# A NOTE ON SOLIDARITY IN BANKRUPTCY PROBLEMS WHEN AGENTS MERGE OR SPLIT

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# A note on solidarity in bankruptcy problems when agents merge or split

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**Abstract**

We explore the relationship between non-manipulability via merging (splitting) and strong non-manipulability via merging (splitting). We show that although, in general, these non-manipulability properties are not equivalent, under the principle of solidarity, fulfilled by a wide range of bankruptcy rules including parametric rules, they coincide.

*Keywords:* non-manipulability via merging or splitting, solidarity, bankruptcy rule
*JEL:* C71

## 1. Introduction

Bankruptcy problems (O'Neill, 1982) deal with situations where a group of agents has claims on a perfectly divisible resource but the available amount to divide is not enough to fulfill all demands. These problems are solved by rules proposing an allocation vector that takes into consideration the specifics of the agents. An important issue in economics is the study of rules that are invariant with respect to merging or splitting operations, that is, to the strategic behavior of the agents by misrepresenting their characteristics. Roughly speaking, a rule is non-manipulable via merging if no group of agents can take advantage from consolidating claims and it is non-manipulable via splitting if no agent can benefit from distributing their claim among a group of agents. A rule is non-manipulable if it is simultaneously unaffected by these two types of misrepresentations. *Non-manipulability* (or *strategy-proofness*) is first considered from an axiomatic perspective by O'Neill (1982) in characterizing the proportional rule in the

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context of bankruptcy problems. O'Neill's result was refined in different ways by Chun (1988), de Frutos (1999), Ju (2003), Ju et al. (2007), and Calleja and Llerena (2022), among others. The implications of non-manipulability in other contexts such as taxation or network problems have been addressed by Ju and Moreno-Ternero (2011) and Ju (2013), respectively.

Another important principle in the axiomatic approach of rules is *solidarity*, a sort of monotonicity condition concerning how a rule is affected by variations in the set of players and in the endowment or resource to distribute. Specifically, it imposes that the arrival of new agents, regardless of whether or not it is accompanied by changes in the available amount to share, should affect all the original agents in the same direction. Solidarity is introduced by Chun (1999) under the name of *population-and-resource monotonicity* and it is equivalent to the combination of two well-established requirements: *resource monotonicity* and *consistency*. Resource monotonicity says that if the amount of resource to be distributed becomes larger, no agent should be worse off. Consistency is an invariant principle with respect to population variations and requires that when a group of agents leaves with its share, then, in the reduced problem, the rule assigns the same amount as originally to the remaining agents.

On the entire domain of bankruptcy problems, Moreno-Ternero (2006) shows that non-manipulability is equivalent to *additivity of claims* (Curiel et al., 1987), or *strong non-manipulability*, requiring that merging or splitting the agents' claims do not affect the amounts received by any other agent involved in the problem. In many situations, and due to legal or practical constraints, only mergers or spin-offs are an option, but not both operations at the same time. Hence, it is worthwhile to study whether or not this reciprocity is preserved between non-manipulability by merging (splitting) and *strong non-manipulability by merging (splitting)*. In this note, we show that, in general, these properties are not equivalent but, under the ethical principle of solidarity, they are tantamount.

The remaining of the paper is organized as follows: Section 2 introduces some preliminaries. Section 3 and Section 4 contain respectively the axioms and the results. Section 5 concludes.

## 2. The model

Let $\mathbb{N} = \{1, 2, \ldots\}$ (the set of natural numbers) represent the set of all potential agents (claimants) and let $\mathcal{N}$ be the collection of all non-empty finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents where $|N| = n$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_T$ denotes the restriction of $x$ to $T$: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$.

