Core stability and other applications of minimal balanced collections

by

Dylan Laplace Mermoud, Michel Grabisch, and Peter Sudhölter

Discussion Papers on Economics
No. 4/2022

FURTHER INFORMATION
Department of Economics
Faculty of Business and Social Sciences
University of Southern Denmark
Campusvej 55, DK – 5230 Odense M
Denmark
Email: astni@sam.sdu.dk
https://www.sdu.dk/en/om_sdu/institutter_centre/oekonomiskinstitut

ISSN 2596-8157

Core stability and other applications of minimal balanced collections

Dylan Laplace Mermoud*†, Michel Grabisch‡, and Peter Sudhölter§

February 2022

Abstract

We describe algorithms and their implementations as computer programs derived from several theoretical results of the theory of cooperative transferable utility (TU) games. We show how to use Peleg's well-known inductive method to explicitly compute all minimal balanced collections of coalitions. The described method is of independent interest and applied in the implementations of (a) the Bondareva-Shapley Theorem, which allows checking whether a TU game is balanced, i.e., has a non-empty core, and (b) a recent result of the second and third author that provides a sufficient and necessary condition for the stability of the core, which allows checking whether a balanced TU game has a core that is a von Neumann-Morgenstern stable set.

Keywords: Core, stable set, minimal balanced collections, cooperative game.

MSC Subject Classification: 91A12
JEL Classification: C71, C44

1 Introduction

The theory of cooperative games aims at defining rational mechanisms (called solutions of the game) to compensate players for their collaboration, by sharing the total benefit generated by the cooperation of the players. In their seminal work, von Neumann and Morgenstern (1944) developed the concept of stable sets as a solution for cooperative games, while fifteen years later, Gillies (1959) popularized the concept of the core.

It rapidly appeared that the concept of stable sets, although appealing and based on a natural notion of domination among payment vectors, was intractable in many respects: some games may have no stable sets, or a continuum of stable sets (Lucas 1969, 1992), they are not

*Centre d'Économie de la Sorbonne, Université Paris 1 Panthéon-Sorbonne.
†Corresponding author (dylan.laplace.mermoud@gmail.com)
‡Paris School of Economics, Université Paris 1 Panthéon-Sorbonne.
§Department of Economics, University of Southern Denmark.

convex in general, and above all, they are extremely difficult to compute, and no condition for the existence of stable sets is known so far. On the other hand, the concept of core was revealed to be much easier to use. When nonempty, it is a convex polyhedron, and as soon as 1963, a necessary and sufficient condition was discovered for its nonemptiness (Bondareva, 1963; Shapley, 1967).

The key point in this condition is the notion of balanced collection. A balanced collection over a finite set N is a collection of subsets of N, each of them with a positive weight so that the sum of the weights for each element of N is equal to 1. Partitions are trivial examples of balanced collections. Minimal balanced collections are those for which no proper subcollection is still balanced, and only these are of interest. It is to be noted that game theory is not the only domain where balanced collections appear. They have arisen in several disciplines, each introducing its terminology. In graph theory, they are known as perfect fractional matching of hypergraphs, but they can be viewed as specific incidence structures, block designs, or set systems. Even if we restrict to minimal balanced collections, their number increases much faster than the number of partitions, and so far no closed form formula exists to enumerate them. As far as we know, nobody has computed minimal balanced collections when $|N| > 4$, despite that Peleg (1965) published a paper describing an inductive method to generate them.

The first aim of this paper is to provide a practical implementation of Peleg's algorithm as a computer program, allowing the generation of all minimal balanced collections on a set N of reasonable size, and consequently to know their number. The second, but more important, aim is to show that minimal balanced collections can be used for checking many properties of cooperative games, and not only nonemptiness of the core: checking properties of coalitions (exactness, strict vital-exactness, extendability), finding the set of effective coalitions, finding regions in the set of imputations, and checking stability of the core. We elaborate on the latter point. Checking if the core of a game, supposing it is nonempty, is a stable set in the sense of von Neumann-Morgenstern, remained for a long time an unsolved problem despite many attempts. There only exist sufficient conditions in the general case: convexity (Shapley, 1971), subconvexity (Sharkey, 1982), extendability (Kikuta and Shapley, 1986), vital-exact extendability, or necessary and sufficient conditions for restricted classes of games: matching games, simple flow games, minimum coloring games (Shellshear and Sudhölter, 2009). A necessary and sufficient condition in the general case was recently found by the second and third authors (2021), using a complex nested balancedness condition. Thanks to our implementation of Peleg's algorithm, it was possible to build a computer program testing core stability.

This nested balancedness condition requires a two-fold generalization of the traditional concept of *minimal balanced collection*. On the one hand side, the game has to be restricted to feasible coalitions. Investigations about the core for games with restricted cooperation are not new. For instance, Pulido and Sánchez-Soriano (2006) consider sets of feasible coalitions that guarantee that the core of the game with restricted cooperation remains bounded, whereas Grabisch and Sudhölter (2014) investigate the case of unbounded cores of games with restricted cooperation in which the sets of feasible coalitions form lattices. However, one of our feasibility conditions requires that the core of the original game must coincide with the core of the game with restricted cooperation (see Section 4). On the other hand, we consider minimal balanced collections of certain derived subsets of the positive orthant of the Euclidean space (see Section 5.2).

The paper is organized as follows. Section 2 introduces the necessary notions on cooperative games and stable sets. Section 3 introduces Bondareva-Shapley Theorem on the nonemptiness of the core and presents the algorithmic implementation of Peleg's inductive method to generate all minimal balanced collections. Section 4 shows how to check via minimal balanced collections several properties of cooperative games (exactness, extendability, etc.), while Section 5 is devoted to the implementation of an algorithm checking core stability, based on the recent publication of Grabisch and Sudhölter (2021), and presents some examples of games on which the algorithm has been tested. Section 6 gives some concluding remarks.

2 Preliminaries

Let N be the finite nonempty set of n players and let 2^N denote its power set, i.e., the set of all subsets of N. In most case, we tacitly assume that $N = \{1, \ldots, n\}$. A nonempty subset of N is called a *coalition*. A *(cooperative TU) game* is a pair (N, v) such that $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$.

Let (N, v) be a game. An *allocation* is an n-dimensional vector $x = (x_i)_{i \in N}$ of real numbers, interpreted as a proposal of how to distribute the payoffs among the players. Denote by $\mathbb{R}^N$ the set of all the allocations, and by $x(S) = \sum_{i \in S} x_i$ what the coalition $S \subseteq N$ receives with allocation x. An allocation is said to be *efficient* if $x(N) = v(N)$, and an efficient allocation is called a *preimputation*. The set of preimputations is denoted by $X(N, v)$. A preimputation x is *individually rational* if $x_i \geq v(\{i\})$ for every player $i \in N$. An individually rational preimputation is called an *imputation*, and the set of imputations is denoted by $I(N, v)$.

A preimputation $x \in X(N, v)$ *dominates via a coalition* $S \subseteq N$ another preimputation y if $x(S) \leq v(S)$ and $x_i > y_i$ for every $i \in S$, written $x \text{ dom}_S\, y$. If there exists a coalition S such that $x \text{ dom}_S\, y$, then x *dominates* y, which is denoted by $x \text{ dom } y$.

Using this notion, von Neumann and Morgenstern (1944) introduced the notion of stable sets for cooperative games. A set $U \subseteq I(N, v)$ is a *stable set* if it satisfies

1. *internal stability*: if $y \in U$ is dominated by $x \in I(N, v)$, then $x \notin U$,
2. *external stability*: $\forall y \in I(N, v) \setminus U, \exists x \in U$ such that $x \text{ dom } y$.

However, stable sets may or may not exist (Lucas, 1969), and are in general difficult to identify. According to Deng and Papadimitriou (1994), the existence of a stable set may be undecidable. These difficulties have led to the development of other solution concepts. According to Gillies (1959), the *core* is defined as the set of *coalitionally rational* preimputations: $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S), \forall S \subseteq N\}$. The core is said to be *stable* if it is a stable set. The computation of the core is relatively easy but expensive, due to the large number of inequalities defining it.

Both stable sets and the core have their own merits as solution concepts. Indeed, the notions of domination and stability are highly intuitive, "coalitional rationality" is a desirable property, and its easy computability supports the core. By definition, the core is contained in each stable set. Hence, if the core is (externally) stable, it must be the unique stable set. Therefore, it is an interesting and important problem to characterize the set of games for which the above-mentioned solution concepts coincide, i.e., to provide necessary and sufficient conditions for external stability of the core. This is what Grabisch and Sudhölter (2021) have achieved.

