariant to v_i, but we often write it as a function of the full profile $\boldsymbol{v}$ for cleaner exposition. For each $x \in \overline{X}$, let $a_i^x(\boldsymbol{v}) \in \mathbb{R}$ be the money allotment such that $(x, a_i^x(\boldsymbol{v})) \in A_i(\boldsymbol{v})$. This is well-defined: If $\{(x,m),(x,t)\} \subseteq A_i(\boldsymbol{v})$ and $m > t$ then, letting $\varphi_i(v_i, \boldsymbol{v}_{-i}) = (x,m)$ and $\varphi_i(v_i', \boldsymbol{v}_{-i}) = (x,t)$, preference monotonicity implies $\varphi_i(v_i, \boldsymbol{v}_{-i})\ P\,[\![v_i']\!]\ \varphi_i(v_i', \boldsymbol{v}_{-i})$, contradicting *strategy-proofness*. Let $\boldsymbol{a}_i(\boldsymbol{v}) = (a_i^x(\boldsymbol{v}))_{x\in\overline{X}}$.

The following lemma summarizes some well-known facts of *strategy-proof* rules.

Lemma (The Invariance Lemma)**.** *Let $w_i \in \mathscr{T}\left(v_i, \varphi_i(\boldsymbol{v})\right)$. Then $\varphi_i\left(w_i, \boldsymbol{v}_{-i}\right)\ I\,[\![v]\!]\ \varphi_i\left(\boldsymbol{v}\right)$ and $\varphi_i\left(w_i, \boldsymbol{v}_{-i}\right)\ I\,[\![w]\!]\ \varphi_i\left(\boldsymbol{v}\right)$. If $w \in \mathscr{T}^{str.}\left(v_i, \varphi_i(\boldsymbol{v})\right)$, then $\varphi_i\left(w_i, \boldsymbol{v}_{-i}\right) = \varphi_i\left(\boldsymbol{v}\right)$.*

Recall the construction of our utility function representation. For $x \in \overline{X}$ and $(y,m) \in \overline{X} \times \mathbb{R}$,

$$\left(x, U^x\left(y, m \middle| v\right)\right) \; I \, [\![v]\!] \; (y, m) \, .$$

As such, the points $\left\{\left(x, U^x\left(y, m \middle| v\right)\right) : x \in \overline{X}\right\}$ form the indifference set of v through (y,m) and, in particular, $U^y\left(y, m \middle| v\right) = m$. The reader may also verify that the following conversions hold for our utility representation when preferences are quasi-linear:

$$\begin{aligned} U^y\left(x, m \middle| u\right) &= U^y\left(x, m \middle| v\right) + v(y) - u(y) + u(x) - v(x) \\ U^y\left(x, m \middle| u\right) &= U^z\left(x, m \middle| u\right) + u(z) - u(y). \end{aligned} \tag{A.1}$$

This in hand, we can prove a few simple but useful lemmas. The first provides some characterizations of Maskin monotonic transforms in terms of their representations.

Lemma 2. *The following are equivalent:*

(1) $u \in \mathscr{T}(v,x)$ *[*$u \in \mathscr{T}^{str.}(v,x)$*].*
(2) There are $u' \simeq u$ *and* $v' \simeq v$ *such that* $u' \leq v'$*, with equality [only] at* x*.*
(3) For each $y \in \overline{X}$*, and each* $m \in \mathbb{R}$*,*

$$U^y\left(x, m \middle| u\right) \geq U^y\left(x, m \middle| v\right),$$

with equality [only] for $y = x$*.*

Proof. By quasilinearity, we are free to choose a money allotment to go with x, so consider the full bundle (x,m).

We first show equivalence of the first two. Assume 1. Let $u' \simeq u$ and $v' \simeq v$ have $u'(x) = v'(x) = 0$. Choose (y,t) with $u'(y) + t = u'(x) + m$. This implies $v'(y) + t \geq v'(x) + m$, conclude

$$v'(y) \geq m - t = u'(y).$$

If $u \in \mathscr{T}^{str.}(v,x)$, then $y \neq x$ implies $v'(y) + t > v'(x) + m$ and we get equality only at x.

Now Assume 2. Let $u'' = u' - u'(x)\boldsymbol{e}$ and $v'' = v' - u'(x)\boldsymbol{e}$. Clearly $u'' \simeq u'$ and $v'' \simeq v'$ and $u'' \leq v''$ with equality [only] at x. Furthermore, $u''(x) = v''(x) = 0$. Then if $u''(y) + t \geq m$, clearly $v''(y) + t \geq m$, and so $u'' \in \mathscr{T}(v'', x)$. Since $u'' \simeq u$ and $v'' \simeq v$, $u \in \mathscr{T}(v,x)$.

Next we show equivalence of 2 and 3. Our conversion formulae (line A.1) yield

$$U^y\left(x,m\middle|u'\right)=U^y\left(x,m\middle|v'\right)+v'(y)-u'(y)+u'(x)-v'(x).$$

As above, we may assume $u'(x)=v'(x)=0$. We then immediately conclude that $U^y\left(x,m\middle|u'\right)\geq[>]U^y\left(x,m\middle|v'\right)$ if and only if $v'(y)\geq[>]u'(y)$. □

Now we show that our utility function representation preserves convexity.

Lemma 3. *For $\theta\in[0,1]$, let $w=\theta u+(1-\theta)v$. Then*

$$U^y\left(x,m\middle|w\right)=\theta U^y\left(x,m\middle|u\right)+(1-\theta)U^y\left(x,m\middle|v\right).$$

Proof. Note that

$$U^y\left(x,m\middle|u\right)=U^y\left(x,m\middle|w\right)+w(y)-u(y)+u(x)-w(x)$$
$$U^y\left(x,m\middle|v\right)=U^y\left(x,m\middle|w\right)+w(y)-v(y)+v(x)-w(x).$$

Taking the convex combination of these two equations yields the claimed equation. □

Lemma 4. *$\mathscr{T}\left(v,(x,m)\right)$ is convex in $\overline{\mathcal{V}}$, and $\mathscr{T}^{str.}\left(v,(x,m)\right)$ is the relative interior of $\mathscr{T}\left(v,(x,m)\right)$.*

Proof. By Lemma 2, $u\in\mathscr{T}^{str.}(v,x)$ if and only if $U^y\left(x,m\middle|u\right)>U^y\left(x,m\middle|v\right)$ for $y\neq x$. If $u'\in\mathscr{T}^{str.}(v,x)$ as well, then by Lemma 3 the required convex combination will preserve this inequality. □

Finally we show that if a preference is a Maskin monotonic transform of several distinct preferences, then it is also a Maskin monotonic transform of every preference in their convex hull.

Lemma 5. *Let $w\in co\left\{v,v'\right\}$. Then*

$$\mathscr{T}\left(v,(x,m)\right)\cap\mathscr{T}\left(v',(x,m)\right)\subseteq\mathscr{T}\left(w,(x,m)\right).$$

Proof. Let $u\in\mathscr{T}\left(v,(x,m)\right)\cap\mathscr{T}\left(v',(x,m)\right)$. Note that by construction $(y,t)\ R\,[\![u]\!]\ (x,m)$ implies that

$$v(y)-v(x)\geq m-t$$
$$v'(y)-v'(x)\geq m-t.$$

The result then follows from taking a convex combination of these two lines. □

We are now equipped to show a continuity and anonymity property of $\boldsymbol{a}$. Restrictions on $\mathcal{V}$ may result in there being certain options in $A_i(\boldsymbol{v})$ that are not chosen for any preference relation. For such options, the value of $a_i^x(\boldsymbol{v})$ is not relevant to the rule. Say that $a_i^x(\boldsymbol{v})$ is in the *relevant domain* of choice for i if there is $u \in \mathring{\mathcal{V}}$ such that $x \in \mathfrak{C}\,(u, \boldsymbol{a}_i(u, \boldsymbol{v}_{-i}))$.

Lemma 6. *For each $x \in \overline{X}$, $a_i^x(\boldsymbol{v})$ is continuous and anonymous in the relevant domain.*

Proof. Let $\boldsymbol{v}' \in \mathring{\mathcal{V}}^N$ be such that there is a bijection $\sigma : N \to N$ with $v'_j = v_i$ for each $j = \sigma(i)$. Suppose that $a_i^x(\boldsymbol{v}) > a_j^x(\boldsymbol{v}')$. Assume $a_j^x(\boldsymbol{v}')$ is in the relevant domain for j. Let $u \in \mathcal{V}$ have $x \in \mathfrak{C}\,(u, \boldsymbol{a}_j(\boldsymbol{v}))$. By *WAnon*,

$$U^x\left(\varphi_i(u, \boldsymbol{v}_{-i})\middle|u\right) = U^x\left(\varphi_j(u, \boldsymbol{v}'_{-j})\middle|u\right) < a_i^x(\boldsymbol{v}) = a_i^x(u, \boldsymbol{v}_{-i}),$$

violating *StP*. Now assume $a_i^x(\boldsymbol{v})$ is in the relevant domain for i and let u have $x \in \mathfrak{C}\,(u, \boldsymbol{a}_i(\boldsymbol{v}))$. Since $u \in \mathring{\mathcal{V}}$, there is $u' \in \mathscr{T}^{str.}(u, x)$. By *WAnon*,

$$a_i^x(u, \boldsymbol{v}_{-i}) = U^x\left(\varphi_i(u, \boldsymbol{v}_{-i})\middle|u\right) = U^x\left(\varphi_j(u, \boldsymbol{v}'_{-j})\middle|u\right) > a_j^x(u, \boldsymbol{v}'_{-j}) = a_j^x(\boldsymbol{v}).$$

By The Invariance Lemma, $\varphi_i(u', \boldsymbol{v}_{-i}) = \varphi_i(u, \boldsymbol{v}_{-i})$. For u' close enough to u, it remains that $x \notin \mathfrak{C}\left(u', a_j^x(\boldsymbol{v}')\right)$, and so letting $y = \xi_j\left(u, \boldsymbol{v}'_{-j}\right)$, we have

$$\begin{aligned} U^x\left(\varphi_j(u', \boldsymbol{v}'_{-j})\middle|u'\right) &= U^y\left(\varphi_j\left(u', \boldsymbol{v}'_{-j}\right)\middle|u'\right) + u'(y) - u'(x) \\ &= U^y\left(\varphi_j(u, \boldsymbol{v}'_{-j})\middle|u'\right) + u'(y) - u'(x) \\ &= U^y\left(\varphi_j\left(u, \boldsymbol{v}'_{-j}\right)\middle|u\right) + u'(y) - u'(x) \\ &= U^x\left(\varphi_j(u, \boldsymbol{v}'_{-j}\middle|u\right) + u(x) - u'(x) - (u(y) - u'(y)) \\ &< a_i^x(u, \boldsymbol{v}_{-i}), \end{aligned}$$

where the second line is again by The Invariance Lemma and the inequality is because $u' \in \mathscr{T}^{str.}(u, x)$. This contradicts *WAnon*. It follows that the relevant domain is independent of identity and so $\boldsymbol{a}$ is anonymous. Continuity then follows immediately from *WCon* and the fact that elements of $\mathring{\mathcal{V}}$ admit strict Maskin monotonic transforms at each bundle. □