A *bankruptcy problem* is a problem of adjudicating claims in which a firm defaults and its available resources are not enough to satisfy its obligations with creditors. This distributive justice problem has been widely studied from O'Neill (1982) and probably the most complete survey is provided by Thomson (2019). Formally, a bankruptcy problem is a triple $(N, E, c)$ where $N \in \mathcal{N}$ represents the set of creditors of the firm going bankrupt; $c \in \mathbb{R}^N_+$ is the vector of claims, being $c_i$ the claim of creditor $i \in N$; and $E \geq 0$ is the net worth or estate of the firm to satisfy its obligations. Additionally, we assume that $\sum_{i \in N} c_i \geq E$. By $\mathcal{B}$ we denote the set of all bankruptcy problems.

A *bankruptcy rule* (hereafter, a rule) is a function $\beta : \mathcal{B} \longrightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ that associates with every $(N, E, c) \in \mathcal{B}$ a unique recommendation $\beta(N, E, c) \in \mathbb{R}^N$ satisfying $\sum\limits_{i \in N} \beta_i(N, E, c) = E$ (*budget balance* (BB)), that is, the sum of all payments should be equal to the estate, and $\beta_i(N, E, c) \leq c_i$ for all $i \in N$ (*claim boundedness* (CB)), requiring that each agent should receive less than her claim. Given a bankruptcy rule $\beta$, its *dual* $\beta^d$ is defined by setting, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $\beta^d_i(N, E, c) = c_i - \beta_i\left(N, \sum_{i \in N} c_i - E, c\right)$. Instances of well-known rules are the *proportional* rule $(P)$, the *constrained equal awards* rule $(CEA)$, and the *constrained equal losses* rule $(CEL)$. The $P$ rule makes awards proportional to the claims and it is probably the most commonly used rule in practice when a firm goes bankrupt. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $P_i(N, E, c) = \lambda\, c_i$ where $\lambda \in \mathbb{R}_+$ is such that $\sum\limits_{j \in N} \lambda\, c_j = E$. The $CEA$ rule rewards equally to all claimants subject to no one receiving more than her claim. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEA_i(N, E, c) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum\limits_{j \in N} \min\{c_j, \lambda\} = E$. In contrast, the $CEL$ rule equalizes the losses of claimants subject to no one receiving a negative amount. That is, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEL_i(N, E, c) = \max\{c_i - \lambda, 0\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum\limits_{j \in N} \max\{c_j - \lambda, 0\} = E$. The $CEA$ and the $CEL$ are dual rules and the $P$ rule is *self-dual*, i.e., $P = P^d$. Most of the classical rules, as $P$, $CEA$, and $CEL$, are members of the so-called *parametric rules* (Young, 1987). Formally, let $[-\infty, +\infty] = \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real line ($-\infty < t < +\infty$ for all $t \in \mathbb{R}$ and $-\infty < +\infty$ by convention) and let $H$ be the set of functions $h : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $a, b \in [-\infty, +\infty], a \leq b$, such that $h$ is *continuous*, *non-decreasing in the first argument*, and for each $\bar{c} \in \mathbb{R}_+$, $h(a, \bar{c}) = 0$ and $h(b, \bar{c}) = \bar{c}$. Then, a rule $\beta$ is *parametric* if there exists $h \in H$ such that for all $(N, E, c) \in \mathcal{B}$ there exists $\lambda \in [a, b]$ satisfying $\beta_i(N, E, c) = h(\lambda, c_i)$ for all $i \in N$ and $\sum_{i \in N} h(\lambda, c_i) = E$. In this case, $h$ is called a *representation* of $\beta$.

## 3. Axioms

In this section, we introduce several axioms for rules. A large fraction of the literature on bankruptcy problems is devoted to study the strategic incentives of claimants to misrepresent claims, either by merging or splitting their respective claims in order to obtain some extra profits. De Frutos (1999) introduces two different "immunity" properties so as to separate these two types of incentives. A rule $\beta$ on $\mathcal{B}$ satisfies

- *non-manipulability via merging* (NMM) if for all $(N, E, c)$, $(N', E, c') \in \mathcal{B}$ with $m \in N' \subset N$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$, then
$$\beta_m(N', E, c') \leq \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c).$$
- *non-manipulability via splitting* (NMS) if for all $(N, E, c)$, $(N', E, c') \in \mathcal{B}$ with $m \in N' \subset N$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$, then
$$\beta_m(N', E, c') \geq \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c).$$

The more demanding axiom of *non-manipulability* (NM) requires NMM and NMS simultaneously.