3 Nonemptiness of the core

3.1 Bondareva-Shapley Theorem

In order to recall the Bondareva-Shapley Theorem, we use the following notation. For $T \subseteq N$, the indicator allocation of T, $\mathbf{1}^T$, is given by $\mathbf{1}^T_i = 1$ if $i \in T$, and $\mathbf{1}^T_i = 0$ otherwise.

Definition 3.1.1. A collection $\mathcal{B}$ of coalitions is a *balanced* if there exists a system of positive weights $(\lambda_S)_{S\in\mathcal{B}}$, called *balancing weights*, such that $\sum_{S\in\mathcal{B}} \lambda \mathbf{1}^S = \mathbf{1}^N$.

A balanced collection is *minimal* if it does not contain a balanced proper subcollection. Denote by $\mathbb{B}(N)$ the set of minimal balanced collections on N. Proofs of the following proposition and theorem can be attributed to Bondareva (1963) and Shapley (1965).

Proposition 3.1.2. *A balanced collection is minimal if and only if it has a unique system of balancing weights.*

Henceforth, if $\mathcal{B}$ is a minimal balanced collection, $\lambda^{\mathcal{B}} = (\lambda^{\mathcal{B}}_S)_{S\in\mathcal{B}}$ denotes its unique system of balancing weights.

Theorem 3.1.3 (Bondareva-Shapley, sharp form)**.** *A game* (N, v) *admits a nonempty core if and only if, for any minimal balanced collection* $\mathcal{B}$*,*

$$\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N).$$

Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$*.*

According to the Bondareva-Shapley Theorem, a game with a nonempty core is said to be *balanced*. The main result of Grabisch and Sudhölter (2021) is also based on minimal balanced collections.

3.2 Peleg's algorithm

Peleg (1965) developed an inductive method to construct, from the minimal balanced collections defined on a set N, all those that are defined on the set $N' = N \cup \{p\}$, with p a new player that was not included in N. In this subsection, his results are introduced from an algorithmic point of view. In the following, the main result is divided into four cases and the fourth one is slightly generalized.

Let $\mathcal{C} = \{S_1, \dots, S_k\}$ be a balanced collection of k coalitions on N. Denote by $[k]$ the set $\{1, \dots, k\}$ for any positive integer k. If λ is a system of balancing weights for $\mathcal{C}$ and $I \subseteq [k]$ is a subset of indices, denote by λ_I the sum $\sum_{i\in I} \lambda_{S_i}$. Also, denote by $A^{\mathcal{C}}$ the $(n \times k)$-matrix defined by $A^{\mathcal{C}}_{ij} = 1$ if $i \in S_j$, $A^{\mathcal{C}}_{ij} = 0$ otherwise. Denote by $\mathrm{rk}(A^{\mathcal{C}})$ the rank of the matrix $A^{\mathcal{C}}$, meaning the dimension of the Euclidean space spanned by its column viewed as k-dimensional vectors.

First case Assume that $\mathcal{C}$ is a minimal balanced collection on N. Take $I \subseteq [k]$ such that $\lambda_I^{\mathcal{C}} = 1$. Denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i \in I}$ contain the new player p as additional member and the other coalitions $\{S_j\}_{j \in [k] \setminus I}$ are kept unchanged.

Lemma 3.2.1. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Because $\mathcal{C}$ is a minimal balanced collection, the equalities $\sum_{S \in \mathcal{C}', S \ni i} \lambda_S^{\mathcal{C}} = 1$ are already satisfied for any player $i \in N$. By definition of I, we also have that $\sum_{S \in \mathcal{C}', S \ni p} \lambda_S^{\mathcal{C}} = 1$. Then $\mathcal{C}'$ is balanced. Because $\mathcal{C}$ is minimal, so is $\mathcal{C}'$. □

Second case We assume that $\mathcal{C}$ is a minimal balanced collection on N. Take $I \subseteq [k]$ such that $\lambda_I^{\mathcal{C}} < 1$. We denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i \in I}$ contain the new player p as additional member, the other coalitions $\{S_j\}_{j \in [k] \setminus I}$ are kept unchanged, and in which the coalition $\{p\}$ is added:

$$\mathcal{C}' = \{S_i \cup \{p\} \mid i \in I\} \cup \{S_i \mid i \in [k] \setminus I\} \cup \{\{p\}\}.$$

Lemma 3.2.2. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Because $\mathcal{C}$ is a minimal balanced collection, the equalities $\sum_{S \in \mathcal{C}', S \ni i} \lambda_S^{\mathcal{C}} = 1$ are already satisfied for any player $i \in N$. Define $\lambda^{\mathcal{C}'}$ such that $\lambda_S^{\mathcal{C}'} = \lambda_S^{\mathcal{C}}$ for $S \in \mathcal{C}$ and $\lambda_{\{p\}}^{\mathcal{C}'} = 1 - \lambda_I^{\mathcal{C}}$. Therefore

$$\sum_{\substack{S \in \mathcal{C}' \\ S \ni p}} \lambda_S^{\mathcal{C}'} = \lambda_{\{p\}}^{\mathcal{C}'} + \sum_{i \in I} \lambda_{S_i}^{\mathcal{C}} = 1 - \sum_{i \in I} \lambda_{S_i}^{\mathcal{C}} + \sum_{i \in I} \lambda_{S_i}^{\mathcal{C}} = 1.$$

Then $\mathcal{C}'$ is balanced. We cannot obtain a balanced subcollection of $\mathcal{C}'$ by discarding one of the $\{S_i\}_{i \in I}$ because $\mathcal{C}$ is minimal, and we cannot either discard coalition $\{p\}$ because $\lambda_I^{\mathcal{C}} < 1$. So $\mathcal{C}'$ is minimal. □

Third case We assume that $\mathcal{C}$ is a minimal balanced collection on N. Take a subset $I \subseteq [k]$ and an index $\delta \in [k] \setminus I$ such that $1 > \lambda_I^{\mathcal{C}} > 1 - \lambda_{S_\delta}^{\mathcal{C}}$. We denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i \in I}$ contain the new player p as additional member, the other coalitions $\{S_j\}_{j \in [k] \setminus I}$ are kept unchanged, and in which the coalition $S_\delta \cup \{p\}$ is added:

$$\mathcal{C}' = \{S_i \cup \{p\} \mid i \in I\} \cup \{S_i \mid i \in [k] \setminus I\} \cup \{S_\delta \cup \{p\}\}.$$

Lemma 3.2.3. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Define $\lambda^{\mathcal{C}'}$ by $\lambda_S^{\mathcal{C}'} = \lambda_S^{\mathcal{C}}$ for $S \in \mathcal{C} \setminus \{S_\delta\}$,

$$\lambda_{S_\delta \cup \{p\}}^{\mathcal{C}'} = 1 - \lambda_I^{\mathcal{C}} \text{ and } \lambda_{S_\delta}^{\mathcal{C}'} = \lambda_{S_\delta}^{\mathcal{C}} - \lambda_{S_\delta \cup \{p\}}^{\mathcal{C}'}.$$

Let $i \in N$ be a player. If $i \notin S_\delta$, by balancedness of $\mathcal{C}$, $\sum_{S \in \mathcal{C}', S \ni i} \lambda_S^{\mathcal{C}'} = 1$. If $i \in S_\delta$, then

$$\sum_{\substack{S \in \mathcal{C}' \\ S \ni i}} \lambda_S^{\mathcal{C}'} = \lambda_{S_\delta \cup \{p\}}^{\mathcal{C}'} + \lambda_{S_\delta}^{\mathcal{C}'} + \sum_{\substack{S \in \mathcal{C} \setminus \{S_\delta\} \\ S \ni i}} \lambda_S^{\mathcal{C}'} = \lambda_{S_\delta}^{\mathcal{C}} + \sum_{\substack{S \in \mathcal{C} \setminus \{S_\delta\} \\ S \ni i}} \lambda_S^{\mathcal{C}} = \sum_{\substack{S \in \mathcal{C} \\ S \ni i}} \lambda_S^{\mathcal{C}},$$

that is equal to 1 by balancedness of $\mathcal{C}$. Concerning player p,

$$\sum_{\substack{S \in \mathcal{C}' \\ S \ni p}} \lambda_S^{\mathcal{C}'} = \lambda_{S_\delta \cup \{p\}}^{\mathcal{C}'} + \lambda_I^{\mathcal{C}'} = 1 - \lambda_I^{\mathcal{C}} + \lambda_I^{\mathcal{C}} = 1.$$

Then $\mathcal{C}'$ is balanced. Because none of the coalitions $S \in \mathcal{C}$ or $S_\delta \cup \{p\}$ can be discarded to obtain a balanced subcollection, the proof is finished. □

Last case In this case, assume that $\mathcal{C}$ is the union of two different minimal balanced collections on N, $\mathcal{C}^1$, and $\mathcal{C}^2$, such that the rank of $A^{\mathcal{C}}$ is $\text{rk}(A^{\mathcal{C}}) = k-1$. Define two systems of balancing weights for $\mathcal{C}$, by

$$\mu_S = \begin{cases} \lambda_S^{\mathcal{C}^1} \text{ if } S \in \mathcal{C}^1, \\ 0 \text{ otherwise.} \end{cases} \qquad \nu_S = \begin{cases} \lambda_S^{\mathcal{C}^2} \text{ if } S \in \mathcal{C}^2, \\ 0 \text{ otherwise.} \end{cases}$$

Take a subset $I \subseteq [k]$ such that $\mu_I \neq \nu_I$ and

$$t^I = \frac{1-\mu_I}{\nu_I - \mu_I} \in]0,1[.$$

Denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i\in I}$ contain the new player p as additional member and the other coalition $\{S_j\}_{j\in[k]\setminus I}$ are kept unchanged.