A.1.2. *Some useful isomorphisms.* A *model* is a list $\mathcal{M} = (\mathcal{A}, \mathcal{Z}, \mathcal{R}, \varphi)$ consisting of a consumption space $\mathcal{A}$, a feasibility constraint $\mathcal{Z} \subseteq \mathcal{A}^N$, preference space $\mathcal{R}$ and a rule

$\varphi : \mathcal{R}^N \to \mathcal{Z}$. We shall induce mappings between models via transformations of the consumption space. Thus, given a bijection $f : \mathcal{A} \to \mathcal{A}'$, we shall define the induced model $\mathcal{M}' = (\mathcal{A}', \mathcal{Z}', \mathcal{R}', \varphi')$ in a fairly obvious way. Allocation $(z_1, \ldots, z_{\overline{n}}) \in \mathcal{Z}'$ if and only if there is an allocation $(a_1, \ldots, a_{\overline{n}}) \in \mathcal{Z}$ with $z_i = f(a_i)$ for each $i \in N$. Preference relation $R_i' \in \mathcal{R}'$ if and only if there is $R_i \in \mathcal{R}$ such that $a\ R\ b$ if and only if $f(a)\ R'\ f(b)$.[10] With abuse of notation, we write $R' = f(R)$. And finally, φ' is defined so that it makes the above operations commute: letting $\varphi(\boldsymbol{R}) = \boldsymbol{z}$, $\varphi'(f(\boldsymbol{R})) = f(\boldsymbol{z})$.

As is customary, we shall be interested in the structures and operations that are preserved under these mappings. It is clear that preference information is preserved, so that upper contour sets map to upper contour sets, and $R' \in \mathscr{T}(R, (x, m))$ if and only if $f(R') \in \mathscr{T}(f(R), f(x, m))$. Thus, we say that any bijection f induces an *ordinal isomorphism* between models.

If f also preserves quasilinearity and convex combinations, then we shall call it a *linear isomorphism.* The simplest of these is permuting the names of the items. Since this can destroy the underlying lattice order on $\overline{X}$, it may not preserve submodularity.

Consider the model $\left(\overline{X} \times \mathbb{R}, Z, \mathcal{V}, \varphi\right)$ that we have been studying. Let $(x^*, m^*) \in \overline{X} \times \mathbb{R}$ and $v_0 \in \overline{\mathcal{V}}$. Consider $f : \overline{X} \times \mathbb{R} \to \overline{X} \times \mathbb{R}$ given by

$$f(x, s) = (x, s + v_0(x)).$$

First, note that for $(x, s') = f(x, s)$ and $(x, t') = f(x, t)$, $s' - t' = s - t$, and so we say that f is *isometric.* Given $v \in \mathcal{V}$ let $R = f(v)$. Then

$$\begin{gathered}(x, s)\ I\ (y, t) \iff \\ v(x) + s - v_0(x) = v(y) + t - v_0(y) \\ u(x) + s = u(y) + t,\end{gathered}$$

where $u(z) = v(z) - v_0(z)$. Thus, u is a representation of R and so f preserves quasilinearity. Moreover, the resulting preference space $f(\mathcal{V})$ is simply $\mathcal{V}$ with v_0 viewed as the origin of the space. That is, f operated on $\mathcal{V}$ is simply a translation, and so maps convex sets to convex sets. The salient feature of this f is that it maps v_0 to the zero vector. This will be very useful.

[10]Viewing R_i as a subset of $\mathcal{A} \times \mathcal{A}$, we note that $(f, f) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}' \times \mathcal{A}'$ provides the necessary bijection between preferences.

A.2. **No nested upper contour sets.** In this section we show that a rule satisfying our conditions cannot admit an allocation at which the upper-contour set of one agent is nested completely within the upper contour set of another. That is, the section is dedicated to proving the following result:

Lemma 7 (The Anti-nesting Lemma)**.** *Assume the rule satisfies 2StP, ETE, and WCon. For each* $\boldsymbol{v} \in \mathring{\mathcal{V}}^N$*, and each pair of agents* $\{i, j\} \subseteq N$*,*

$$\boldsymbol{U}\left[v_i, \varphi_i(\boldsymbol{v})\right] \bigcap \boldsymbol{L}\left[v_j, \varphi_j(\boldsymbol{v})\right] \neq \emptyset.$$

Our proof is by contradiction, so throughout this subsection, we make the following assumption: there is an economy $\boldsymbol{v} \in \mathcal{V}^N$ such that

$$\boldsymbol{U}\left[v_1, \varphi_1(\boldsymbol{v})\right] \bigcap \boldsymbol{L}\left[v_2, \varphi_2(\boldsymbol{v})\right] = \emptyset. \qquad \text{(Contradiction Hypothesis)}$$

If a condition such as this holds for a given problem, we say that agent 1 nests agent 2 and that the allocation has nested upper contour sets. Choosing agents 1 and 2 is without loss of generality for the purposes of our proof and we suppress notation for the other agents for the rest of this section. Note that this relieves us from having to use the $\cdot_{-i}$ operator, and it is unambiguous that when we write $\varphi(v_2, v_1)$, it means that agent 1 is reporting v_2 and agent 2 is reporting v_1; such arguments will be important. It will be useful to note that (Contradiction Hypothesis) has the following, equivalent formulation:

$$\forall z \in \overline{X}, U^z\left(\varphi_1(\boldsymbol{v})\middle|v_1\right) > U^z\left(\varphi_2(\boldsymbol{v})\middle|v_2\right). \tag{A.2}$$

If there are nested upper-contour sets for a non-regular problem, then by *welfare continuity* and since regular problems are dense, there are nested upper-contour sets for a regular problem. Thus, for our contradiction argument, it is sufficient to assume $\boldsymbol{v}$ is regular, that is, $\boldsymbol{v} \in \mathring{\mathcal{V}}^N$.

We limit our arguments to operations preserved by linear isomorphisms, and thus we are free to make any of these transformations that suits our purposes. Thus, assume that $v_2 = \mathbf{0}$, and choose a representation for v_1 that is non-negative. Moreover, note that regular preferences map to regular preferences.

Let $\varphi_1 = (\xi_1, \mu_1) = \varphi_1(\boldsymbol{v})$. Since $v_2 = \mathbf{0}$, the indifference set of v_2 through any bundle (z, s) is $\{(y, t) : t = s\}$. Thus, line A.2 implies

$$\forall z \in \overline{X}, U^z\left(\varphi_1(\boldsymbol{v})\middle|v_1\right) > \mu_2, \tag{A.3}$$

and in particular, for $z = \xi_1$, $\mu_1 > \mu_2$. Note that *strategy-proofness* implies that, for each $y \in \overline{X}$, $U^y\left(\varphi_i(\boldsymbol{v})\middle|v_i\right) \geq a_i^y(\boldsymbol{v})$, and so

$$\max_{y \in \overline{X}} a_2^y(v_1) \leq \mu_2. \tag{A.4}$$

Lemma 8. *There is a problem $\boldsymbol{u} \in \mathring{\mathcal{V}}^N$ such that agent 1 nests agent 2 at $\boldsymbol{u}$, and $u_1 \in \mathscr{T}\left(u_2, \varphi_1(\boldsymbol{u})\right)$.*

Proof. Suppose not. Let $u_1 \in \mathscr{T}^{str.}\left(v_1, \xi_1\right) \cap \mathscr{T}\left(v_2, \xi_1\right)$ be regular. By The Invariance Lemma,

$$\varphi_1(u_1, v_2) = \varphi_1. \tag{A.5}$$

If $\mu_2(u_1, v_2) < \mu_1$, then $\boldsymbol{L}\left[v_2, \varphi_2(u_1, v_2)\right] = \{(y,t) : t \leq \mu_2(u_1, v_2)\}$ has an empty intersection with $\boldsymbol{U}\left[u_1, \varphi_1(u_1, v_2)\right]$ and so, since u_1 and v_2 are both regular, (u_1, v_2) is the problem required for the lemma. Assume, therefore, that

$$\mu_2(u_1, v_2) \geq \mu_1. \tag{A.6}$$

For each $\theta \in [0,1]$, let $v_2^\theta = \theta v_1 + (1-\theta) v_2$. As $v_1, v_2 \in \mathring{\mathcal{V}}$ and this is a convex set, $v_2^\theta \in \mathring{\mathcal{V}}$. Given $\delta < \mu_1 - \mu_2$, there is $\theta > 0$ sufficiently small so that, for each $z \in \overline{X}$,

$$U^z\left(\varphi_2(v_1, v_2^\theta)\middle|v_2^\theta\right) \leq U^z\left(\varphi_2(\boldsymbol{v})\middle|v_2\right) + \delta = \mu_2 + \delta < \mu_1. \tag{A.7}$$

Claim. $U^{\xi_1}\left(\varphi_1\left(v_1, v_2^\theta\right)\middle|v_1\right) = \mu_1$

Proof. Let $(y,t) = \varphi_1(v_1, v_2^\theta)$. Lines A.6 and A.7 imply that 2 prefers the joint manipulation from (v_1, v_2^θ) to (u_1, v_2). If $U^z\left(y,t\middle|v_1\right) < U^z\left(\varphi_1\middle|v_1\right)$, then 1 also prefers the joint manipulation, by line A.5.

Suppose instead that $U^z\left(y,t\middle|v_1\right) > U^z\left(\varphi_1\middle|v_1\right)$. Then, line A.3 yields

$$\forall z \in \overline{X}, U^z\left(y,t\middle|v_1\right) > U^z\left(\varphi_1\middle|v_1\right) > \mu_2. \tag{A.8}$$

Now let $w_1^\theta \in \mathscr{T}^{str.}\left(v_1, (y,t)\right) \cap \mathscr{T}\left(v_2^\theta, (y,t)\right)$ be regular (recalling that v_1 and v_2 are regular, so this intersection is non-empty). By *strategy-proofness*, $\varphi_1\left(w_1^\theta, v_2^\theta\right) = (y,t)$. If there are nested upper contour sets at (w_1^θ, v_2^θ) then the lemma is shown.