While NMM stipulates that no group of claimants can take advantage from consolidating claims; NMS, on the contrary, guarantees that no claimant can benefit from dividing its claim into claims of a group of claimants. NM imposes that agents merging or splitting receive exactly the same as initially. These properties can be, indeed, reformulated taking into account the effects on the agents that do not misrepresent their claims. We might require that the merge of some agents in a single one or the split of an agent in a multiplicity of them, affect all agents whose claims do not change in the same direction. We will, indeed, impose that all these agents receive at least as much as initially. A rule $\beta$ on $\mathcal{B}$ satisfies

- *strong non-manipulability via merging* (SNMM) if for all $(N, E, c)$, $(N', E, c') \in \mathcal{B}$ with $m \in N' \subset N$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$, then
$$\beta_j(N', E, c') \geq \beta_j(N, E, c) \text{ for all } j \in N' \setminus \{m\}.$$
- *strong non-manipulability via splitting* (SNMM) if for all $(N, E, c)$, $(N', E, c') \in \mathcal{B}$ with $m \in N' \subset N$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$, then
$$\beta_j(N', E, c') \leq \beta_j(N, E, c) \text{ for all } j \in N' \setminus \{m\}.$$

Curiel et al. (1987) define *additivity of claims*, renamed as *strong non-manipulability* (SNM) by Moreno-Ternero (2006), requesting both SNMM and SNMS at the same time. While SNMM and SNMS impose that each of the agents not involved in the mergers or spin offs is not worse off, SNM enforces they are compensated exactly as initially. Clearly, under BB, these are stronger versions of NMM, NMS, and NM, respectively. Moreover, as Moreno-Ternero (2006) shows, SNM and NM are equivalent requirements. It is well known that the $CEA$ rule satisfies NMM while the $CEL$ rule satisfies NMS. Furthermore, the $P$ rule satisfies both, and it has been characterized as the unique rule satisfying NM (or equivalently, SNM) together with the mild requirement of *non-negativity* (requiring awards to be non-negative), and without imposing CB, by de Frutos (1999).[1]

Another important property in our analysis is *solidarity*, which demands that the arrival of new agents affects all the original agents in the same direction: either all gain or all lose. Chun (1999) shows that solidarity is equivalent to the standard requirement of *resource monotonicity* and *consistency.* The former says that no one should be worse off when the firm's assets increase and the later requires that in the reduced bankruptcy problem, which arises when some players leave with their share, each of the remaining players receives the same amount as in the original problem. A rule $\beta$ on $\mathcal{B}$ satisfies

- *solidarity* (SOL) if for all $(N, E, c)$, $(N', E', c') \in \mathcal{B}$ such that $N' \subseteq N$, if $c' = c_{N'}$, then either $\beta(N'E', c') \geq \beta_{N'}(N, E, c)$ or $\beta(N'E', c') \leq \beta_{N'}(N, E, c)$;
- *resource monotonicity* (RM) if for all pair $(N, E, c), (N, E', c) \in \mathcal{B}$ with $E' > E$, $\beta_i(N, E', c) \geq \beta_i(N, E, c)$ for all $i \in N$;
- *consistency* (CONS) if for all $(N, E, c) \in \mathcal{B}$ and all $\emptyset \neq N' \subseteq N$, $\beta_{N'}(N, E, c) = \beta\left(N', \sum_{i \in N'} \beta_i(N, E, c), c_{N'}\right)$.

## 4. Results

In the following, we address the question on how big is the gap between NMM and NMS and their strong counterparts. In order to study classes of rules for which SNMM and SNMS do not make a difference with their weak formulations, we show that these are dual properties. We say that two *properties* P and P* are *dual* if, whenever a rule $\beta$ satisfies P then its dual $\beta^d$ satisfies P*. If, moreover, P coincides with P*, then P is *self-dual.* As shown by de Frutos (1999), NMM and NMS are dual to each other. In the next proposition we show that SNMM and SNMS are dual properties.