Lemma 3.2.4. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Define $\lambda = (\lambda_S)_{S\in\mathcal{C}'}$ by $\lambda_S = (1-t^I)\mu_S + t^I\nu_S$. Because λ is a convex combination of two systems of balancing weights of $\mathcal{C}$, $\sum_{S\in\mathcal{C}', S\ni i}\lambda_S = 1$ for all the players $i \in N$. Concerning player p,

$$\sum_{\substack{S\in\mathcal{C}'\\ S\ni p}} \lambda_S = \lambda_I = (1-t^I)\mu_I + t^I\nu_I = \mu_I + t^I(\nu_I - \mu_I) = \mu_I + 1 - \mu_I = 1.$$

We conclude that $\mathcal{C}'$ is a balanced collection. Now, let us prove the minimality of $\mathcal{C}'$ as a balanced collection. The set of systems of balancing weights for $\mathcal{C}$ is the convex set of μ and ν, and therefore the set of systems of balancing weights for $\mathcal{C}'$ is a subset of this. More precisely, it is the subset $\{\lambda \in \text{conv}(\mu,\nu) \mid \lambda_I = 1\}$, equivalently $\{t \in [0,1] \mid (1-t)\mu_I + t\nu_I = 1\} = T$, and therefore the condition is on the variable t. By assumption, $\mu_I \neq \nu_I$, and then $\mu_I < 1 \leq \nu_I$ without loss of generality. Because the map $f : t \mapsto (1-t)\mu_I + t\nu_I$ is linear and $f(0) < 1$ and $f(1) \geq 1$, there is a unique $t^* \in T$ such that $f(t^*) = 1$, then this unique t^* must be t^I. □

3.3 Final algorithm

It is now possible to construct, from the set of minimal balanced collections on a set N, the set of minimal balanced collections on another set $N' = N \cup \{p\}$ (see Algorithm 1).

Algorithm 1 AddNewPlayer

Require: A set of minimal balanced collection $\mathbb{B}(N)$ on a set N
Ensure: A set of minimal balanced collection $\mathbb{B}(N')$ on a set $N' = N \cup \{p\}$

procedure AddNewPlayer($\mathbb{B}_N, p$)
  for $(\mathcal{C}^1, \mathcal{C}^2) \in \mathbb{B}_N \times \mathbb{B}_N$ **do**
    $\mathcal{C} \leftarrow \mathcal{C}^1 \cup \mathcal{C}^2$ and $k \leftarrow |\mathcal{C}|$
    if $\text{rank}(A^{\mathcal{C}}) = k-1$ **then**
      for $I \subseteq [k]$ **such that** $t^I \in]0,1[$ **do**
        for $i \in I$ **do** add $S_i \cup \{p\}$ with weights $(1-t^I)\mu_{S_i} - t^I\nu_{S_i}$ to $\mathcal{C}'$
        for $i \notin I$ **do** add S_i with weights $(1-t^I)\mu_{S_i} - t^I\nu_{S_i}$ to $\mathcal{C}'$
        add $\mathcal{C}'$ to $\mathbb{B}(N')$
  for $\mathcal{C} \in \mathbb{B}_N$ **do**

$k \leftarrow |\mathcal{C}|$
for $I \subseteq [k]$ **such that** $\lambda_I^{\mathcal{C}} \leq 1$ **do**
  $C' \leftarrow \emptyset$
  for $i \in I$ **do** add $S_i \cup \{p\}$ with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
  for $i \notin I$ **do** add S_i with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
  if $\lambda_I^{\mathcal{C}} < 1$ **then** add $\{p\}$ with weight $1 - \lambda_I^{\mathcal{C}}$ to $\mathcal{C}'$
  add $\mathcal{C}'$ to $\mathbb{B}(N')$
  for $\delta \in [k] \setminus I$ **such that** $\lambda_{S_\delta} > 1 - \lambda_I^{\mathcal{C}}$ **do**
    $C' \leftarrow \emptyset$
    for $i \in I \setminus \{\delta\}$ **do** add $S_i \cup \{p\}$ with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
    for $i \notin I \cup \{\delta\}$ **do** add S_i with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
    add $S_\delta \cup \{p\}$ with weight $1 - \lambda_I^{\mathcal{C}}$ to $\mathcal{C}'$
    add S_δ with weight $\lambda_{S_\delta}^{\mathcal{C}} + \lambda_I^{\mathcal{C}} - 1$ to $\mathcal{C}'$
    add $\overline{\mathcal{C}}$ to $\mathbb{B}(N')$
return $\mathbb{B}(N')$

Theorem 3.3.1. *The algorithm* AddNewPlayer*, which takes as an input the set of all minimal balanced collections on a set N, generates all the minimal balanced collections on the set $N' = N \cup \{p\}$.*

Proof. Thanks to the four previous lemmas, the algorithm generates only minimal balanced collections on N'. It remains to prove that every minimal collection is generated by this algorithm. Let $\mathcal{B}$ be a minimal balanced collection on N'. If the player p is removed from each coalition of $\mathcal{B}$, the collection is still balanced. Denote by $\mathcal{B}_{-p}$ this new collection.

- If $\{p\} \in \mathcal{B}$: since $\mathcal{B}$ has a unique system of balancing weights, $\mathcal{B}_{-p}$ has only one system of balancing weights, and so it is a minimal balanced collection, and $\mathcal{B}$ is generated by the second case.
- If $\{p\} \notin \mathcal{B}$ and there are two identical coalitions in $\mathcal{B}_{-p}$: the minimality of $\mathcal{B}$ implies the minimality of $\mathcal{B}_{-p}$ when the two identical coalitions are merged and their weights added. Then $\mathcal{B}$ is generated by the third case.
- Assume now that there is no singleton $\{p\}$ in $\mathcal{B}$, and that no pair of coalitions in $\mathcal{B}_{-p}$ contains the same coalition twice.
  - ▷ If $\mathcal{B}_{-p}$ is a minimal balanced collection, $\mathcal{B}$ is generated by the first case.
  - ▷ Assume now that $\mathcal{B}_{-p}$ is not a minimal balanced collection. Because $\mathcal{B}$ is a minimal balanced collection of k coalitions, $\mathrm{rk}(A^{\mathcal{B}}) = k$, and therefore $\mathrm{rk}(A^{\mathcal{B}_{-p}}) = k - 1$. Consequently, the set of solutions of the following system of inequalities

$$A^{\mathcal{B}_{-p}}\lambda = \mathbf{1}^N, \quad \lambda_i \geq 0, \forall i \in [k] \tag{1}$$

is one-dimensional and has the form $\lambda = \lambda_0 + t\lambda_1$, where λ_0 is a system of balancing weights for $\mathcal{B}_{-p}$, t is a real number and λ_1 is a nonzero solution of the homogeneous system

$$A^{\mathcal{B}_{-p}}\lambda = 0, \quad \lambda_i \geq 0, \forall i \in [k].$$

The set of solutions of (1) being bounded and one-dimensional, it is a non-degenerate segment $[\alpha, \beta]$ that consists of all the solutions of the above system. Let $U_\alpha = \{i \mid \alpha_i > 0\}$ and $U_\beta = \{i \mid \beta_i > 0\}$. Clearly, U_α and U_β are proper subsets of $\{1, \ldots, k\}$ and $U_\alpha \cup U_\beta = \{1, \ldots, k\}$. Denote $\mathcal{B}^\alpha = \{B_i \in \mathcal{B} \mid i \in U_\alpha\}$ and $\mathcal{B}^\beta = \{B_i \in \mathcal{B} \mid i \in U_\beta\}$. α^*, the restriction of α to U_α, is a system of balancing weights for $\mathcal{B}^\alpha$, and β^*, the restriction of β to U_β, is a system of balancing weights for $\mathcal{B}^\beta$. Since α and β are extremal solutions of the system (1), $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ must be minimal balanced collections. Then $\mathcal{B}$ is the union of $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$, and is generated by the fourth case.