Otherwise, there is an item x where 2's indifference crosses above 1's. Then

$$\begin{aligned}
U^y\left(\varphi_2(w_1^\theta, v_2^\theta)\middle|v_2^\theta\right) - U^y\left(y,t\middle|w_1^\theta\right) &= U^x\left(\varphi_2(w_1^\theta, v_2^\theta)\middle|v_2^\theta\right) - U^x\left(y,t\middle|w_1^\theta\right) \\
&\quad + v_2^\theta(x) - v_2^\theta(y) - \left(w_1^\theta(x) - w_1^\theta(y)\right) \\
&\geq w_1^\theta(y) - w_1^\theta(x) - \left(v_2^\theta(y) - v_2^\theta(x)\right) \geq 0,
\end{aligned}$$

where the last inequality is by our characterization of Maskin monotonic transforms (Lemma 2). Thus,

$$U^y\left(\varphi_2\left(w_1^\theta, v_2^\theta\right)\middle|v_2^\theta\right) \geq U^y\left(y, t\middle|w_1^\theta\right) = t = U^y\left(y, t\middle|v_2^\theta\right).$$

Now for any $z \in \overline{X}$,

$$\begin{aligned} U^z\left(\varphi_2\left(w_1^\theta, v_2^\theta\right)\middle|v_2^\theta\right) &\geq U^z\left(y, t\middle|v_2^\theta\right) \\ \text{by Lemma 3} \quad &= \theta U^z\left(y, t\middle|v_1\right) + (1-\theta)U^z\left(y, t\middle|v_2\right) \\ &\geq \min\left\{U^z\left(y, t\middle|v_1\right), U^z\left(y, t\middle|v_2\right)\right\}. \end{aligned}$$

Note that $U^z\left(y, t\middle|v_2\right) = t$ because $v_2 = \mathbf{0}$. If we use $y = z$ in line A.8 we get $t > \mu_2$. Line A.8 also directly gives $U^z\left(y, t\middle|v_1\right) > \mu_2$ and so these observations allow us to continue the above string of inequalities as

$$> \mu_2 = U^z\left(\varphi_2(v_1, v_2)\middle|v_2\right).$$

In sum, for each $z \in \overline{X}$, $U^z\left(\varphi_2\left(w_1^\theta, v_2^\theta\right)\middle|v_2^\theta\right) > U^z\left(\varphi_2(v_1, v_2)\middle|v_2\right)$. With $z = \xi_2\left(w_1^\theta, v_2^\theta\right)$, we conclude that

$$U^z\left(\varphi_2(w_1^\theta, v_2^\theta)\middle|v_2\right) = \mu_2\left(w_1^\theta, v_2^\theta\right) = U^z\left(\varphi_2\left(w_1^\theta, v_2^\theta\right)\middle|v_2^\theta\right) > U^z\left(\varphi_2(v_1, v_2)\middle|v_2\right)$$

and so agent 2, at $\boldsymbol{v}$, prefers joint deviation (w_1^θ, v_2^θ). Recalling $\varphi_1\left(w_1^\theta, v_2^\theta\right) = (y, t)$, line A.8 implies that agent 1 also prefers this joint deviation, violating *2StP*. □

Claim. $a_1^{\xi_1}(v_2^\theta) \geq \mu_1$.

Proof. Suppose $a_1^{\xi_1}(v_2^\theta) < \mu_1$. Since v_1 and u_1 are regular, and the domain is convex, given ε, we may find $v_1' \in \mathscr{T}^{str.}(v_1, \varphi_1)$ with $\|v_1 - v_1'\| < \varepsilon$ and $u_1' \in \mathscr{T}^{str.}(u_1, \varphi_1)$ with $\|u_1 - u_1'\| < \varepsilon$. Since $\varphi_1(u_1, v_2) = \varphi_1$ (see the first paragraph), $\varphi_1(u_1', v_2) = \varphi_1$. Since $U^{\xi_1}\left(\varphi_1(v_1, v_2^\theta)\middle|v_1\right) = \mu_1$, $\xi_1 \notin \mathfrak{C}(v_1, \boldsymbol{a}(v_2^\theta))$ and so by The Invariance Lemma, for ε sufficiently small,

$$U^{\xi_1}\left(\varphi_1(v_1', v_2^\theta)\middle|v_1'\right) < \mu_1.$$

By lines A.6 and A.7, and *welfare continuity*, we can choose ε small enough that

$$\max_{y \in \overline{X}} U^y\left(\varphi_2(v_1', v_2^\theta)\middle|v_2^\theta\right) < \mu_1 - \delta \leq \mu_1(u_1', v_2).$$

Thus we find a joint manipulation from $\left(v_1', v_2^\theta\right)$ to (u_1', v_2). □

Thus, the two claims yield

$$a_1^{\xi_1}(v_2^\theta) = U^{\xi_1}\left(\varphi_1\left(v_1, v_2^\theta\right)\middle|v_1\right) = \mu_1.$$

If there are nested upper contour sets at (u_1, v_2^θ), then, recalling that $u_1 \in \mathscr{T}^{str.}(v_1, \xi_1) \cap \mathscr{T}(v_2, \xi_1)$, The Invariance Lemma and Lemma 5, yield what is desired.

Let

$$\theta^* = \max\left\{\theta \in [0,1] : U^{\xi_1}\left(\varphi_1(v_1, v_2^\theta)\middle|v_1\right) = a_1^{\xi_1} = \mu_1\right\}.$$

Our previous arguments imply the above set is non-empty, and *welfare continuity* ensures that it is closed, and so the maximum is well-defined and $\theta^* > 0$. There remain nested upper contour sets at $(v_1, v_2^{\theta^*})$. To see why, letting $y \in \overline{X}$ have $v_1(y) = \max_{x\in\overline{X}} v_1(x)$, if there were $z \in \overline{X}$ with

$$\begin{aligned}
0 &\le U^z\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right) - U^z\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)\\
&= U^y\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right) - U^y\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)\\
&\quad + \theta^*\left(v_1(y) - v_1(z)\right) + (1-\theta^*)(\cancelto{0}{v_2(y)} - \cancelto{0}{v_2(z)}) - \left(v_1(y) - v_1(z)\right)\\
&= U^y\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right) - U^y\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)\\
&\quad - (1-\theta^*)\left(v_1(y) - v_1(z)\right),
\end{aligned}$$

then $U^y\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right) \ge U^y\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)$. Since $v_2 = \mathbf{0}$, y is also a maximal item for $v_2^{\theta^*}$, so letting $z = \xi_2(v_1, v_2^{\theta^*})$, line A.4 yields

$$\begin{aligned}
\mu_2 &\ge U^z\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right)\\
&= U^y\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right) + v_2^{\theta^*}(y) - v_2^{\theta^*}(z)\\
&\ge U^y\left(\varphi_2(v_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right).
\end{aligned}$$

Thus, $\mu_2 \ge U^y\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)$, implying via line A.3 that $U^y\left(\varphi_1\middle|v_1\right) > U^y\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)$. Switching reference item from y to ξ_1, this gives $\mu_1 > U^{\xi_1}\left(\varphi_1(v_1, v_2^{\theta^*})\middle|v_1\right)$, contradicting *welfare continuity*.

Thus, at $(v_1, v_2^{\theta^*})$, it remains that 1 nests 2. Suppose $\theta^* < 1$. Consider the transformation given by $f(x, s) = (x, s + v_2^{\theta^*}(x) - v_2^{\theta^*}(\xi_1))$. This takes us to a problem with the same features that we started with and so we can repeat the argument. In particular, it maps $v_2^{\theta^*}$ to the constant vector $v_2^{\theta^*}(\xi_1)\boldsymbol{e}$, which is equivalent to $\mathbf{0}$, and $f(\xi_1, \mu_1) = (\xi_1, \mu_1)$. Post transformation, it remains that $U^{\xi_1}\left(\varphi_1\left(v_1, v_2^{\theta^*}\right)\middle|v_1\right) = \mu_1$, and since Lemma 5 implies $u_1 \in \mathscr{T}\left(v_2^{\theta}, \varphi_1\right)$,

$$\boldsymbol{U}\left[u_1, \varphi_1\right] \subseteq \boldsymbol{U}\left[v_2^{\theta}, \varphi_1\right] = \{(y,t) : t \geq \mu_1\}.$$

Since there are not nested upper contour sets at $(u_1, v_2^{\theta^*})$ (two paragraphs ago), we have

$$U^{\xi_1}\left(\varphi_1(u_1, v_2^{\theta^*})\middle|v_2^{\theta^*}\right) \geq \mu_1.$$

Therefore, we can apply the foregoing arguments to $(v_1, v_2^{\theta^*})$ and find $\theta > \theta^*$ with

$$U^{\xi_1}\left(\varphi_1(v_1, v_2^{\theta})\middle|v_1\right) = a_1^{\xi_1}(v_2^{\theta}) = \mu_1,$$

in contradiction to the assumption that θ^* is the maximal such parameter. Conclude that $\theta^* = 1$, which together with line A.4, yields a contradiction to *equal treatment of equals.* □

The lemma therefore allows us to complete the proof assuming that at $\boldsymbol{v}$, agent 1 nests agent 2 and $v_1 \in \mathscr{T}\left(v_2, \varphi_1(\boldsymbol{v})\right)$. Let $\varphi_1 = \varphi_1(\boldsymbol{v})$ and $(\xi_2, \mu_2) = \varphi_2(\boldsymbol{v})$. As before, we are also free to assume $v_2 = \mathbf{0}$, which then implies that

$$\forall z \in \overline{X}, a_2^z(v_1) \leq U^z\left(\xi_2, \mu_2\middle|v_2\right) = \mu_2, \tag{A.9}$$

and that, since v_1 is a Maskin monotonic transform of v_2 at φ_1,

$$\forall z \in \overline{X}, U^z\left(\xi_1, \mu_1\middle|v_1\right) \geq \mu_1.$$

For each $\theta \in [0,1]$ and each $i \in \{1,2\}$, let $v_i^{\theta} = \theta v_1 + (1-\theta)v_2$. We shall construct a net $(v_1^{\lambda}, v_2^{\theta})$ with first element $\boldsymbol{v}$. The first component of the net will then have λ decreasing from 1 and the second component will have θ increasing from 0. We shall maintain $\lambda \geq \theta$ and we shall show that the indifference sets of the two agents remain bounded away from each other. This will then give a contradiction to *equal treatment of equals*, as the net will have limit $\boldsymbol{u}$ with $u_1 = u_2$.