[1] Ju et al. (2007) employ the weaker axioms of *one-sided boundedness*, stating that payoffs should be bounded from either above or below, and *pairwise non-manipulability.*

**Proposition 1.** SNMM *and* SNMS *are dual to each other.*

*Proof.* Let $\beta$ be a rule satisfying SNMM and $(N, E, c)$, $(N', E, c') \in \mathcal{B}$ with $m \in N' \subset N$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$. Then, by SNMM of $\beta$ applied to $(N', \sum_{i \in N'} c'_i - E, c')$ and $(N, \sum_{i \in N} c_i - E, c)$ being $\sum_{i \in N'} c'_i - E = \sum_{i \in N} c_i - E \geq 0$ it holds that, for all $j \in N' \setminus \{m\}$

$$\begin{array}{rcll} \beta_j(N', \sum_{i \in N'} c'_i - E, c') & \geq & \beta_j(N, \sum_{i \in N} c_i - E, c) & \iff \\ c'_j - \beta_j(N', \sum_{i \in N'} c'_i - E, c') & \leq & c_j - \beta_j(N, \sum_{i \in N} c_i - E, c) & \iff \\ \beta^d_j(N', E, c') & \leq & \beta^d_j(N, E, c), & \end{array}$$

which means that $\beta^d$ satisfies SNMS.

Analogously, simply changing the direction of the chain of inequalities, it can be shown that if $\beta$ satisfies SNMS then its dual rule $\beta^d$ fulfills SNMM. □

Even though, in general, NMM and NMS are not respectively equivalent to SNMM and SNMS, as we will show in Theorem 1, for the class of rules that meet solidarity they are.

**Proposition 2.** *Let* $\beta$ *be a bankruptcy rule satisfying* SOL. *Then,* $\beta$ *satisfies* NMM (NMS) *if and only if it satisfies* SNMM (SNMS).

*Proof.* Since NMM (SNMM) and NMS (SNMS) are dual properties to each other (Proposition 1), it is enough to see that, under SOL, a bankruptcy rule $\beta$ satisfies NMM if and only if it satisfies SNMM. Clearly, SNMM implies NMM. To show the reverse implication, consider a bankruptcy rule $\beta$ satisfying SOL and NMM. Let $(N, E, c)$, $(N', E, c') \in \mathcal{B}$ such that $N' \subset N$ and there is $m \in N'$ with $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c_j = c'_j$ for all $j \in N' \setminus \{m\}$. By NMM, $\beta_m(N'E, c') \leq \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c)$ or, equivalently,

$$E - \beta_m(N', E, c') \geq E - \sum_{j \in \{m\} \cup N \setminus N'} \beta_j(N, E, c). \tag{1}$$

From (1), and taking into account that $c_i = c'_i$ for all $i \in N' \setminus \{m\}$, by SOL, which implies RM and CONS, we obtain

$$\begin{aligned} \beta_i(N', E, c') & \underset{\text{CONS}}{=} \beta_i\left(N' \setminus \{m\}, E - \beta_m(N', E, c'), c'_{N' \setminus \{m\}}\right) \\ & \underset{\text{RM}}{\geq} \beta_i\left(N' \setminus \{m\}, E - \sum_{j \in \{m\} \cup N \setminus N'} \beta_j(N, E, c), c_{N' \setminus \{m\}}\right) \\ & \underset{\text{CONS}}{=} \beta_i(N, E, c), \end{aligned}$$

which proves SNMM of $\beta$.

Note that BB and CB ensure that the reduced problems are well defined. $\square$

The well-established class of *parametric rules* is characterized by Young (1987) making use of CONS, together with *endowment continuity* and *equal treatment of equals*. While endowment continuity guarantees that small changes on the estate do not provoke large changes on the rule, equal treatment of equals impose that agents with the same claim are rewarded equally. These rules also satisfy RM and thus SOL. Hence, a direct consequence of Proposition 2 is the following.