□

With the procedure AddNewPlayer used recursively, all the minimal balanced collections on any fixed set N are generated from the ones on $\{1, 2\}$. This is achieved by the procedure Peleg (see Algorithm 2). Table 1 gives the number of minimal balanced collections as computed by Peleg up to $n = 6$.

Number of players	Number of minimal balanced collections
3	6
4	42
5	1292
6	201 076

Table 1: Number of minimal balanced collections according to the number of players

Once the minimal balanced collections are generated, checking the balancedness of a game amounts to checking a set of linear inequalities (one per minimal balanced collection).

Algorithm 2 Minimal balanced collections computation

Require: A number of players $n \geq 3$
Ensure: The set of minimal balanced collections on the set $[n]$

procedure Peleg(n)
  $\mathbb{B}(\{1, 2\}) \leftarrow \{\{1, 2\}, \{1\}, \{2\}\}$
  for $i \in \{3, \ldots, n\}$ **do**
    $\mathbb{B}([i]) \leftarrow$ AddNewPlayer($\mathbb{B}([i-1])$, i)
  return $\mathbb{B}([n])$

It is possible to adapt Algorithm 1 to compute the minimal balanced collections on every set system on which the game is defined. The only difference for the implementation is to check, when a new minimal balanced collection is created, that every coalition is a subset of an element of the set system. If it is not the case, just ignore the newly created collection and continue the computation.

3.4 Example

Let $N = \{a, b, c, d\}$ and $N' = N \cup \{e\}$. Let $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{a, d\}$ and $S_4 = \{b, c, d\}$. Then $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$ is a minimal balanced collection with the following system of balancing weights $\lambda = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

First case Remark that the set $I = \{1, 4\}$ satisfies the equation $\lambda_I = 1$. Therefore, a minimal balanced collection can be constructed as follows:

$$\mathcal{C}' = \{\{a,b,e\},\{a,c\},\{a,d\},\{b,c,d,e\}\}, \text{ with } \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\right).$$

Second case Let $I = \{4\}$. Then $\lambda_I = \frac{2}{3} < 1$. Therefore,

$$\mathcal{C}' = \{\{a,b\},\{a,c\},\{a,d\},\{b,c,d,e\},\{e\}\}, \text{ with } \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3}\right).$$

is a minimal balanced collection on N'.

Third case Let $I = \{1, 2\}$ and $\delta = 4$. Then $\lambda_I = \frac{2}{3}$ and $1 - \lambda_{S_\delta} = \frac{1}{3}$. Therefore, $1 > \lambda_I > 1 - \lambda_{S_\delta}$ and the following minimal balanced collection can be constructed:

$$\mathcal{C}' = \{\{a,b,e\},\{a,c,e\},\{a,d\},\{b,c,d\},\{b,c,d,e\}\}, \text{ with } \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right).$$

Last case For the last case, consider another framework. Let $N = \{a, b\}$, and $\mathcal{C}^1 = \{\{a\},\{b\}\}$, $\mathcal{C}^2 = \{\{a,b\}\}$ be the only two minimal balanced collections on N. Let $\mathcal{C}$ be the union $\mathcal{C} = \{\{a\},\{b\},\{a,b\}\}$.

$$\mu = (1,1,0) \text{ and } \nu = (0,0,1).$$

Observe that

$$\text{rk}(A^{\mathcal{C}}) = \text{rk}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2 = k - 1.$$

Finally, let $I = \{1, 2\}$. Then $\mu_I = 2$, $\nu_I = 0$, and

$$t^I = \frac{1 - \mu_I}{\nu_I - \mu_I} = \frac{1}{2} \in]0, 1[.$$

The following collection may therefore be constructed:

$$\mathcal{C}' = \{\{a,c\},\{b,c\},\{a,b\}\}, \text{ with}$$

$$\begin{aligned}
\lambda^{\mathcal{C}'}_{\{a,c\}} &= (1-t^I)\mu_{\{a,c\}} + t^I\nu_{\{a,c\}} = \tfrac{1}{2}\mu_{\{a,c\}} &= \tfrac{1}{2}, \\
\lambda^{\mathcal{C}'}_{\{b,c\}} &= (1-t^I)\mu_{\{b,c\}} + t^I\nu_{\{b,c\}} = \tfrac{1}{2}\mu_{\{b,c\}} &= \tfrac{1}{2}, \\
\lambda^{\mathcal{C}'}_{\{a,b\}} &= (1-t^I)\mu_{\{a,b\}} + t^I\nu_{\{a,b\}} = \tfrac{1}{2}\nu_{\{a,b\}} &= \tfrac{1}{2}.
\end{aligned}$$

4 Properties of coalitions and collections

There already exist several sufficient or necessary conditions for core stability in the general case (Kikuta and Shapley, 1986), or restricted classes of games: matching games, simple flow games, or minimum coloring games (Shellshear and Sudhölter, 2009). Similarly to the foregoing authors, we also need the notions of *strictly vital-exact* or *exact* coalitions, *extendability*, and *feasible* collections. For recalling the precise definitions, some notation is needed. Throughout the section let (N, v) be a balanced game, S be a coalition, and

$S^c := \{i \in N \mid i \notin S\}$. Denote by H_S the hyperplane of the set of preimputations defined by

$$H_S = \{x \in X(N,v) \mid x(S) = v(S)\}.$$

Denote by (S,v) the subgame on S, in which only the subcoalitions of S are considered, and by (N, v^S) the game that may differ from (N,v) only inasmuch as $v^S(S^c) = v(N) - v(S)$. This definition can be extended to a collection of coalitions $\mathcal{S}$, with $v^{\mathcal{S}}(S^c) = v(N) - v(S)$ for all $S \in \mathcal{S}$ and $v^{\mathcal{S}}(T) = v(T)$ otherwise.

4.1 Strict vital-exactness

Let (N,v) be a balanced game and S a coalition.

Definition 4.1.1. A coalition S is *exact* (at v) if there exists a core element $x \in C(N,v)$ such that $x(S) = v(S)$. In this case, we say that S is *effective* for x. Denote by $\mathcal{E}(N,v)$ the set of coalitions that are effective for all core elements.

Hence, a coalition S is exact if and only if the hyperplane H_S intersects the core. Moreover, $S \in \mathcal{E}(N,v)$ if and only if $C(N,v) \subseteq H_S$. The following result permits us to build an algorithm that checks exactness.

Proposition 4.1.2. *Let (N,v) be a balanced game. A coalition S is exact if and only if (N,v^S) is balanced.*

Proof. Assume that (N,v^S) is balanced. Then, for all $x \in C(N,v^S)$, $x(N) = v(N)$ and $x(S^c) \geq v(N) - v(S)$. It implies that $x(S) = x(N) - x(S^c) \leq v(S)$. But, because x belongs to the core of (N,v^S), it follows that $x(S) \geq v(S)$, and therefore $x(S) = v(S)$.
Assume now that S is exact. Therefore, there exists $x \in C(N,v)$ such that $x(S) = v(S)$. Because $x(N) = v(N)$, then $x(S^c) = v(N) - v(S)$, and $x \in C(N,v^S)$. □

Recall that the core is said to be *stable* if it is a stable set. Thanks to exactness, Gillies (1959) found a necessary condition for a game to have a stable core.

Proposition 4.1.3 (Gillies, 1959)**.** *A balanced game has a stable core only if each singleton is exact.*

Now that the set of exact coalitions can be computed, the necessary condition of Gillies can be checked. Another interesting consequence of this result is the expansion of the space in which the core is externally stable if it is a stable set. Indeed, if a balanced game satisfies this necessary condition, for all player $i \in N$, the core element $x \in C(N,v)$ such that $x_i = v(\{i\})$ dominates every element y of $X(N,v)$ such that $y_i < v(\{i\})$, via $\{i\}$. Therefore, in the definition of stability (see Sect. 2) for the core, the set $I(N,v)$ may be replaced by $X(N,v)$.

To formulate a novel necessary condition for core stability, the definition of strict vital-exactness is recalled.

Definition 4.1.4 (Grabisch and Sudhölter, 2021)**.** A coalition S is *strictly vital-exact* if there exists a core element $x \in C(N,v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Denote by $\mathcal{VE}(N,v)$ the set of strictly vital-exact coalitions.

In particular, an exact singleton is strictly vital-exact.