Given $\frac{1}{n}$, assume $\theta > 0$ is such that

$$\max_{y \in \overline{X}} U^y\left(\varphi_2(v_1, v_2^{\theta})\middle|v_2^{\theta}\right) < \mu_2 + \frac{1}{n}. \tag{A.10}$$

Suppose that

$$U^z\left(\varphi_1(v_1, v_2^{\theta})\middle|v_1\right) < U^z\left(\varphi_1(\boldsymbol{v})\middle|v_1\right). \tag{A.11}$$

Let

$$\lambda^1 = \inf\left\{\lambda \in [0,1] : \mathfrak{C}\left(v_1^{\lambda}, \boldsymbol{a}_1(v_2)\right) \subseteq \mathfrak{C}\left(v_1, \boldsymbol{a}_1(v_2)\right)\right\}.$$

As the set has a non-empty interior, $\lambda^1 < 1$. Now if there are $\lambda > \lambda^1$ and $y \in \overline{X}$ such that

$$U^y\left(\varphi_2(v_1^\lambda, v_2)\middle|v_2\right) \geq \mu_2 + \frac{1}{n},$$

then $v_2 = \mathbf{0}$ implies $\mu_2(v_1^\lambda, v_2) \geq \mu_2 + {}^1\!/_n$. Our assumption on θ then ensures that agent 2, at (v_1, v_2^θ), prefers the joint manipulation (v_1^λ, v_2). Since $\lambda > \lambda^1$, for $z = \xi_1(v_1^\lambda, v_2) \in \mathfrak{C}\left(v_1^\lambda, \boldsymbol{a}_1(v_2)\right)$,

$$U^z\left(\varphi_1(v_1^\lambda, v_2)\middle|v_1\right) = a_1^z(v_2) = U^z\left(\varphi_1(\boldsymbol{v})\middle|v_1\right).$$

Together with line A.11, we conclude that agent 1 also prefers the joint manipulation. If for each n sufficiently large, there is θ satisfying lines A.10 and A.11, then we say that $U^z\left(\varphi_1(v_1, v_2^\theta)\middle|v_1\right)$ is locally decreasing in θ at 0. Thus we have shown that, if $U^z\left(\varphi_1(v_1, v_2^\theta)\middle|v_1\right)$ is locally decreasing in θ at 0, then for each $\lambda \geq \lambda^1$,

$$\max_{y\in\overline{X}} U^y\left(\varphi\left(v_1^\lambda, v_2\right)\middle|v_2\right) \leq \mu_2,$$

where the weak inequality in $\lambda \geq \lambda^1$ is by *welfare continuity*.

Note that $v_1^{\lambda^1} \in \mathscr{T}(v_2, \varphi_1)$, since this set contains v_2 and is convex by Lemma 4. Suppose that, for $\theta > 0$, small enough

$$U^{\xi_1}\left(\varphi_1(v_1^{\lambda^1}, v_2^\theta)\middle|v_1^{\lambda^1}\right) < U^{\xi_1}\left(\varphi_1(v_1^{\lambda^1}, v_2)\middle|v_1^{\lambda^1}\right). \tag{A.12}$$

Then we can repeat the previous argument and construct a sequence λ^n. Note that this sequence proceeds from one element to another by changing $\mathfrak{C}\left(v_1^{\lambda^n}, \boldsymbol{a}_1(v_2)\right)$, so there are at most $n^* \in \mathbb{N}$ elements. Assume $\lambda^{n^*} = 0$. Since 2's welfare is not increasing along this sequence,

$$U^{\xi_1}\left(\varphi_2(v_1^{\lambda^{n^*}}, v_2)\middle|v_2\right) \leq \mu_2 < \mu_1 = a_1^{\xi_1}(v_2) \leq U^{\xi_1}\left(\varphi_1\left(v_1^{\lambda^{n^*}}, v_2\right)\middle|v_1^{\lambda^{n^*}}\right),$$

a contradiction to *equal treatment of equals*. Thus, our construction reaches some λ^n for which $U^{\xi_1}\left(\varphi_1\left(v_1^{\lambda^n}, v_2\right)\middle|v_1^{\lambda^n}\right)$ is *not* locally decreasing in θ. Let $v_1^1 = v_1^{\lambda^n}$.

For $\theta > 0$ sufficiently small,

$$U^{\xi_1}\left(\varphi_1\left(v_1^1, v_2^\theta\right)\middle|v_1^1\right) \geq U^{\xi_1}\left(\varphi_1\left(v_1^1, v_2\right)\middle|v_1^1\right) \geq \mu_1.$$

Let $(y,t) = \varphi_2(v_1^1, v_2^\theta)$. Since $v_1 \in \mathscr{T}(v_2, \varphi_1)$ and $v_2 = \mathbf{0}$, Lemma 2 implies that for each $z \in \overline{X}$, $v_1(\xi_1) \geq v_1(z)$. Since $U^{\xi_1}\left(\varphi_2(v_1^1, v_2)\middle|v_2\right) \leq \mu_2$, *strategy-proofness* and

$v_2 = \mathbf{0}$ imply that $\boldsymbol{a}_2(v_1^1) \le \mu_2 \boldsymbol{e}$, and so $a_2^y(v_1^1) = t \le \mu_2$. Thus we calculate

$$\begin{aligned} U^{\xi_1}\left(y, t \middle| v_2^\theta\right) &= U^y\left(y, t \middle| v_2^\theta\right) + v_2^\theta(y) - v_2^\theta(x^*) \\ &= t + \theta\left(v_1(y) - v_1(x^*)\right) \\ &\le t \le \mu_2, \end{aligned}$$

We now apply the mapping $f(x, s) = (x, s + v_2^\theta(x) - v_2^\theta(\xi_1))$, which sends v_2^θ to the equivalent of $\mathbf{0}$ and (ξ_1, r) to (ξ_1, r) for each $r \in \mathbb{R}$. Therefore, post transformation,

$$U^{\xi_1}\left(\varphi_2(v_1^1, v_2^\theta) \middle| v_2^\theta\right) = t \le \mu_2 < \mu_1 \le U^{\xi_1}\left(\varphi_1\left(v_1^1, v_2^\theta\right) \middle| v_1^1\right). \tag{A.13}$$

Do this for each θ at which $U^{\xi_1}\left(\varphi_1\left(v_1^1, v_2^\theta\right) \middle| v_1^1\right)$ is locally non-decreasing in θ. Either v_2^θ gets to v_1^1 or we get to θ with $U^{\xi_1}\left(\varphi_1\left(v_1^1, v_2^\theta\right) \middle| v_1^1\right)$ locally decreasing in θ. In the former case, line A.13 gives a contradiction to *equal treatment of equals* and in the latter we simply repeat the construction of v_1^λ to get v_1^2.

Thus we have our net (v_1^n, v_2^θ), which is monotone in the order $\left(v_1^n, v_2^\theta\right) \prec \left(v_1^m, v_2^{\theta'}\right)$ if $m > n$ or $\theta' > \theta$. Only one component of the net advances at a time. The first component is discrete and tends toward v_2, and the second is a continuous path that tends toward v_1. Line A.13 is preserved all along the net. The limit $\boldsymbol{u}$ of this net has $u_1 = u_2$; if not, then by construction it remains that $u_1 \in \mathscr{T}(u_2, \varphi_1)$ and we can repeat the arguments above. However, since $u_1 = u_2$ but line A.13 remains true, we have a contradiction to *equal treatment of equals*.

A.3. **Quasi-Equilibrium.** In this section, we prove Theorems 1, 3, and 4. As in the previous section, it suffices to focus on agents 1 and 2, with our default problem being $\boldsymbol{v} = (v_1, v_2)$, and the default allocation given by $(\xi, \mu) = \varphi = \varphi(\boldsymbol{v})$.

We first uncover a lemma that will allow us to take care of the case when real items come in multiple copies.

Lemma 9. *Assume real items come in copies and the rule satisfies, 2StP, WAnon, WCon and neutral tie-breaking. Let* $\boldsymbol{w} \in \mathcal{V}$ *have* $\xi_2(\boldsymbol{w}) = \xi_1(\boldsymbol{w})$ *and* $w_2 \in \mathscr{T}^{str.}(w_1, \xi_1(\boldsymbol{w}))$. *If* $\varphi_1(\boldsymbol{w})\ P\ [\![w_2]\!]\ \varphi_2(\boldsymbol{w})$, *then* $\xi_1(\boldsymbol{w}) = 0$.

Proof. Suppose $\xi_1(\boldsymbol{w}) \in X$. As $w_2 \in \mathscr{T}^{str.}(w_1, \xi_1(\boldsymbol{w}))$, $\mathfrak{C}(w_2, A_1(w_2)) = \{\varphi_1(\boldsymbol{w})\}$. Since $\boldsymbol{a}_i$ is anonymous, this holds for agent 2 as well, so for each $i \in \{1, 2\}$, $\varphi_i(w_2, w_2) = \varphi_1(\boldsymbol{w})$. Given $\lambda \in [0, 1]$, let $u^\lambda = \lambda w_2 + (1 - \lambda) w_1$ and let λ^* be the maximal number such that there is $(x, m) \ne \varphi_1(\boldsymbol{w})$ with $(x, m) \in \mathfrak{C}\left(u^\lambda, A_1(u^\lambda)\right)$. We just deduced that $\lambda^* < 1$. If $\lambda^* = 0$, then $\varphi_1(\boldsymbol{w}) \in A_1(u^0) = A_2(w_1)$, a contradiction to *strategy-proofness*. Thus, $\lambda^* \in]0,1[$. By *WCon*, since A_i is continuous, $\varphi_1(\boldsymbol{w}) \in \mathfrak{C}\left(u^{\lambda^*}, A_i(u^{\lambda^*})\right)$. Then by invoking *neutral tie-breaking*, we may assume $\varphi_2(u^{\lambda^*}, u^{\lambda^*}) = \varphi_1(\boldsymbol{w})$ and $\xi_1(u^{\lambda^*}, u^{\lambda^*}) \neq \xi_1(\boldsymbol{v})$. Clearly, at profile $\boldsymbol{w}$, 2 prefers the joint deviation $(u^{\lambda^*}, u^{\lambda^*})$. For 1, *WAnon* implies $\varphi_1(u^{\lambda^*}, u^{\lambda^*})\ I\ [\![u^{\lambda^*}]\!]\ \varphi_1(\boldsymbol{w})$, and since $u^{\lambda^*} \in \mathscr{T}^{str.}(w_1, \varphi_1(\boldsymbol{w}))$, $\varphi_1(u^{\lambda^*}, u^{\lambda^*})\ P\ [\![v_1]\!]\ \varphi_1(\boldsymbol{w})$. □

Lemma 10. *Assume that φ either satisfies the hypotheses of Theorem 1 or of Theorem 3. It does not matter whether items come in copies or not. If $\varphi_1(\boldsymbol{v})\ P\ [\![v_2]\!]\ \varphi_2(\boldsymbol{v})$, then letting $\{i,j\} = \{1,2\}$, there is a problem $\boldsymbol{w}$ with $\xi_i(\boldsymbol{w}) = \xi_j(\boldsymbol{w}) = \xi_1(\boldsymbol{v})$, $w_j \in \mathscr{T}^{str.}(w_i, \xi_i(\boldsymbol{w}))$ and $\varphi_i(\boldsymbol{w})\ P\ [\![w_j]\!]\ \varphi_j(\boldsymbol{w})$.*

Proof. As usual, let $\varphi = (\xi, \mu) = \varphi(\boldsymbol{v})$. For most of this proof, we require only *2Stp*, *ETE*, and *WCon*. The only exceptions come in Claim 2 below, where we separately treat the two cases, one in which the rule satisfies the extra conditions of Theorem 3, but not *WAnon*, and the other in which the rule satisfies *WAnon*.