**Corollary 1.** *A parametric rule is* NMM (NMS) *if and only if it is* SNMM (SNMS).

Thus, the $CEA$ and the $CEL$ rules satisfy SNMM and SNMS, respectively, while the $P$ rule meets both properties. For a characterization of the set of parametric bankruptcy rules that are NMM or NMS we refer readers to Proposition 1 in Ju (2003).

Finally, we show that removing solidarity in the analysis, non-manipulability via merging or splitting are not equivalent to their strong counterpart. To do it, we first introduce a rule that will play a significant role: the *low claims priority rule*, that gives priority to agents with small claims.

Let $\delta = (N, E, c)$ be a bankruptcy problem and $\prec^\delta$ be the strict priority order on $N$, defined by setting for all $i, j \in N$,

$$i \prec^\delta j \text{ if either } c_i < c_j \text{ or } c_i = c_j \text{ and } i < j. \tag{2}$$

The order $\prec^\delta$ places agents with low claims first and, in case two agents have the same claim, then the one indexed with a lower natural number first. Thus, the low claims priority rule first fully honor the highest priority claimant, if the estate is enough; if not, she receives the entire estate. Second, it fully compensate the second highest priority claimant if possible; if not, she receives what is left of the estate; and so on. Formally,

**Definition 1.** *The* low claims priority rule*, $P^{\prec^\delta}$, is defined as follows: let $\delta = (N, E, c) \in \mathcal{B}$ and $\prec^\delta$ be the corresponding order as defined in (2),*

$$P_i^c(N, E, c) = \max\left\{0, \min\left\{E - \sum_{\{j \in N \mid j \prec^\delta i\}} c_j,\ c_i\right\}\right\} \tag{3}$$

*for all $i \in N$.*

The $P^{\prec^\delta}$ rule has the flavour of sequential priority rules (see Thomson, 2019). While sequential priority rules are defined taking a fixed order for all bankruptcy problems, the low claims priority rule, on the contrary, is introduced according to the endogenous order $\prec^\delta$ determined by agents' claims.

**Theorem 1.** *Neither* NMM *implies* SNMM*, nor* NMS *implies* SNMS*.*

*Proof.* Since SNMM and SNMS are dual properties (Proposition 1), it is enough to prove that NMM does not imply SNMM.

Let us consider the following subclass of bankruptcy problems:

$$\mathcal{C}^* = \left\{ \begin{array}{c} \delta = (N, E, c) \in \mathcal{B} \text{ such that } \{1,2\} \subset N, E = 1, c_1 = c_2 = c_{k^\delta} = 1 \\ \text{for some } k^\delta \in N \setminus \{1,2\}, \text{ and } c_i = 0 \text{ for all } i \in N \setminus \{1,2,k^\delta\} \end{array} \right\}.$$

Now define the rule $\beta^*$ by setting, for all $\delta = (N, E, c) \in \mathcal{B}$ and all $i \in N$,

$$\beta_i^*(\delta) = \begin{cases} P_i^c(\delta) & \text{if} \quad \delta \notin \mathcal{C}^* \\ 1 & \text{if} \quad \delta \in \mathcal{C}^* \text{ and } i = k^\delta \\ 0 & \text{if} \quad \delta \in \mathcal{C}^* \text{ and } i \neq k^\delta. \end{cases} \tag{4}$$

Note that, for all $\delta = (N, E, c) \in \mathcal{B}$ and all $i \in N$, $\beta_i^*(\beta) \geq 0$ (non negativity) and, if $c_i = 0$, then $\beta_i^*(\delta) = 0$.

First, we show that $\beta^*$ is strong manipulable by merging.