Proposition 4.1.5. *Let (N,v) be a balanced game. The core is stable only if $\mathcal{VE}(N,v)$ is* *core-describing, i.e.,*

$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S), \forall S \in \mathcal{VE}(N, v)\}.$$

Proof. Assume that the core is stable, and let $y \in X(N, v) \setminus C(N, v)$. Because the core is stable, there exists $x \in C(N, v)$ such that x dom y. Choose a minimal (w.r.t. inclusion) coalition S such that $x \text{ dom}_S\, y$. Then, $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Therefore, S is strictly vital-exact. □

***Remark* 4.1.6.** Let (N, v) be a balanced game. Because the core is convex, for any collection $\mathcal{S}$ of coalitions such that $\mathcal{S} \cap \mathcal{E}(N, v)$ is empty, there exists a core element $x^{\mathcal{S}}$ such that $x^{\mathcal{S}}(S) > v(S)$, for all $S \in \mathcal{S}$. Indeed, for every coalition $S \in \mathcal{S}$, there exists a core element x^S such that $x^S(S) > v(S)$ because $S \notin \mathcal{E}(N, v)$. Then, by taking the convex midpoint $\frac{1}{|\mathcal{S}|} \sum_{S \in \mathcal{S}} x^S$, the desired $x^{\mathcal{S}}$ is defined, and it belongs to the core by convexity.

By Remark 4.1.6, we deduce that the minimal (w.r.t. inclusion) coalitions of $\mathcal{E}(N, v)$ are strictly vital-exact. The following result allows checking whether a coalition is effective for all core elements.

Lemma 4.1.7. *$\mathcal{E}(N, v)$ is the union of all minimal balanced collections $\mathcal{B}$ such that*

$$\sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N).$$

Proof. Let $\mathcal{B}$ be a minimal balanced collection such that $\sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N)$ and x be a core element. Then

$$v(N) = x(N) = \sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} x(S) \geq \sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N).$$

As $\lambda_S^{\mathcal{B}} > 0$, $x(S) = v(S)$ for all $S \in \mathcal{B}$, i.e., $\mathcal{B} \subseteq \mathcal{E}(N, v)$.
For the other inclusion, let $S \in \mathcal{E}(N, v)$. As $\{N\}$ is a minimal balanced collection, it may be assumed that $S \neq N$. It remains to show that S is contained in some minimal balanced collection $\mathcal{B}$ that satisfies $\sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N)$. Assume the contrary. Then, by Theorem 3.1.3, there exists $\varepsilon > 0$ such that the game (N, v^{ε}) that differs from (N, v) only inasmuch as $v^{\varepsilon}(S) = v(S) + \varepsilon$ is still balanced. Hence, for $x \in C(N, v^{\varepsilon})$, it follows $x(S) > v(S)$ and $x \in C(N, v)$, then the desired contradiction has been obtained. □

In view of Lemma 4.1.7, the following proposition shows that it is possible to compute the set of strictly vital-exact coalitions.

Proposition 4.1.8. *A coalition S is strictly vital-exact if and only if it is exact and*

$$\mathcal{E}(N, v^S) \setminus \{S\} \subseteq \{R \in 2^N \mid R \cap S^c \neq \emptyset\}.$$

Proof. Assume that S is strictly vital-exact. Then there exists $x \in C(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Therefore, no coalition $T \in 2^S \setminus \{\emptyset, S\}$ is included in $\mathcal{E}(N, v^S)$.
Conversely, assume that S is exact and $\mathcal{E}(N, v^S) \setminus \{S\} \subseteq \{R \in 2^N \mid R \cap S^c \neq \emptyset\}$. Thanks to Proposition 4.1.2, $C(N, v^S)$ is nonempty. The collection $2^S \setminus \{S\}$ does not intersect $\mathcal{E}(N, v^S)$ by hypothesis. Hence, by Remark 4.1.6 there exists an element $x \in C(N, v^S)$ such that $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. □

Algorithm 3 Strict vital-exactness checking algorithm

Require: A coalition S, a balanced game (N, v)
Ensure: The Boolean value: 'S is strictly vital-exact'

procedure IsStrictlyVitalExact(S, (N, v))
    $\mathbb{B}(N) \leftarrow$ Peleg($|N|$)
    for $\mathcal{B} \in \mathbb{B}(N)$ **do**
        if $\sum_{T \in \mathcal{B}} \lambda_T^{\mathcal{B}} v^S(T) > v(N)$ **then**
            return False
        else if $\sum_{T \in \mathcal{B}} \lambda_T^{\mathcal{B}} v^S(T) = v(N)$ **then**
            for $T \in \mathcal{B}$ **do**
                if $T \cap S^c = \emptyset$ **then**
                    return False
    return True

4.2 Feasibility

Let (N, v) be a balanced game and $\mathcal{F} \subseteq 2^N$ be a core-describing collection of coalitions, i.e.,

$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S), \forall S \in \mathcal{F}\}.$$

External stability requires that, for every element $y \in X(N, v) \setminus C(N, v)$, there exists a core element that dominates y. A necessary condition for a preimputation to be dominated is that there exists a coalition that has interest to leave the grand coalition to gain more than with the current preimputation. Formally, it is necessary to have a coalition $S \in \mathcal{F}$ such that $x(S) < v(S)$. Let us denote by $\mathcal{S} \subseteq \mathcal{F}$ the set of such coalitions, and define *regions* as

$$X_{\mathcal{S}} = X_{\mathcal{S}}^{\mathcal{F}} = \left\{ x \in X(N, v) \;\middle|\; \begin{array}{l} x(S) < v(S) \text{ for all } S \in \mathcal{S}, \\ x(T) \geq v(T) \text{ for all } T \in \mathcal{F} \setminus \mathcal{S} \end{array} \right\}.$$

The collection $\mathcal{S}$ is *$\mathcal{F}$-feasible* if the corresponding region $X_{\mathcal{S}}^{\mathcal{F}}$ is nonempty. The regions form a partition of $X(N, v)$, with $C(N, v) = X_{\{\emptyset\}}$. If no ambiguity occurs, the collection is simply said to be feasible, and the region is simply denoted by $X_{\mathcal{S}}$. Here are some properties about the feasible collections.

Lemma 4.2.1 (Grabisch and Sudhölter, 2021)**.** *Let $\mathcal{S} \subseteq \mathcal{F}$. The following holds.*

1. *If $\mathcal{S}$ is feasible, then it does not contain a balanced collection.*
2. *For $S, S' \in \mathcal{S}$ such that $S \cup S' = N$, no $x \in X_{\mathcal{S}}$ is dominated via S or S'.*

A collection $\mathcal{S}$ that contains only two coalitions satisfying condition (ii) above is called a *blocking feasible collection*. A characterization that can be translated into an algorithm is needed to compute the set of feasible collections. In the sequel, denote $\mathcal{S}^c = \{S^c \mid S \in \mathcal{S}\}$.

Lemma 4.2.2. *A collection $\mathcal{S} \subseteq \mathcal{F}$ is feasible (w.r.t. $\mathcal{F}$) if and only if for every minimal balanced collections $\mathcal{B} \subseteq \mathcal{F}' = (\mathcal{F} \setminus \mathcal{S}) \cup \mathcal{S}^c$,*

$$\sum_{T \in \mathcal{B}} \lambda_T^{\mathcal{B}} v^{\mathcal{S}}(T) \begin{cases} \leq v(N), & \\ < v(N), & \text{if } \mathcal{B} \cup \mathcal{S}^c \neq \emptyset. \end{cases} \quad (2)$$

Proof. For $\varepsilon, \alpha \in \mathbb{R}$ define $(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ by, for all coalitions T,

$$v^{\mathcal{S}}_{\varepsilon,\alpha}(T) = \begin{cases} v^{\mathcal{S}}(T) + \varepsilon & \text{if } T \in \mathcal{S}^c, \\ v(T) & \text{if } T \in \mathcal{F} \setminus (\mathcal{S} \cup \mathcal{S}^c) \text{ or } T = N, \\ \alpha & \text{otherwise.} \end{cases}$$

A collection $\mathcal{S}$ is feasible if and only if there exists $x \in \mathbb{R}^N$ and $\varepsilon > 0$ such that $x(S) \geq v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{S}$, $x(N) = v(N)$, and $x(P) \leq v(P) - \varepsilon$, i.e., $x(N \setminus P) = x(N) - x(P) = v(N) - x(P) \geq v(N) - v(P) + \varepsilon$ for all $P \in \mathcal{S}$. Therefore, for $\alpha \leq \min_{R \in 2^N} x(R)$, $x \in C(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ so that if part of the proof is finished by Theorem 3.1.3.
For the only if part we again employ Theorem 3.1.3. Indeed, by (2), there exist $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ small enough such that $(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ is balanced. The existence of a core element of $(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ guarantees that $\mathcal{S}$ is feasible. □

Algorithm 4 Feasibility checking algorithm

Require: The collection $\mathcal{F}$, a collection $\mathcal{S} \subseteq \mathcal{F}$, a balanced game (N, v)
Ensure: The Boolean value: '$\mathcal{S}$ is feasible'

```
procedure IsFeasible(S, F, (N,v))
    B(N) ← Peleg(|N|)
    for B ∈ B(N) such that B ⊆ F' do
        if B ∩ S^c ≠ ∅ and Σ_{S∈B} λ_S^B v^S(S) ≥ v(N) then
            return False
        else if B ∩ S^c = ∅ and Σ_{S∈B} λ_S^B v^S(S) > v(N) then
            return False
    return True
```

4.3 Extendability

Kikuta and Shapley (1986) provide a sufficient condition for a game to have a stable core, a property they called extendability.