By linear isomorphism, we assume $v_2 = \mathbf{0}$. This further implies that

$$\forall y \in \overline{X}, \mu_2 = U^y\left(\varphi_2 \middle| v_2\right) < U^y\left(\varphi_1 \middle| v_2\right) = \mu_1. \tag{A.14}$$

By richness, since $\boldsymbol{v}$ is regular, there is $u_1 \in \mathscr{T}^{str.}(v_1, \xi_1) \cap \mathscr{T}^{str.}(v_2, \xi_1)$. Then by Lemma 4, for each $\theta \in [0,1]$ and $\lambda \in [0,1]$,

$$v_1^\theta = (1-\theta)v_1 + \theta u_1 \in \mathscr{T}^{str.}(v_1, \xi_1), \tag{A.15}$$

$$v_2^\lambda = (1-\lambda)v_2 + \lambda u_1 \in \mathscr{T}^{str.}(v_2, \xi_1) \tag{A.16}$$

The Invariance Lemma implies that, for each $\theta \in [0,1]$,

$$\varphi_1\left(v_1^\theta, v_2\right) = \varphi_1. \tag{A.17}$$

By Lemma 2 we may choose a representation for u_1 with $u_1 \leq v_2 = \mathbf{0}$, with equality only at ξ_1. Thus,

$$\forall y \in \overline{X}, U^y\left(\varphi_1 \middle| u_1\right) = U^{\xi_1}\left(\varphi_1 \middle| u_1\right) + u_1(\xi_1) - u_1(y) \geq \mu_1.$$

Therefore, by The Anti-nesting Lemma, the fact that $v_2 = \mathbf{0}$, and line A.14,

$$\forall y \in \overline{X}, U^y\left(\varphi_2(u_1, v_2) \middle| v_2\right) \geq \mu_1 > \mu_2 = U^y\left(\varphi_2 \middle| v_2\right). \tag{A.18}$$

By *welfare continuity* then, there is θ_1 satisfying

$$\forall y \in \overline{X}, U^y\left(\varphi_1\left(v_1^{\theta_1}, v_2\right)\middle| v_2\right) = \frac{\mu_1+\mu_2}{2}. \tag{A.19}$$

Henceforth, when no ambiguity may arise, we may replace v_i^η by η, as we will be working primarily with the two dimensional subset of profiles of the form (v_1^θ, v_2^η). Let

$$S^1 = \left\{\lambda \in [0,1] : \forall y \in \overline{X}, U^y\left(\varphi_2(\theta_1,\lambda)\middle| v_2^\lambda\right) > \mu_2\right\}.$$

Claim 1. For each $\lambda \in S^1$, $\varphi_1(\theta_1,\lambda) = \varphi_1$.

Proof. Let $\hat{S}^1$ be the subset of S^1 in which 1's welfare has not changed. That is, it is the set of λ such that $\varphi_1(\theta_1,\lambda)\ I\ [\![v_1^{\theta_1}]\!]\ \varphi(\theta_1,v_2) = \varphi_1$. Suppose there is $\lambda \in \hat{S}^1$ with $\xi_1(\theta_1,\lambda) = y \neq \xi_1$. Then since $\theta_1 > 0$, by line A.15 and the definition of a strict Maskin monotonic transform, $\varphi_1(\theta_1,\lambda)\ P\ [\![v_1]\!]\ \varphi_1$. Further, the definition of S^1 and v_2 yield that

$$\begin{aligned} U^{\xi_2(\theta_1,\lambda)}\left(\varphi_2(\theta_1,\lambda)\middle| v_2\right) &= U^{\xi_2(\theta_1,\lambda)}\left(\varphi_2(\theta_1,\lambda)\middle| v_2^\lambda\right) \\ &> \mu_2 \\ &= U^{\xi_2(\theta_1,\lambda)}\left(\varphi_2\middle| v_2\right). \end{aligned}$$

In sum, we have a joint manipulation from $\boldsymbol{v}$ to (θ_1,λ).

Now suppose that there were $\lambda \in S^1$ for which $\varphi_1(\theta_1,\lambda)\ P\ [\![v_1^{\theta_1}]\!]\ \varphi_1$. By lines A.15 and A.17, $\varphi_1(\theta_1,\lambda)\ P\ [\![v_1]\!]\ \varphi_1$. The calculations above again hold for agent 2 and we have a joint manipulation from $\boldsymbol{v}$ to (θ_1,λ).

Let $\eta = \inf S^1 \setminus \hat{S}^1$. Note that S^1 is non-empty and open, and $\hat{S}^1$ is closed, so $S^1 \setminus \hat{S}^1$ is open. If $S^1 \setminus \hat{S}^1$ is empty, our proof would be done. As a boundary point of $\hat{S}^1$, $\eta \in \hat{S}^1$. Note that, since $\varphi_1(\theta_1,v_2) = \varphi_1\ P\ [\![v_2]\!]\ \varphi_2(\theta_1,v_2)$ (see line A.19), if $\varphi_2(\theta_1,\eta)\ R\ [\![v_2^\eta]\!]\ \varphi_1$, then by line A.16,

$$\varphi_2(\theta_1,\eta)\ P\ [\![v_2]\!]\ \varphi_1\ P\ [\![v_2]\!]\ \varphi_2(\theta_1,v_2)$$

a violation of *strategy-proofness*. Therefore, $\varphi_1\ P\ [\![v_2^\eta]\!]\ \varphi_2(\theta_1,\eta)$, and so given $\delta > 0$, there is an open set $W \ni \eta$ such that if $\eta' \in W$ then

$$(\xi_1, \mu_1 - \delta)\ P\ \left[\!\left[v_2^{\eta'}\right]\!\right]\ \varphi_2(\theta_1,\eta'). \tag{A.20}$$

Next we claim that for each such $\eta' \in W$,

$$\varphi_2(u_1,\eta)\ R\left[\!\left[v_2^{\eta'}\right]\!\right](\xi_1,\mu_1-\delta). \tag{A.21}$$

Since $\eta \in \hat{S}^1$, $\varphi_1(\theta_1,\eta)=\varphi_1$, and so by The Invariance Lemma, $\varphi_1(u_1,\eta)=\varphi_1$. Assume $\varphi_1\ P\,[\![v_2^\eta]\!]\ \varphi_2(u_1,\eta)$. Equivalently, $\boldsymbol{U}[v_2^\eta,\varphi_1]\subseteq int\,(\boldsymbol{U}[v_2^\eta,\varphi_2(u_1,\eta)])$. Since $u_1\in\mathscr{T}(v_2^\eta,\xi_1)$, we get $\boldsymbol{U}[u_1,\varphi_1]\subseteq\boldsymbol{U}[v_2^\eta,\varphi_1]$. In sum,

$$\boldsymbol{U}[u_1,\varphi_1(u_1,\eta)]=\boldsymbol{U}[u_1,\varphi_1]\subseteq\boldsymbol{U}[v_2^\eta,\varphi_1]\subseteq int\,(\boldsymbol{U}[v_2^\eta,\varphi_2(u_1,\eta)])\,,$$

violating The Anti-nesting Lemma. Conclude that $\varphi_2(u_1,\eta)\ R\,[\![v_2^\eta]\!]\ \varphi_1$ and the result follows by choosing W small enough.

Combining lines A.20 and A.21 yields, for each $\eta'\in W$, $\varphi_2(u_1,\eta)\ P\left[\!\left[v_2^{\eta'}\right]\!\right]\varphi_2(\theta_1,\eta')$. Since we have shown that $U^y\left(\varphi_1(\theta_1,\lambda)\middle|v_1^{\theta_1}\right)$ cannot increase on S^1, and since we assume $S^1\setminus\hat{S}^1$ is non-empty, there is a sequence $\eta^n\in S^1\setminus\hat{S}^1$, $\eta^n\to\eta$, such that $\varphi_1\ P\,[\![v_1^{\theta_1}]\!]\ \varphi_1(\theta_1,\eta^n)$. As η^n is eventually in W, (u_1,η) is a successful joint deviation from (θ_1,η^n). □

As in the argument immediately following line A.17, since $u_1\in\mathscr{T}^{str.}(v_2,\xi_1)$, for each $y\in\overline{X}$, any (z,t), and any $\lambda\in[0,1]$,

$$\begin{aligned}U^y\left(z,t\middle|v_2^\lambda\right)&=U^{\xi_1}\left(z,t\middle|v_2^\lambda\right)+v_2^\lambda(\xi_1)-v_2^\lambda(y)\\&=U^{\xi_1}\left(z,t\middle|v_2^\lambda\right)+(1-\lambda)\left(\cancel{v_2(\xi_1)}-\cancel{v_2(y)}\right)+\lambda\left(u_1(\xi_1)-u_1(y)\right)\\&\geq U^{\xi_1}\left(z,t\middle|v_2^\lambda\right).\end{aligned}$$

Thus, the indifference set of v_2^λ at any bundle takes its minimum at ξ_1. Let $\lambda_1=\sup S^1$. Suppose there is $\lambda<\lambda_1$ such that $\xi_2(\theta_1,\lambda)=\xi_1$. By definition, $U^{\xi_1}\left(\varphi_2(\theta_1,\lambda)\middle|v_2^\lambda\right)>\mu_2$ and so since $v_2^{\lambda'}\in\mathscr{T}^{str.}\left(v_2^\lambda,\xi_1\right)$ when $\lambda'>\lambda$, $\varphi_2(\theta_1,\lambda')=\varphi_2(\theta_1,\lambda)$ and $U^{\xi_1}\left(\varphi_2(\theta_1,\lambda)\middle|\lambda'\right)>\mu_2$. It follows that $S^1=[0,1]$ and so by the claim we arrive at $\varphi_1(\theta_1,u_1)=\varphi_1$, $\xi_2(\theta_1,u_1)=\xi_1$, and $\varphi_1(\theta_1,u_1)\ P\,[\![u_1]\!]\ \varphi_2(\theta_1,u_1)$. This is the conclusion of the Lemma, so in this case, we are done.

Assume, therefore, that for each $\lambda<\lambda_1$, $\xi_2(\theta_1,\lambda)\neq\xi_1$.