**Claim 1**: $\beta^*$ does not satisfy SNMM.

Consider the bankruptcy problem $\delta = (N, E, c)$ with set of players $N = \{1, 2, 3, 4\}$, estate $E = 1$, and vector of claims $c = (1, 0, 1, 1)$. Now let $\delta' = (N', E, c')$ with $N' = \{1, 2, 3\}$, where agents 2 and 4 have merged into agent 2, and the vector of claims is $c' = (1, 1, 1)$. Since $c_2 = 0$, $\delta \notin \mathcal{C}^*$ and thus $\beta^*(\delta) = P^{\prec^\delta}(\delta) = (1, 0, 0, 0)$. On the other hand, since $\delta' \in \mathcal{C}^*$, $\beta^*(\delta') = (0, 0, 1)$. Hence, $\beta_1^*(\delta') = 0 < \beta_1^*(\delta) = 1$, and $\beta^*$ does not satisfy SNMM.

Next, we show that $\beta^*$ is non-manipulable by merging.

**Claim 2**: $\beta^*$ satisfies NMM.

Let $\delta = (N, E, c)$ and $\delta' = (N', E, c')$ be two bankruptcy problems such that $N' \subset N$ and there is $m \in N'$ with $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$, for all $j \in N' \setminus \{m\}$. We consider the following cases:

Case 1: $\delta, \delta' \notin \mathcal{C}^*$.

Then, $\beta^*(\delta) = P^c(\delta)$ and $\beta^*(\delta') = P^c(\delta')$. Observe that when players in $\{m\} \cup N \setminus N'$ merge into $m$, then for a given $j \in N' \setminus \{m\}$ the position according to $\prec^{\delta'}$ is less than or equal to the position according to $\prec^\delta$ and, moreover, she does not have any new predecessor. Additionally, $\sum_{\{k \in N' \mid k \prec^{\delta'} j\}} c'_k \leq \sum_{\{k \in N \mid k \prec^\delta j\}} c_k$. So, for all $j \in N' \setminus \{m\}$, we have $\beta_j^*(\delta') \geq \beta_j^*(\delta)$ and thus, by BB, $\beta_m^*(\delta') \leq \beta_m^*(\delta) + \sum_{j \in N \setminus N'} \beta_j^*(\delta)$ and thus NMM holds.

Case 2: $\delta, \delta' \in \mathcal{C}^*$.

- If $m \neq k^{\delta'}$, then $\beta^*_m(\delta') = 0$ and hence, by non-negativity of $\beta^*$, $0 = \beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.
- If $m = k^{\delta'}$, then $\beta^*_m(\delta') = 1$. To see that $\beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta) = 1$ it is enough to check that $k^\delta \in \{m\} \cup N \setminus N'$. Indeed, if not, $k^\delta \in N' \setminus \{m\}$ with $k^\delta \neq k^{\delta'}$ and hence $c_{k^{\delta'}} = c_{k^\delta} = 1$, in contradiction with $\delta' \in \mathcal{C}^*$.

  Thus, in any case NMM holds.

Case 3: $\delta \notin \mathcal{C}^*$ and $\delta' \in \mathcal{C}^*$.

- If $m \neq k^{\delta'}$, by non-negativity of $\beta^*$, $0 = \beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.
- If $m = k^{\delta'}$, then $\beta^*_m(\delta') = 1$. Since $\delta' \in \mathcal{C}^*$,

$$1 = c'_{k^{\delta'}} = c_{k^{\delta'}} + \sum_{j \in N \setminus N'} c_j, \tag{5}$$

  Let $A = \{j \in \{k^{\delta'}\} \cup N \setminus N' \text{ such that } c_j > 0\}$. By (5), $|A| \geq 1$.