Definition 4.3.1. A coalition S is called *extendable* (w.r.t. (N, v)) if, for any $x \in C(S, v)$, there exists $y \in C(N, v)$ such that $x = y_S$. A game (N, v) is *extendable* if all coalitions are extendable.

Theorem 4.3.2 (Kikuta and Shapley, 1986)**.** *An extendable game has a nonempty and stable core.*

To check whether a coalition is extendable, by convexity of the core, it is sufficient to check if each vertex of $C(S, v)$ can be extended to a core element. To this end, the reduced game property of the core is used. Let S be a coalition, and $z \in \mathbb{R}^{S^c}$. Recall that the traditional *reduced game* (Davis & Maschler, 1965) of (N, v) w.r.t. S and z, $(S, v_{S,z})$, is the game defined by

$$v_{S,z}(T) = \begin{cases} v(N) - z(S^c), & \text{if } T = S, \\ \max_{Q \subseteq S^c} v(T \cup Q) - z(Q), & \text{if } \emptyset \neq T \subsetneq S. \end{cases}$$

According to Peleg (1986) the core satisfies the *reduced game property*, i.e., if $x \in C(N, v)$, then $x_S \in C(S, v_{S,x_{S^c}})$.

Lemma 4.3.3. *Let (N, v) be a balanced game, S be a coalition and $y \in C(S, v)$. Then there exists $x \in C(N, v)$ such that $x_S = y$ if and only if $(S^c, v_{S^c,y})$ is balanced.*

Proof. The only if part is due to the reduced game property. For the if part choose an arbitrary $z \in C(S^c, v_{S^c,y})$. It suffices to show that the only allocation $x \in \mathbb{R}^N$ such that $x_S = y$ and $x_{S^c} = z$ belongs to the core. Assume, on the contrary, that $x \notin C(N, v)$. As $x(S^c) = v_{S^c,y}(S^c) = v(N) - x(S)$ by definition, $x(N) = v(N)$. Therefore, there exists $T \subsetneq N$ such that $x(T) < v(T)$. As (N, v) is balanced, $v(S^c) \leq v(N) - v(S) = v(N) - x(S)$, so that $T \neq S^c$. Moreover, as $y \in C(S, v)$, $T \cap S^c \neq \emptyset$. Therefore, $v_{S^c,y}(T \cap S^c) = \max_{Q \subseteq S} v((T \cap S^c) \cap Q) - x(Q) \geq v((T \cap S^c) \cup (T \cap S)) - x(T \cap S)$. Hence, $x(T \cap S^c) < v(T) - x(T \cap S) \leq v_{S^c,y}(T \cap S^c)$, which contradicts $x_{S^c} = z \in C(S^c, v_{S^c,y})$. □

Lemma 4.3.3 gives us a necessary and sufficient condition for the existence of an extension of an element of $C(S, v)$ to an element of $C(N, v)$, based upon a balancedness check. If there exists an extension for each extreme point of $C(S, v)$, by convexity of the core, any element of $C(S, v)$ can be extended. The following algorithm checks for each extreme point whether the reduced game of (N, v) w.r.t. the complement of S and the currently considered extreme point is balanced.

Algorithm 5 Extendability checking algorithm

Require: A coalition S, a balanced game (N, v)
Ensure: The Boolean value: 'S is extendable'

```
procedure IsExtendable(S, (N, v))
    B(S^c) ← Peleg(|S^c|)
    for ξ ∈ ext(C(S, v)) do
        define the reduced game v_{S^c,ξ}
        for 𝓑 ∈ B(S^c) do
            if Σ_{T∈𝓑} λ_T^𝓑 v_{S^c,ξ}(T) > v(N) − v(S) then
                return False
    return True
```

We conclude this section by remarking that extendability can be weakened as follows. A balanced game has a stable core if it is *weakly* extendable (w.r.t. a core-describing collection $\mathcal{F}$) in the sense that each feasible collection of coalitions of $\mathcal{F}$ contains a minimal (w.r.t. inclusion) coalition that is extendable.

5 Stability of the core

The algorithm checking whether a game has a stable core is based on Theorem 5.3.1, which is the main result of Grabisch and Sudhölter (2021).

5.1 Association, admissibility, and outvoting

Let (N, v) be a balanced game. All results and definitions in this section are due to Grabisch and Sudhölter (2021). We first recall their definition of *outvoting*, a transitive subrelation of domination, that was inspired by a definition given by Kulakovskaja (1973). In view of Proposition 4.1.5, throughout we assume that (N, v) is a balanced game for which the collection of strictly vital-exact coalitions is core-describing.

Definition 5.1.1. A preimputation y *outvotes* another preimputation x via $P \in \mathcal{VE}(N, v)$, written $y \succ_P x$, if y dom$_P$ x and $y(S) \geq v(S)$ for all $S \notin 2^P$. Also, y outvotes x, ($y \succ x$) if there exists a coalition $P \in \mathcal{VE}(N, v)$ such that $y \succ_P x$.

Denote by $M(v) = \{x \in X(N, v) \mid y \not\succ x, \forall y \in X(N, v)\}$ the set of preimputations that are maximal w.r.t. outvoting.

Proposition 5.1.2. *Let (N, v) be a balanced game. Then $C(N, v) = M(v)$ if and only if $C(N, v)$ is stable.*

All results are based on this new characterization. To present the main result, some definitions are needed.

Definition 5.1.3. Let S be a strictly vital-exact coalition and $\mathcal{B}$ be a minimal balanced collection. $\mathcal{B}$ is *associated with* S if there exists $i \in S$ such that $\{i\} \in \mathcal{B}$ and

$$\mathcal{B} \subseteq \{\{j\} \mid j \in S\} \cup \{S^c\} \cup (\mathcal{VE}(N, v) \setminus 2^S).$$

Denote by $\mathbb{B}^S(N)$ the set of minimal balanced collections on N associated with S.

Example 5.1.4. Let $N = \{1, 2, 3, 4\}$, $\mathcal{VE}(N, v) = 2^N \setminus \{\emptyset\}$ and $S = \{1, 2\}$. Therefore, the minimal balanced collection $\mathcal{B} = \{\{1\}, \{2\}, \{3, 4\}\}$ is included in $\mathbb{B}^S(N)$. Indeed, the coalitions $\{1\}$ and $\{2\}$ are singletons of S, and $\{3, 4\}$ is the complement of S. Moreover, $\mathcal{B}$ is also associated with $\{1, 2, 3\}$ for example.

Let S be a strictly vital-exact coalition and $\mathcal{B}$ be a minimal balanced collection associated with S. Denote by $\mathcal{B}^*_S$ the collection $\mathcal{B}^*_S = \mathcal{B} \setminus \{\{i\} \mid i \in S\}$. Thanks to the notions of association and outvoting, the following result holds.

Theorem 5.1.5. *Let x be a preimputation, and S a strictly vital-exact coalition. Then x is outvoted by some preimputation via S if and only if*

$$\forall \mathcal{B} \in \mathbb{B}^S(N), \qquad \sum_{\substack{i \in S \\ \{i\} \in \mathcal{B}}} \lambda^{\mathcal{B}}_{\{i\}} x_i + \sum_{T \in \mathcal{B}^*_S} \lambda^{\mathcal{B}}_T v^S(T) < v(N). \tag{3}$$

This result can be sharpened, with the use of the following notion.

Definition 5.1.6. Let $\mathcal{S}$ be a nonempty collection of strictly vital-exact coalitions, $S \in \mathcal{S}$, and $\mathcal{B}$ be a minimal balanced collection associated with S. $\mathcal{B}$ is *admissible* for $\mathcal{S}$ if $\mathcal{B}^*_S \cap \mathcal{S} \neq \emptyset$ or $\mathcal{B}^*_S \cap \mathcal{S}^c = \emptyset$. Denote by $\mathbb{B}^S_{\mathcal{S}}(N)$ the set of minimal balanced collections associated with S and admissible for $\mathcal{S}$.

Example 5.1.7. Let $N = \{1, 2, 3, 4\}$, $\mathcal{VE}(N, v) = 2^N \setminus \{\emptyset\}$ and $S = \{1, 4\}$. Therefore, the collection $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$ is associated with S, and $\mathcal{B}^*_S = \mathcal{B} \setminus \{\{4\}\}$. Let $\mathcal{S} = \{\{2, 3\}, \{1, 4\}\}$. The first condition of the definition is satisfied: $\mathcal{B}^*_S \cap \mathcal{S} = \{\{2, 3\}\} \neq \emptyset$, therefore $\mathcal{B}$ is admissible for $\mathcal{S}$.