Claim 2. We claim that

$$U^{\xi_1}\left(\varphi_2(\theta_1,\lambda_1)\middle|v_2^{\lambda_1}\right)=\mu_2. \tag{A.22}$$

Proof. If line A.22 is false, then for each $y\in\overline{X}$, $U^y\left(\varphi_2(\theta_1,\lambda_1)\middle|v_2^{\lambda_1}\right)>\mu_2$. It follows that $\lambda_1=1$, so $v_2^{\lambda_1}=u_1$. By Claim 1 and *StP*, for each $\theta'>\theta_1$, $\varphi_1(\theta',u_1)=\varphi_1$ and by *ETE*, eventually, $\varphi_2(\theta', u_1)\ R\,[\![u_1]\!]\ \varphi_1$. By *WCon*, for each $\varepsilon > 0$, there is $\theta > \theta_1$ with $U^{\xi_1}\left(\varphi_2(\theta, u_1)\middle|u_1\right) = \mu_1 - \varepsilon$. Assume that for each such ε, $\xi_2(\theta, u_1) \neq \xi_1$, as otherwise we have again shown what is required.

Case 1. φ satisfies the conditions of Theorem 3.

Given $\overline{m} \in \mathbb{R}$, we could have chosen u_1 so that, for each $y \neq \xi_1$, $U^y\left(\varphi_1\middle|u_1\right) > \overline{m}$. It follows then that for ε sufficiently small,

$$\begin{aligned}
\mu_2(\theta, u_1) &= U^{\xi_2(\theta,u_1)}\left(\varphi_2(\theta, u_1)\middle|u_1\right)\\
&= U^{\xi_1}\left(\varphi_2(\theta, u_1)\middle|u_1\right) + u_1(\xi_1) - u_1\left(\xi_2(\theta, u_1)\right)\\
&= \mu_1 - \varepsilon + u_1(\xi_1) - u_1\left(\xi_2(\theta, u_1)\right)\\
&= U^{\xi_1}\left(\varphi_1\middle|u_1\right) + u_1(\xi_1) - u_1\left(\xi_2(\theta, u_1)\right) - \varepsilon\\
&= U^{\xi_2(\theta,u_1)}\left(\varphi_1\middle|u_1\right) - \varepsilon > \overline{m},
\end{aligned}$$

so φ cannot be bounded.

Case 2. φ satisfies *WAnon*.

Then for ε small enough, letting $z = \xi_2(\theta, u_1)$,

$$\begin{aligned}
U^{\xi_1}\left(\varphi_2(\theta, u_1)\middle|v_1\right) &= U^{z}\left(\varphi_2(\theta, u_1)\middle|v_1\right) + v_1(z) - v_1(\xi_1)\\
&= U^{z}\left(\varphi_2(\theta, u_1)\middle|u_1\right) + v_1(z) - v_1(\xi_1)\\
&= U^{\xi_1}\left(\varphi_2(\theta, u_1)\middle|u_1\right) + u_1(\xi_1) - u_1(z) - \left(v_1(\xi_1) - v_1(z)\right)\\
&= \mu_1 - \varepsilon + u_1(\xi_1) - v_1(\xi_1) + v_1(z) - u_1(z)\\
&> U^{\xi_1}\left(\varphi_1\middle|v_1\right),
\end{aligned}$$

where the last inequality is again because $u_1 \in \mathscr{T}^{str.}(v_1, \xi_1)$ (with reference to Lemma 2) and by choosing ε sufficiently small. Thus, 1 and 2 would like to swap bundles at (v_1^θ, u_1). If they switch preferences, then since $\mathfrak{C}\left(v_1^\theta, \boldsymbol{a}_1(u_1)\right) = \{\varphi_1\}$ (as $\theta > \theta_1$), *WAnon* implies agent 2 is better off. Agent 1 only fails to be better off if $\xi_1(u_1, \theta) = \xi_1$, and we have yet again shown the required conclusion.

□

Given line A.19, there is $y \in \overline{X}$ with $a_2^y(\theta_1) = \frac{\mu_1+\mu_2}{2}$. Thus, by *strategy-proofness*,

$$U^{y}\left(\varphi_2(\theta_1, \lambda_1)\middle|v_2^{\lambda_1}\right) - U^{\xi_1}\left(\varphi_2\left(\theta_1, \lambda_1\right)\middle|v_2^{\lambda_1}\right) \geq \frac{\mu_1+\mu_2}{2} - \mu_2 = \frac{\mu_1-\mu_2}{2}.$$

Applying our conversion formulae, the left hand side of this inequality becomes

$$v_2^{\lambda_1}(\xi_1) - v_2^{\lambda_1}(y) = \lambda_1 \left(u_1(\xi_1) - u_1(y)\right).$$

Recalling again that $u_1 \leq v_2 = \mathbf{0}$, with equality at ξ_1, we derive that

$$\lambda_1 \geq \underline{\lambda} \coloneqq \min_{y \neq \xi_1} \frac{\mu_2 - \mu_1}{2\,|u_1(y)|}.$$

Now since $\varphi_1(\theta_1, \lambda) = \varphi_1$ for all $\lambda \in S^1$, by *WCon*, $a_1^{\xi_1}(v_2^{\lambda_1}) = \mu_1$ and $\xi_1 \in \mathfrak{C}\left(v_1^{\theta_1}, \boldsymbol{a}(v_2^{\lambda_1})\right)$. Consider the linear isomorphism $f(x,s) = \left(x, s + v_2^{\lambda_1}(x) - v_2^{\lambda_1}(\xi_1)\right)$. This sends $v_2^{\lambda_1}$ to the zero preference relation, and sends each (ξ_1, t) to (ξ_1, t). Though it need not be the case that $\xi_1\left(v_1^{\theta_1}, v_2^{\lambda_1}\right) = \xi_1$, it will hold for $v_1^{\theta} \in \mathscr{T}^{str.}\left(v_1^{\theta_1}, \xi_1\right)$, and we can repeat all the foregoing arguments, constructing a sequence (θ_n, λ_n). For each n,

$$U^{\xi_1}\left(\varphi_2\left(v_1^{\theta_n}, v_2^{\lambda_n}\right)\big|v_2^{\lambda_n}\right) = \mu_2.$$

Since f is isometric, each $\lambda_n - \lambda_{n-1} \geq \underline{\lambda}$, and so λ_n will reach 1 in finitely many steps. Thus, repeating our arguments, we either arrive at the conclusion, or there is a joint manipulation. □

Proof of Theorem 3. Assume the rule satisfies all the hypothesis of the theorem *except* that

$$\xi_i(\boldsymbol{v}) = \xi_j(\boldsymbol{v}) = 0 \implies \mu_i(\boldsymbol{v}) = \mu_j(\boldsymbol{v}). \tag{A.23}$$

We argue that, in this case, *not* no envy implies *not* line A.23, and thus we have the proof by contrapositive. As real items do not come in copies, for this case, Lemma 10 implies $\xi_1(\boldsymbol{v}) = 0$ directly. The conclusion of the lemma further implies that $\mu_1(\boldsymbol{w}) \neq \mu_2(\boldsymbol{w})$, contradicting line A.23. □

We now, very briefly, show that the rule satisfies

No Envy of Real Items: If $\varphi_i(\boldsymbol{v})\ P\,[\![v_j]\!]\ \varphi_j(\boldsymbol{v})$, then $\xi_i(\boldsymbol{v}) = 0$.

Lemma 11. *The rule satisfies no envy of real items.*

Proof. Assume $\varphi_1(\boldsymbol{v})\ P\,[\![v_2]\!]\ \varphi_2(\boldsymbol{v})$. If real items do not come in copies, Lemma 10 implies $\xi_1(\boldsymbol{v}) = 0$ and so the proof is complete. In the case that real items do come in copies, Lemmas 9 and 10 together imply $\xi_1(\boldsymbol{v}) = 0$. □

To complete the proof, we need one final lemma.

Lemma 12. *Assume the rule satisfies 2StP, WAnon, WCon, and no envy of real items. Let $\boldsymbol{v} \in \mathcal{V}$ have $\xi_1(\boldsymbol{v}) = 0$ and $v_2 \in \mathscr{T}^{str.}(v_1, 0)$. Then $\varphi_2(\boldsymbol{v})\ R\,[\![v_2]\!]\ \varphi_1(\boldsymbol{v})$.*

Proof. Fix $\varphi = (\xi, \mu) = \varphi(\boldsymbol{v})$. Suppose $\varphi_1\ P\,[\![v_2]\!]\ \varphi_2$. As usual, we may, via isomorphism, assume $v_1 = \mathbf{0}$. By Lemma 2, we may choose a representation for v_2 such that $v_2(y) < 0$ for $y \neq \xi_1 = 0$ and $v_2(\xi_1) = 0$. For each $\theta \in [0, 1]$ and each $i \in \{1, 2\}$, let $v_i^\theta = (1-\theta)v_1 + \theta v_2 = \theta v_2$. We shall work on this one-dimensional subspace of preferences for much of this proof, so for cleaner notation, we may replace v_i^θ with θ where there can be no confusion. Note that since $v_2 \in \mathscr{T}^{str.}(v_1, 0)$, $\{\varphi_1\} = \mathfrak{C}(v_2, A_i(v_2))$. Thus, by *WAnon*, $\varphi_1(1, 1) = \varphi_2(1, 1) = \varphi_1$. It follows then that, for each θ and each i, $\varphi_i(\theta, \theta)\ R\,[\![\theta]\!]\ \varphi_1$, as otherwise there is a joint deviation from (θ, θ) to $(1, 1)$.

Let θ^* be the maximal parameter such that $\mathfrak{C}(\theta^*, \boldsymbol{a}_i(\theta^*)) \cap X \neq \emptyset$. Since for each $\theta > \theta^*$, $\mathfrak{C}(\theta, \boldsymbol{a}_i(\theta)) = \{0\}$, then to avoid a joint manipulation, $a_i^0(\theta) = \mu_1$. By *WCon*, $\varphi_1 \in \mathfrak{C}(\theta^*, A_i(\theta^*))$, so it must be that $\theta^* > 0$ since, by assumption, $\varphi_1 \notin A_2(v_1)$. Note that for $y \in \mathfrak{C}(\theta^*, \boldsymbol{a}_i(\theta^*))$, not 0,

$$\begin{aligned} a_i^y(\theta^*) &= U^y\left(\varphi_i(\theta^*, \theta^*)\middle|\theta^*\right) \\ &= U^0\left(\varphi_i(\theta^*, \theta^*)\middle|\theta^*\right) - \theta^* v_2(y) \\ &= \mu_1 - \theta^* v_2(y). \end{aligned} \tag{A.24}$$

Thus, since $v_2(y) < 0$ and $\theta^* > 0$, $a_i^y(\theta^*) > \mu_1$.