  Suppose that $|A| = 1$. In this situation, $\{1, 2\} \subseteq N' \setminus \{m\}$, $c_1 = c'_1 = 1$, $c_2 = c'_2 = 1$, and there is a unique $k \in A$ with $c_k = 1$. Moreover, from the definition of $\delta$ and $\delta'$, $c_i = 0$ for all $i \in N \setminus \{1, 2, k\}$. But then, $\delta \in \mathcal{C}^*$ getting a contradiction. Consequently, $|A| \geq 2$ and, in view of (5), $0 < c_j < 1$ for all $j \in A$. Since, $c_1 = c'_1 = 1$, $c_2 = c'_2 = 1$, and $c'_i = c_i = 0$ for all $i \in N' \setminus \{1, 2, k^{\delta'}\} = N \setminus A \cup \{1, 2\}$, the order $\prec^\delta$ on $N$ place all players in $A$ immediately after zero-claimants and hence, by (5), we obtain $\beta^*_{k^{\delta'}}(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta) = 1 = \beta^*_m(\delta')$.

  Thus, in any case NMM holds.

Case 4: $\delta \in \mathcal{C}^*$ and $\delta' \notin \mathcal{C}^*$.

- If $k^\delta \in \{m\} \cup N \setminus N'$, since $\delta \in \mathcal{C}^*$, $\beta^*_{k^\delta}(\delta) = 1$ and, for all $i \in N \setminus \{k^\delta\}$, $\beta^*_i(\delta) = 0$. Thus, $1 = \beta^*_{k^\delta}(\delta) = \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta) \geq \beta^*_m(\delta')$, where the inequality comes from BB and non-negativity of $\beta^*$.
- If $k^\delta \notin \{m\} \cup N \setminus N'$. Recall that, since $\delta \in \mathcal{C}^*$, $c_j = 0$ for all $j \in N \setminus \{1, 2, k^\delta\}$. We distinguish the following sub-cases:
  (a) $1, 2 \notin \{m\} \cup N \setminus N'$.
  Hence, $m \notin \{1, 2, k^\delta\}$. Consequently, $c'_m = c_m + \sum_{j \in N \setminus N'} c_j = 0$. By non-negativity of $\beta^*$, we conclude that $0 = \beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.
  (b) $1, 2 \in \{m\} \cup N \setminus N'$.
  Here, $c'_m = c_1 + c_2 = 2$, $c'_{k^\delta} = c_{k^\delta} = 1$, and $c'_j = 0$ for all $j \in N' \setminus \{m, k^\delta\}$. Hence, $\beta^*_m(\delta') = P^c_m(\delta') = 0$, and by non-negativity of $\beta^*$, we conclude that $\beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

(c) $1 \in \{m\} \cup N \setminus N'$ but $2 \notin \{m\} \cup N \setminus N'$.
In this case, $c'_m = c_1 = 1$, $c'_2 = c_2 = 1$, $c'_{k^\delta} = c_{k^\delta} = 1$, and $c'_j = 0$ for all $j \in N' \setminus \{m, 2, k^\delta\}$. Since $\delta' \notin \mathcal{C}^*$, $m \neq 1$ and thus $m \geq 3$. Hence, $\beta^*_m(\delta') = P^c_m(\delta') = 0$ and, by non-negativity of $\beta^*$, we obtain $\beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

(d) $1 \notin \{m\} \cup N \setminus N'$ but $2 \in \{m\} \cup N \setminus N'$.
In this situation, $c'_m = c_2 = 1$, $c'_1 = c_1 = 1$, $c'_{k^\delta} = c_{k^\delta} = 1$, and $c'_j = 0$ for all $j \in N' \setminus \{1, m, k^\delta\}$. Since $\delta' \notin \mathcal{C}^*$, $m \neq 2$ and thus $m \geq 3$. Hence, $\beta^*_m(\delta') = P^c_m(\delta') = 0$ and, by non-negativity of $\beta^*$, we obtain $\beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.
Thus, in any case NMM holds.

□

## 5. Final comments

In the setting of bankruptcy problems, we have shown that non-manipulability by merging or splitting are not equivalent to their strong counterpart; however, in the presence of the principle of solidarity, they are. This implies that for a wide range of rules, including parametric rules (Young, 1987), both properties are coincident. To conclude, let us stress that the rule used to prove Theorem 1 is neither RM nor CONS. Therefore, an interesting open question for future research is to investigate whether or not solidarity can be weakened into resource monotonicity or continuity.


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