For each nonempty collection $\mathcal{S}$ of strictly vital-exact coalitions, denote

$$\mathbb{C}(\mathcal{S}) = \bigtimes_{S \in \mathcal{S}} \mathbb{B}^S_{\mathcal{S}}(N).$$

The concept of admissibility allows to sharpen the previous result, and then to reduce the algorithmic complexity of the core stability checking, thanks to the following result.

Corollary 5.1.8. *Let $\mathcal{S}$ be a feasible collection. $M(v) \cap X_{\mathcal{S}} \neq \emptyset$ if and only if there exists a system of balanced collections $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$ and $x \in X_{\mathcal{S}}$ such that*

$$\forall S \in \mathcal{S}, \qquad \sum_{\substack{i\in S\\ \{i\}\in\mathcal{B}}} \lambda^{\mathcal{B}}_{\{i\}} x_i + \sum_{T\in\mathcal{B}^*_S} \lambda^{\mathcal{B}}_T v^S(T) \geq v(N).$$

5.2 Minimal balanced subsets

In this section, we recall (see Grabisch and Sudhölter, 2021) how the notion of balancedness, defined for collections of coalitions till now, is extended to finite subsets of the positive orthant of the Euclidean space. Throughout this subsection let N be a finite nonempty set and Z be a finite subset of $\mathbb{R}^N_+ \setminus \{0\}$.

Definition 5.2.1. Let $Z \subseteq \mathbb{R}^N_+ \setminus \{0\}$ be a finite set. Z is balanced if there exists a system $(\lambda_z)_{z\in Z}$ of positive weights (called *balancing weights*) such that

$$\sum_{z\in Z} \lambda_z z = \mathbf{1}^N.$$

A balanced set is *minimal* if it does not contain a proper subset that is balanced. Note that a balanced set is minimal if and only if it has a unique system of balancing weights. Let $F(Z) = \{\lambda \in \mathbb{R}^Z_+ \mid \sum_{z\in Z} \lambda_z z = \mathbf{1}^Z, \lambda_z \geq 0, \forall z \in Z\}$. Note that $F(Z)$ is a convex polytope. Note that $\lambda \in F(Z)$ is an extreme point of $F(Z)$ if and only if $\{z \in Z \mid \lambda > 0\}$ is a minimal balanced set. A minimal balanced set must be linearly independent. Hence, it contains at most n elements.

In Sect. 3, the minimal balanced collections on a set N are constructed recursively, by induction on the cardinality of N, finding a proper way to add the new players among the existing coalitions. The idea here for the new algorithm is completely different because there are not only indicator functions but also ordinary real vectors. To this end, a linear program is solved to find a unique system of balancing weights. To avoid useless calculations, we restrict our attention to sets Z the elements of which span a linear space that contains $\mathbf{1}^N$ and are linearly independent, i.e., the rank of the matrix the columns of which are the vectors of Z must be equal to its number of columns, and the expansion by the column $\mathbf{1}^N$ must not change the rank of the matrix. After this restriction, we simply check if the coefficients of the linear combinations are positive.

For the study of core stability, minimal balanced subsets of a specific set must be computed. Let $\mathcal{S}$ be a feasible collection and $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$. For $S \in \mathcal{S}$, let $z^S \in \mathbb{R}^N$ be given by

$$z^S_j = \begin{cases} \lambda^{\mathcal{B}}_{\{i\}}, & \text{if } j = i \text{ for some } i \in S \text{ such that } \{i\} \in \mathcal{B}_S,\\ 0, & \text{for all other } j \in N. \end{cases}$$

Define the set

$$\Omega(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) = \left\{\mathbf{1}^{S^c} \mid S \in \mathcal{S}\right\} \cup \left\{\mathbf{1}^T \mid T \in \mathcal{VE}(N,v) \setminus \mathcal{S}\right\} \cup \left\{z^S \mid S \in \mathcal{S}\right\}.$$

Let LinAlgSolve be a procedure that takes as an input a $(n \times k)$-matrix A and returns a k-dimensional vector λ such that $A\lambda = \mathbf{1}^N$. Denote by $\mathbb{B}(\Omega)$ the set of minimal balanced subsets of Ω.

Algorithm 6 Minimal balanced sets computation algorithm

Require: A set Ω
Ensure: The set $\mathbb{B}(\Omega)$

procedure IsMinimalBalanced(Z)
  if $\text{rk}(A^Z) = \text{rk}(A^Z_{\mathbf{1}}) = |Z|$ **then**
    $\lambda \leftarrow$ LinAlgSolve(A^Z)
    if $\lambda > 0$ **then**
      return True
  return False
procedure BalancedSets(Ω)
  for $Z \subseteq \Omega$ **such that** $|Z| \leq n$ **do**
    if IsMinimalBalanced(Z) **then**
      add Z to $\mathbb{B}(\Omega)$
  return $\mathbb{B}(\Omega)$

5.3 Final algorithm

Finally (see Grabisch and Sudhölter, 2021), for each $z \in \Omega$, we define $a_z = a_z(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) = \max(A \cup B \cup C)$, where

$$\begin{aligned} A &= \{v(N) - v(S) \mid S \in \mathcal{S}, \mathbf{1}^{S^c} = z\}, \\ B &= \{v(T) \mid T \in \mathcal{VE}(N,v) \setminus \mathcal{S}, \mathbf{1}^T = z\}, \\ C &= \{v(N) - \sum_{T\in\mathcal{B}^*_S} \lambda_T^{\mathcal{B}_S} v^S(T) \mid S \in \mathcal{S}, z = z^S\}. \end{aligned}$$

Note that A and B are empty or singletons, but C can be multi-valued because distinct coalitions can generate the same z. Let $N = \{1,2,3\}$, $S = \{1,2\}$, $T = \{1,3\}$ and $\mathcal{B}_S = \mathcal{B}_T = \{\{1\},\{2,3\}\}$. Then, $z^S = (1,0,0) = z^T$. To summarize,

$$a_z = \begin{cases} \max C & \text{if } C \neq \emptyset = A, \\ \max\{A, C\} & \text{if } C \neq \emptyset \neq A, \\ v(N) - v(S) & \text{if } z = \mathbf{1}^{S^c} \text{ for some } S \in \mathcal{S}, C = \emptyset, \\ v(T) & \text{if } z = \mathbf{1}^T \text{ for some } T \in \mathcal{VE}(N,v) \setminus \mathcal{S}, A = \emptyset = C. \end{cases}$$

Recall that $\mathbb{B}(\Omega)$ is the set of all minimal balanced sets $Z \subseteq \Omega$ and denote by $\mathbb{B}_0(\Omega)$ the subset of $\mathbb{B}(\Omega)$ such that, for all $Z \in \mathbb{B}_0(\Omega)$, there exists $S \in \mathcal{S}$ such that $z = \mathbf{1}^{S^c} \in Z$ and $a_z = v(N) - v(S)$.

Theorem 5.3.1 (Grabisch and Sudhölter, 2021)**.** *A balanced game (N, v) has a stable core if and only if, for every feasible collection $\mathcal{S}$ and for every $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$,*

$$\exists Z \in \mathbb{B}(\Omega) \setminus \mathbb{B}_0(\Omega) \textit{ such that } \sum_{z\in Z} \lambda_z^Z a_z > v(N), \textit{ or}$$
$$\exists Z \in \mathbb{B}_0(\Omega) \textit{ such that } \sum_{z\in Z} \lambda_z^Z a_z \geq v(N).$$

For each $Z \in \mathbb{B}(\Omega)$, let $\psi(Z) = \sum_{z\in Z} \lambda_z^Z a_z$. The following algorithm checks whether a game has a stable core.

Algorithm 7 Nested balancedness checking algorithm

Require: A game (N, v)
Ensure: The Boolean value: '(N, v) has a stable core'

procedure $\textsc{IsNestedBalanced}(N, v)$
  for $\mathcal{S} \subseteq \mathcal{VE}(N, v)$ that is feasible **do**
    for $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$ **do**
      $\Omega \leftarrow \{\mathbf{1}^{S^c} \mid S \in \mathcal{S}\} \cup \{\mathbf{1}^T \mid T \in \mathcal{VE}(N, v) \setminus \mathcal{S}\} \cup \{z^S \mid S \in \mathcal{S}\}$
      if $\max_{Z \in \mathbb{B}(\Omega)} \psi(Z) \leq v(N)$ **and** $\arg\max_{Z \in \mathbb{B}(\Omega)} \psi(Z) \notin \mathbb{B}_0(\Omega)$ **then**
        return False
      if $\max_{Z \in \mathbb{B}(\Omega)} \psi(Z) < v(N)$ **and** $\arg\max_{Z \in \mathbb{B}(\Omega)} \psi(Z) \in \mathbb{B}_0(\Omega)$ **then**
        return False
  return True

5.4 Examples

Computing device: Apple M1 chip, CPU 3.2 GHz, 16 GB RAM.