Given θ and $\delta > 0$, let $u_2^{\theta,\delta}(0) = v_2^\theta(0) + \delta = \delta$ and for all $x \in X$, $u_2^{\delta,\theta}(x) = v_2^\theta(x)$. Fix δ and let $u_2 = u_2^{\theta^*,\delta}$. Suppose $a_1^0(u_2) = m > \mu_1$ and let $\varphi' = (\xi', \mu') = \varphi(u_2, u_2)$. For each $x \in \overline{X}$ and each $i \in \{1, 2\}$,

$$\begin{aligned} U^x\left(\varphi_i'\middle|u_2\right) &= U^0\left(\varphi_i'\middle|u_2\right) + u_2(0) - u_2(x) \\ &\geq m + \delta - v_i^{\theta^*}(x) \\ &> \mu_1 + \delta - v_i^{\theta^*}(x) \\ &= U^0\left(\varphi_i(\theta^*, \theta^*)\middle|v_i^{\theta^*}\right) + \delta - v_i^{\theta^*}(x) \\ &= U^x\left(\varphi_i(\theta^*, \theta^*)\middle|v_i^{\theta^*}\right) + \delta. \end{aligned} \tag{A.25}$$

Since x is arbitrary, and $\mu_i' = U^{\xi_i'}\left(\varphi_2'\middle|u_2\right)$, this shows a joint deviation. Conclude that

$$\exists\bar{\delta}\forall\delta \in]0, \bar{\delta}[\left(a_1^0(u_2) \leq \mu_1\right). \tag{A.26}$$

Since $\varphi_1 \in \mathfrak{C}(\theta^*, A_2(\theta^*))$, $\{\varphi_1\} = \mathfrak{C}(u_2, A_2(\theta^*))$. If $U^0\left(\varphi_1(\theta^*, u_2)\,\middle|\theta^*\right) < \mu_1$, then into equation A.25, insert φ_1 for φ_i', and $\varphi_2(\theta^*, u_2)$ for $\varphi_i(\theta^*, \theta^*)$. The inequalities become equalities and we have another violation of The Anti-nesting Lemma. So $U^0\left(\varphi_1(\theta^*, u_2)\middle|\theta^*\right) \geq \mu_1$. Now assume there is $x \in X$ in $\mathfrak{C}(\theta^*, \boldsymbol{a}_1(u_2))$. The same argument as in line A.24 yields $a_1^x(u_2) > \mu_1$. There is $v_1' \in \mathscr{T}^{str.}(\theta^*, x)$, close enough to $v_1^{\theta^*}$ so it remains that $\mathfrak{C}(u_2, \boldsymbol{a}_2(v_1')) = \{0\}$ and $a_2^0(v_1') > \mu_2$. Then we have a joint deviation from $\boldsymbol{v}$ to (v_1', u_2). Conclude that, 1 chooses only 0 at (θ^*, u_2) and $U^0\left(\varphi_1(\theta^*, u_2)\,\middle|\theta^*\right) = a_1^0(u_2) = \mu_1$.

In sum, $\mathfrak{C}(\theta^*, A_i(u_2)) = \mathfrak{C}(u_2, A_i(\theta^*)) = \{\varphi_1\}$. By *WCon*, there is an open set W of problems of the form $\left(v_1^\theta, u_2^{\theta,\delta}\right)$, such that, for each $\boldsymbol{w} \in W$, $\mathfrak{C}(w_1, A_1(w_2)) = \mathfrak{C}(w_2, A_2(w_1)) = \{\varphi_1\}$. We may find a further open subset $W^* \subseteq W$ such that, for each $\boldsymbol{w} \in W^*$ and each $y \in \mathfrak{C}(\theta^*, \boldsymbol{a}_i(\theta^*))$, $U^y\left(\varphi_2(\boldsymbol{w})\middle|w_2\right) > a_i^y(\theta^*)$. For each $\boldsymbol{w} \in W^*$ and each $y \in \mathfrak{C}(\theta^*, \boldsymbol{a}_i(\theta^*))$, line A.24 yields

$$\begin{aligned} U^y\left(\varphi_1(\boldsymbol{w})\middle|w_1\right) &= U^0\left(\varphi_1(\boldsymbol{w})\middle|w_1\right) + w_1(0) - w_1(y) \\ &= \mu_1 - w_1(\xi_1(\boldsymbol{w})) \\ &= a_1^y(\theta^*) + \theta^* v_2(y) - v_1^\theta(y) \\ &= a_1^y(\theta^*) + (\theta^* - \theta)\, v_2(y) \\ &< a_1^y(\theta^*), \end{aligned} \tag{A.27}$$

where, for the last inequality we recall that $v_2(y) < 0$. Now for $\nu \in [0,1]$, let w_2^ν be in the convex hull of w_2 and $v_2^{\theta^*}$ with $w_2^0 = w_2$, and let $\varphi^\nu = \varphi(w_1, w_2^\nu)$. Then $a_1^y(w_2^\nu) \to a_1^y(\theta^*)$ as $\nu \to 1$. By *StP* there is $\nu > 0$, such that $U^y\left(\varphi_1^\nu\middle|w_1\right) > U^y\left(\varphi_1^0\middle|w_1\right)$. Let ν^* be the infimum of such values, so that $U^y\left(\varphi_1^{\nu^*}\middle|w_1\right) = U^y\left(\varphi_1^0\middle|w_1\right)$, but the inequality holds for at least a sequence of $\nu^n \searrow \nu^*$. We perform a similar calculation to line A.25: for $x \in X$, $w_2^{\nu^*}(x) < 0$, so

$$\begin{aligned} U^x\left(\varphi_2^{\nu^*}\middle|w_2^{\nu^*}\right) &= U^0\left(\varphi_2^{\nu^*}\middle|w_2\right) + w_2^{\nu^*}(0) - w_2^{\nu^*}(x) \\ &= \mu_1 - w_2^{\nu^*}(x) \\ &= U^0\left(\varphi_1^{\nu^*}\middle|w_1\right) - w_2^{\nu^*}(x) \\ &> U^x\left(\varphi_1^{\nu^*}\middle|v_i^{\theta^*}\right). \end{aligned}$$

Thus, by Lemma 11, $U^x\left(\varphi_2^{\nu^*}\middle|w_2^{\nu^*}\right) > a_2^x(w_1)$. It follows that for $\nu > \nu^*$ on the aforementioned sequence and sufficiently close to ν^*, $U^x\left(\varphi_2^\nu\middle|w_2^\nu\right) > a_2^x(w_1)$. In particular,

$\varphi_2^\nu = \varphi_1$. Now recalling $v_1 = \mathbf{0}$ and $v_2 \le 0$ with $v_2(0) = 0$, we calculate

$$\begin{aligned} U^{\xi_1^\nu}\left(\varphi_1^\nu \middle| v_1\right) &= U^y\left(\varphi_1^\nu \middle| v_1\right) \\ &= U^y\left(\varphi_1^\nu \middle| w_1\right) + w_1(y) - w_1(\xi_1^\nu) \\ &> U^y\left(\varphi_1^0 \middle| w_1\right) + w_1(y) - w_1(\xi_1^\nu) \\ &= U^0\left(\varphi_1^0 \middle| w_1\right) + w_1(0) - w_1(\xi_1^\nu) \\ &= \mu_1 + \theta v_2(0) - \theta v_2(\xi_1^\nu) \\ &\ge \mu_1. \end{aligned}$$

Thus, agent 1 prefers the joint report $\boldsymbol{w}^\nu$when the truth is $\boldsymbol{v}$. Since $\varphi_2^\nu = \varphi_1$, agent 2 agrees, and we have a joint deviation. □

Proof of Theorems 1 and 4. Suppose $\varphi_1(\boldsymbol{v})\ P\,[\![v_2]\!]\ \varphi_2(\boldsymbol{v})$. By Lemma 11, the rule satisfies *no envy for real items*, so $\xi_1(\boldsymbol{v}) = 0$. By Lemma 10, there is a problem $\boldsymbol{w}$ with $\xi_i(\boldsymbol{w}) = \xi_j(\boldsymbol{w}) = 0$, $w_j \in \mathscr{T}^{str.}(w_i, 0)$ and $\varphi_i(\boldsymbol{w})\ P\,[\![w_j]\!]\ \varphi_j(\boldsymbol{w})$. However, Lemma 12 implies that, for this problem, we have $\varphi_j(\boldsymbol{w})\ R\,[\![w_j]\!]\ \varphi_i(\boldsymbol{w})$. □

Appendix B. Proof of Theorem 2

Let us strengthen *pairwise weak monotonicity* to

Joint Monotonicity: If for each $i \in N$, $R_i' \in \mathscr{T}(R_i, \varphi_i(\boldsymbol{R}))$, then for each $i \in N$, $\varphi_i(\boldsymbol{R}')\ R_i'\ \varphi_i(\boldsymbol{R})$.

In our model, we can strengthen the theorem of Barberà et al. (2016) (BBM) into the following form:

Proposition 2. *If a rule on a rich domain, $\varphi : \mathcal{R}^N \to Z^N$, satisfies strategy-proofness and joint-monotonicity, then it satisfies weak group-strategy-proofness.*

Proof. Given the results of BBM, we need only to show that *strategy-proofness* and *joint-monotonicity* in this model imply *respectfulness.* Fix $\boldsymbol{R} \in \overline{\mathcal{R}}^N$ and assume that R_i' has $\boldsymbol{U}[R_i', \varphi_i(\boldsymbol{R})] = \boldsymbol{U}[R_i, \varphi_i(\boldsymbol{R})]$. Note that this implies the equality of the lower-contour and indifference sets as well. Now if $\varphi_i(R_i', \boldsymbol{R}_{-i})\ P_i'\ \varphi_i(\boldsymbol{R})$, then $\varphi_i(R_i', \boldsymbol{R}_{-i})\ P_i\ \varphi_i(\boldsymbol{R})$ in violation of *strategy-proofness*. We can make a symmetric observation about R_i and therefore conclude that both R_i and R_i' are indifferent between $\varphi_i(R_i', \boldsymbol{R}_{-i})$ and $\varphi_i(\boldsymbol{R})$. This is the first hypothesis of *respectfulness*. It also implies that $R_i' \in \mathscr{T}(R_i, \varphi_i(\boldsymbol{R}))$ and $R_i \in \mathscr{T}(R_i', \varphi_i(R_i', \boldsymbol{R}_{-i}))$; this is the remaining hypothesis of *respectfulness.* Let $j \in N$. Then we invoke *joint-monotonicity* in two directions, from $\boldsymbol{R}$ to $(R_i', \boldsymbol{R}_{-i})$ and the opposite, to conclude that $\varphi_j(R_i', \boldsymbol{R}_{-i})\ I_j\ \varphi_j(\boldsymbol{R})$, which is the conclusion of *respectfulness.* □