4-player game Let (N, v) be the game defined by $N = \{1, 2, 3, 4\}$ and $v(S) = 0.6$ if $|S| = 3$, $v(N) = 1$ and $v(T) = 0$ otherwise. The algorithm returns that the set $\mathcal{E}(N, v)$ only contains N. The set of strictly vital-exact coalitions is $\mathcal{VE}(N, v) = \{\{i\} \mid i \in N\} \cup \{N \setminus \{i\} \mid i \in N\}$. The collection $\{\{1, 3, 4\}, \{1, 2, 3\}\}$ is a blocking feasible collection, so by Lemma 4.2.1, the core is not stable. The CPU time for this example is 0.1 second.

5-player game Let (N, v) be the game defined by Biswas et al. (1999), defined on $N = \{1, 2, 3, 4, 5\}$ by $v(S) = \max\{x(S), y(S)\}$ with $x = (2, 1, 0, 0, 0)$ and $y = (0, 0, 1, 1, 1)$. For this game, the set of effective proper coalitions is

$$\mathcal{E}(N, v) \setminus \{N\} = \{\{2, 3\}, \{2, 4\}, \{2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}.$$

The set of strictly vital-exact coalitions is $\mathcal{VE}(N, v) = \mathcal{E}(N, v) \cup \{\{i\} \mid i \in N\}$. The feasible collections that do not contain a minimal extendable coalition are the nonempty subsets of $\{\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}$, so there are 7 feasible collections. The collection $\{\{1, 3, 5\}, \{1, 4, 5\}\}$ does not satisfy the condition of Theorem 5.3.1, therefore the core of the game is not stable. The CPU time for this example is 1.5 seconds.

Let (N, v) be the same game, but with $v(N) = 3.1$. The set $\mathcal{E}(N, v)$ becomes $\{N\}$. The set of strictly vital-exact coalitions now contains 14 coalitions, while the previous game had 11 strictly vital-exact coalitions. The additional ones are $\{1, 3\}, \{1, 4\}, \{1, 5\}$. The set of feasible collections that do not contain a minimal extendable coalition considerably increases, with 51 feasible collections, but no blocking feasible collection. The largest feasible collection contains 6 strictly vital-exact coalitions. The estimated time for the algorithm to check if this specific collection satisfies the condition of Theorem 5.3.1 is greater than 200 hours, due to the cardinality of the set $\mathbb{C}(\mathcal{S})$ with $\mathcal{S}$ denoting the specific collection.

6-player game Let (N, v) be the game defined by Studený and Kratochvíl (2021), defined on $N = \{1, 2, 3, 4, 5, 6\}$ by

$$
\begin{aligned}
v(S) &= 2 \text{ for } S = \begin{cases} \{2,5\}, \{3,5\}, \{1,2,5\}, \{2,3,5\}, \{2,4,5\}, \{2,5,6\}, \{1,2,4,5\} \\ \{1,2,4,6\}, \{1,2,5,6\}, \{2,4,5,6\} \text{ and } \{1,2,4,5,6\}, \end{cases} \\
v(S) &= 3 \text{ for } S = \{3,4,5\}, \\
v(S) &= 4 \text{ for } \ S = \begin{cases} \{3,6\}, \{1,3,5\}, \{1,3,6\}, \{3,4,6\}, \{3,5,6\}, \{1,2,3,5\}, \\ \{1,3,4,5\}, \{1,3,4,6\}, \{1,3,5,6\}, \{2,3,4,5\} \text{ and } \{1,2,3,4,5\}, \end{cases} \\
v(S) &= 6 \text{ for } S = \begin{cases} \{2,3,6\}, \{1,2,3,6\}, \{2,3,4,6\}, \{2,3,5,6\}, \\ \{1,2,3,4,6\} \text{ and } \{1,2,3,5,6\}, \end{cases} \\
v(S) &= 8 \text{ for } S = \{3,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \\
v(N) &= 10 \text{ and } v(T) = 0 \text{ otherwise.}
\end{aligned}
$$

The set $\mathcal{E}(N, v)$ is only $\{N\}$. The set of strictly vital-exact coalitions is

$$\{\{i\} \mid i \in N\} \cup \{\{2,5\}, \{3,6\}, \{1,3,5\}, \{2,3,6\}, \{1,2,4,6\}, \{2,3,4,5\}, \{3,4,5,6\}\}$$

and the feasible collections that do not contain a minimal extendable coalition are the nonempty subsets of $\{\{1,3,5\}, \{3,4,5,6\}, \{2,3,4,5\}\}$. The feasible collection $\{\{1,3,5\}, \{3,4,5,6\}\}$ does not satisfy the condition of Theorem 5.3.1, therefore the core of the game is not stable. The CPU time for this example is 18 minutes and 12 seconds, among which 43 seconds for computing the set of minimal balanced collections on a set of 6 players.

6 Concluding remarks

We have shown in this paper that minimal balanced collections are a central notion in cooperative game theory, as well as in other areas of operations research and graph theory. As a balanced collection is merely the expression of a sharing of one unit of resource among subsets, we believe that many more applications should be possible.

Just focusing on the domain of cooperative games, the consequences of our results appear to be of primary importance for the computability of many notions like exactness, extendability, etc. Indeed, a blind application of the definition of these notions leads to difficult problems related to polyhedra, limiting their practical applicability. Thanks to our results, provided minimal balanced collections are generated beforehand (which is possible since they do *not* depend on the considered game), these notions can be checked very easily and quickly, as most of the tests to be done reduce to checking simple inequalities.

Generating minimal balanced collections has also permitted implementing an algorithm testing core stability. The examples in Section 5 have shown that, even if for many cases, the answer can be obtained quickly, there are instances where the computation time goes beyond tractability, due to the highly combinatorial character of the condition of core stability. Still, further research is needed to overcome this limitation.

References

[1] Aumann, R. J., & Peleg, B. (1960). Von Neumann-Morgenstern solutions to cooperative games without side payments. Bulletin of the American Mathematical Society, 66(3), 173–179.

[2] Biswas, A. K., Parthasarathy, T., Potters, J. A. M., & Voorneveld, M. (1999). Large Cores and Exactness. *Games and Economic Behavior, 28*(1), 1–12.

[3] Bondareva, O. N. (1963). Several applications of linear programming methods to the theory of cooperative games. *Problemi Kibernetiki*, 119–139.

[4] Deng, X., & Papadimitriou, C. H. (1994). On the Complexity of Cooperative Solution Concepts. *Mathematics of Operations Research, 19*(2), 257–266.

[5] Gillies, D. B. (1959). 3. Solutions to General Non-Zero-Sum Games. *Contributions to the Theory of Games (AM-40), IV*, 47–86.

[6] Grabisch, M., & Sudhölter, P. (2021). Characterization of TU games with stable cores by nested balancedness. *Mathematical Programming.*

[7] Grabisch, M., & Sudhölter, P. (2014). On the restricted cores and the bounded core of games on distributive lattices. *European Journal of Operational Research, 235*(3), 709-717.

[8] Kulakovskaja T. E. (1973). Solutions for some classes of binary relations and cooperative games. *Ph.D. Thesis (Russian)*, Leningrad State University.

[9] Lucas, W. F. (1969). The proof that a game may not have a solution. *Transactions of the American Mathematical Society, 137*(0), 219–229.

[10] Lucas, W. F. (1992). Von Neumann-Morgenstern stable sets. *Handbook of Game Theory with Economic Applications, I*, 543–590.

[11] Peleg, B. (1965). An inductive method for constructing minimal balanced collections of finite sets. *Naval Research Logistics Quarterly, 12*(2), 155–162.

[12] Pulido, M. A., & Sánchez-Soriano, J. (2006). Characterization of the core in games with restricted cooperation. *European Journal of Operational Research, 175*(2), 860-869.

[13] Shapley, L. S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly, 14*(4), 453–460.

[14] Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory, 1*(1), 11–26.

[15] Sharkey, W. W. (1982). Cooperative games with large cores. *International Journal of Games Theory, 11*(3-4), 175-182.

[16] Shellshear, E., & Sudhölter, P. (2009). On core stability, vital coalitions, and extendability. *Games and Economic Behavior, 67*(2), 633–644.

[17] Studený, M., & Kratochvíl V. (2021). Facets of the cone of exact games. arXiv:2103.02414

[18] von Neumann, J., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*, Princeton University Press.