Let φ be a rule and consider the induced two-person rule ψ given by $\psi(\boldsymbol{R}) = \varphi_{\{1,2\}}(\boldsymbol{R}, \boldsymbol{R}^0_{N\setminus\{1,2\}})$. We may apply Proposition 2 to ψ. In fact, if φ is *strategy-proof* and *pairwise weakly monotonic*, we can apply it for each $\boldsymbol{R}^0_{N\setminus\{1,2\}} \in \mathcal{R}^{N\setminus\{1,2\}}$ and conclude that

Corollary 7. *If a rule on a rich domain, $\varphi : \mathcal{R}^N \to Z^N$, satisfies strategy-proofness and pairwise weak monotonicity, then it satisfies weak pairwise strategy-proofness.*

The corollary and Theorem 1 then imply that a rule satisfying the conditions of Theorem 2 must be a quasi-equilibrium when restricted to the quasilinear domain, $\mathcal{V}$, induced by $\mathcal{R}$. To show the proof on the entire domain, we argue via induction. In particular, assume that φ satisfies no envy when restricted to $\mathcal{R}^{\{1,\ldots,k-1\}} \times \mathcal{V}^{\{k,\ldots,n\}}$. Suppose now that $R_k \in \mathcal{R} \setminus \mathcal{V}$ and $\varphi_i(\boldsymbol{R})\ P_k\ \varphi_k(\boldsymbol{R})$. Note that it cannot be that i and k both have quasilinear preferences as the two-person rule $\varphi\left(\cdot, \boldsymbol{R}_{-\{i,k\}}\right)$ is a quasi-equilibrium on the quasilinear domain. Thus, we can ignore agents $j \notin \{i,k\}$ for this argument so we suppress notation for their preferences.

Let $R_i' \in \mathscr{T}^{str.}(R_i, \varphi_i(\boldsymbol{R}))$ be close to R_i in terms of the uniform topology on the utility function representation. By The Invariance Lemma, $\varphi_i(R_i', R_k) = \varphi_i(\boldsymbol{R})$. By *welfare continuity* it remains that $\varphi_i(R_i', R_k)\ P_k\ \varphi_k(R_i', R_k)$. Let v_k be the quasilinear preference induced by R_k at $\varphi_k(R_i', R_k)$ and v_i' be the quasilinear preference induced by R_i' at $\varphi_i(R_i', R_k)$. Then by two invocations of respectfulness,

$$\varphi_i(v_i', v_k)\ I_i'\ \varphi_i(R_i', R_k), \tag{B.1}$$

$$\varphi_k(v_i', v_k)\ I_k'\ \varphi_k(R_i', R_k) \tag{B.2}$$

By the induction hypothesis, to prevent envy, we have $\varphi_i(R_i', v_k) \neq \varphi_i(R_i', R_k)$ and so $\varphi_i(v_i', v_k)\ P_i\ \varphi_i(\boldsymbol{R})$. Now let $v_i'' \in \mathscr{T}^{str.}(v_i', \varphi_i(v_i', v_k))$. By The Invariance Lemma, $\varphi_i(v_i'', v_k) = \varphi_i(v_i', v_k)$. By *joint monotonicity*, $\varphi_k(v_i'', v_k)\ R[\![v_k]\!]\ \varphi_k(v_i', v_k)$, and if this is strict, we have a joint manipulation from $\boldsymbol{R}$ to (v_i'', v_k) . Otherwise, it is an indifference relation, so we can invoke respectfulness again to get

$$\varphi_i(v_i'', R_k)\ I[\![v_i'']\!]\ \varphi_i(v_i'', v_k) = \varphi_i(v_i', v_k). \tag{B.3}$$

If $\varphi_i(v_i'', R_k) = \varphi_i(v_i', v_k)$, then since $\varphi_i(v_i', v_k)\ P_i\ \varphi_i(\boldsymbol{R})$, we get a violation of *strategy-proofness*. Otherwise, line B.3 implies $\varphi_i(v_i'', R_k)\ P\,[\![v_i']\!]\ \varphi_i(v_i', v_k)$, so the foregoing analysis yields

$$\begin{aligned} \varphi_i(v_i'', R_k) &\in int\,(\boldsymbol{U}[v_i', \varphi_i(v_i', v_k)]) \\ \text{by line B.1} \qquad &= int\,(\boldsymbol{U}[R_i', \varphi_i(R_i', R_k)]) \\ \text{since } \varphi_i(R_i', R_k) = \varphi_i(\boldsymbol{R}) \qquad &= int\,(\boldsymbol{U}[R_i', \varphi_i(\boldsymbol{R})]) \\ &\subseteq int\,(\boldsymbol{U}[R_i, \varphi_i(\boldsymbol{R})])\,, \end{aligned}$$

another violation of *strategy-proofness*.

References

Abdulkadiroglu, A., P. Pathak, A. E. Roth, and T. Sonmez (2006): "Changing the Boston School Choice Mechanism," Working Paper 11965, NBER.

Adachi, T. (2014): "Equity and the Vickrey allocation rule on general preference domains," *Social Choice and Welfare*, 42, 813–830.

Alva, S. and V. Manjunath (2019): "Strategy-proof Pareto-improvement," *Journal of Economic Theory*, 181, 121–142.

Andersson, T., L. Ehlers, L.-G. Svensson, and R. Tierney (2022): "Gale's Fixed Tax for Exchanging Houses," *Mathematics of Operations Research*.

Andreoni, J. and J. Miller (2002): "Giving according to GARP: An experimental test of the consistency of preferences for altruism," *Econometrica*, 70, 737–753.

Ashlagi, I. and S. Serizawa (2012): "Characterizing Vickrey allocation rule by anonymity," *Social Choice and Welfare*, 38, 531–542.

Baisa, B. (2020): "Efficient multiunit auctions for normal goods," *Theoretical Economics*, 15, 361–413.

Baisa, B. and J. Burkett (2019): "Efficient ex post implementable auctions and English auctions for bidders with non-quasilinear preferences," *Journal of Mathematical Economics*, 82, 227–246.

Barberà, S., D. Berga, and B. Moreno (2016): "Group strategy-proofness in private good economies," *American Economic Review*, 106, 1073–99.

Bikhchandani, S. and J. M. Ostroy (2002): "The package assignment model," *Journal of Economic theory*, 107, 377–406.

Demange, G. and D. Gale (1985): "The Strategy Structure of Two-Sided Matching Markets," *Econometrica*, 53, 873–888.

Dubins, L. E. and D. A. Freedman (1981): "Machiavelli and the Gale-Shapley algorithm," *The American Mathematical Monthly*, 88, 485–494.

Fleurbaey, M. and F. Maniquet (1997): "Implementability and Horizontal Equity Imply No-Envy," *Econometrica*, 65, 1215–1219.

Fujishige, S. and Z. Yang (2003): "A Note on Kelso and Crawford's Gross Substitutes Condition," *Mathematics of Operations Research*, 28, 463–469.

Gul, F. and E. Stacchetti (1999): "Walrasian equilibrium with gross substitutes," *Journal of Economic Theory*, 87, 95–124.

Hatfield, J. W. and F. Kojima (2009): "Group incentive compatibility for matching with contracts," *Games and Economic Behavior*, 67, 745–749.

Hirata, D. and Y. Kasuya (2017): "On stable and strategy-proof rules in matching markets with contracts," *Journal of Economic Theory*, 168, 27–43.

Holmström, B. (1979): "Groves' Scheme on Restricted Domains," *Econometrica*, 47, 1137–1144.

Kazumura, T., D. Mishra, and S. Serizawa (2020a): "Mechanism design without quasilinearity," *Theoretical Economics*, 15, 511–544.

——— (2020b): "Strategy-proof multi-object mechanism design: Ex-post revenue maximization with non-quasilinear preferences," *Journal of Economic Theory*, 188, 105036.

Kazumura, T. and S. Serizawa (2016a): "Efficiency and strategy-proofness in object assignment problems with multi-demand preferences," *Social Choice and Welfare*, 47, 633–663.

——— (2016b): "Efficiency and strategy-proofness in object assignment problems with multi-demand preferences," *Social Choice and Welfare*, 47, 633–663.

Kelso Jr, A. S. and V. P. Crawford (1982): "Job matching, coalition formation, and gross substitutes," *Econometrica*, 1483–1504.

Leonard, H. B. (1983): "Elicitation of Honest Preferences for the Assignment of Individuals to Positions," *Journal of Political Economy*, 91, 461–479.

Li, S. (2017): "Obviously strategy-proof mechanisms," *American Economic Review*, 107, 3257–87.

MALIK, K. AND D. MISHRA (2021): "Pareto efficient combinatorial auctions: Dichotomous preferences without quasilinearity," *Journal of Economic Theory*, 191, 105128.

MCKELVEY, R. D. AND T. R. PALFREY (1992): "An experimental study of the centipede game," *Econometrica*, 803–836.

MORIMOTO, S. AND S. SERIZAWA (2015): "Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule," *Theoretical Economics*, 10, 445–487.

MUKHERJEE, C. (2013): "Fair and group strategy-proof good allocation with money," *Social Choice and Welfare*, 42, 289–311.

MUROTA, K. (2003): *Discrete Convex Analysis*, SIAM.

MUROTA, K. AND A. SHIOURA (1999): "M-Convex Function on Generalized Polymatroid," *Mathematics of Operations Research*, 24, 95–105, publisher: INFORMS.

MYERSON, R. B. (1981): "Optimal auction design," *Mathematics of Operations Research*, 6, 58–73.

PÁPAI, S. (2003): "Groves sealed bid auctions of heterogeneous objects with fair prices," *Social Choice and Welfare*, 20, 371–385.

SAITOH, H. AND S. SERIZAWA (2008): "Vickrey allocation rule with income effect," *Economic Theory*, 35, 391–401.

SAKAI, T. (2007): "Second price auctions on general preference domains: two characterizations," *Economic Theory*, 37, 347–356.

——— (2012): "An equity characterization of second price auctions when preferences may not be quasilinear," *Review of Economic Design*, 17, 17–26.

SHAPLEY, L. S. AND M. SHUBIK (1971): "The assignment game I: The core," *International Journal of Game Theory*, 1, 111–130.

SVENSSON, L.-G. (2009): "Coalitional strategy-proofness and fairness," *Economic Theory*, 40, 227–245.

TIERNEY, R. (2019): "The problem of multiple commons: A market design approach," *Games and Economic Behavior*, 114, 1–27.

VAN LANGE, P. A. (1999): "The pursuit of joint outcomes and equality in outcomes: An integrative model of social value orientation." *Journal of Personality and Social Psychology*, 77, 337.