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Stable source connection and assignment problems as multi-period shortest path problems

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Abstract

We extend the familiar shortest path problem by supposing that agents have demands over multiple periods. This potentially allows agents to combine their paths if their demands are complementary; for instance if one agent only needs a connection to the source in the summer while the other requires it only in the winter.

We show that the resulting cost sharing problem always has a non-empty core, regardless of the number of agents and periods, the cost structure or the demand profile.

We then exploit the fact that the model encompasses many well-studied problems to obtain or reobtain non-vacuity results for the cores of source-connection problems, (m-sided) assignment problems and minimum coloring problems.

JEL classification numbers: C71, D63.

Keywords: shortest path, demand over multiple periods, cooperative game, core, source-connection, assignment.

1 Introduction

Shortest path problems are well-studied in operations research and economics. While they are often used to determine, for instance, the quickest route for a truck making a delivery from A to B, we are interested in applications in which capacity has to be built to connect agents to a source, the capacity is not easily adjustable and the cost is linearly increasing with capacity. Gas and oil pipelines, as well as rail networks, are some examples that fit the bill. Following Rosenthal (2013) and Bahel and Trudeau (2014), we are interested in the cost sharing problem generated by these

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situations. We, however, generalize the problem by supposing that there are multiple periods, and that agents have demands that vary over these periods. Imagine two small cities, A and B, located respectively to the northeast and southeast of the capital city. City A is a ski destination popular in the winter, while city B is a beach destination popular in the summer. When designing a rail network that will connect both cities to the capital, we could connect them both directly to the capital, but it probably makes more sense to connect only one, say A, while connecting B to A, to take advantage of the complementary demand to both cities. We could say the same of gas pipelines to cities A and B, for which the gas is either used for heating in the winter or cooling in the summer.

The model in itself, that we call the *multi-period shortest path* (MSP) problem, has not been studied in economics, and our first contribution is to show that the core of the resulting cooperative game is always non-empty. Thus, we are always able to share the cost in a way that makes sure that no group of agents could jointly connect to the source at a cost cheaper than the amount they are assigned. The non-vacuity of the core thus extends from the classic, one-period shortest path problem (Rosenthal (2013)), and does not require any condition on the network structure, number of periods, number of agents or demand profile. In fact, it holds for any subgame as well. Our proof is strongly inspired by Quant et al. (2006), who study a (one-period) network flow problem where on each edge of the network there is a cost function that is convex with respect to the flow. The non-vacuity result is also valid whether we suppose that a coalition of agents can connect through (non-cooperating) neighbors or not. These variants are discussed for the related minimum cost spanning tree problem in Trudeau and Vidal-Puga (2019). Our non-vacuity result also extends to private games (agents can prevent others from using their nodes) where the cost functions on all edges are convex.

Our second contribution is to show how general the model is, and how it encompasses many well-known problems. Source-connection problems are an obvious group of problems that can be rewritten as MSP problems, such as the aforementioned (one-period) shortest path and the minimum cost spanning tree (mcst) problems (Bird (1976)). Mcst problems are such that on any edge the first unit of capacity is costly, but others are free.

Less obviously, MSP problems also encompass assignment problems (Shapley and Shubik (1971)), where we need to match agents belonging to different sides of the market, with the classic example being a housing market. As in the example described above, we can construct a MSP problem in which we have some cities demanding in the summer and some others in the winter, and matching them allows to save on cost. It is well-known that the core of an assignment game is always non-empty, and we show that all assignment problems can be written as a MSP problem. Extensions of the assignment problem to m-sides, with $m > 2$, is where the connection to MSP problems is more useful. Many variants have been proposed and we study the strict m-sided assignment problem of Quint (1991) and the generalized version of Atay et al. (2016). In the former, value is created only when we match a group that contains one player from each side, while in the latter there is also some value created when we match smaller groups of agents from different sides. In both cases, the core of the corresponding game can be empty, and sufficient conditions for its non-vacuity have been proposed. Some but not all of these m-sided assignment problems have a corresponding MSP representation, but by our stability result, if an assignment problem can be represented as a MSP problem, it has a non-empty core. We provide sufficient conditions for representability as MSP problems, and thus for non-vacuity of the core, and show that the resulting set of stable games are distinct from those described by Quint (1991) and Atay et al. (2016).

We also examine compatibility problems, most notably the minimum coloring problem, studied as a cost sharing problem by Okamoto (2008). In those problems, the set of agents has to be partitioned in groups, such that agents in the same group have no conflict with each other. The conflicts are represented in a graph, with an edge between i and j on the graph indicating that i and j are in conflict and cannot be assigned to the same group. A sufficient condition for the non-vacuity of the core is provided, requiring that for any subset of agents, the number of groups needed to avoid conflicts (the chromatic number of the graph) is equal to the size of the largest group of agents all in conflict with each other (the size of the largest clique in the graph). A simpler version where the condition is always verified is presented by Bahel and Trudeau (2019) where agents have time-sensitive jobs to be processed on a machine, and in which we are trying to determine the smallest number of machines required to process all jobs without conflict. We show that when the condition of Okamoto (2008) is verified, we have representability as a MSP problem. We are also able to use our stability result in a different way to extend the set of problems with non-empty cores. If we weaken the condition of Okamoto (2008) so that it only needs to hold for the grand coalition, then there exists a MSP problem which has the same cost for the grand coalition and a cost that might be smaller but not strictly larger for any other coalition. The core of this MSP problem is thus a subset of the core of the minimum coloring problem. By our non-vacuity result for the MSP problem, the core of the minimum coloring problem is also non-empty.

The rest of the paper is divided as follows. Section 2 describes the MSP problem and the associated cooperative game. Section 3 is devoted to the non-emptiness of the core. In Section 4 we show applications to source connection, assignment and compatibility problems.

2 The model

Let $N = \{1, ..., n\}$ be the set of *agents*. Let $M = \{1, ..., m\}$ be the set of *periods*. Both n and m are finite, with $n \geq 2$ and $m \geq 1$.

For all $i \in N$, let $q_i = (q_{i1}, ..., q_{im}) \in \mathbb{R}_+^M$ be the *demand profile* for agent i. In particular, for $t \in M$, q_{it} is the demand of agent i at time t. Let $Q = (q_1, ..., q_n)$ be the demand profiles for all agents. For all $t \in M$, let $N^t(Q) = \{i \in N | q_{it} > 0\}$ be the set of agents with a strictly positive demand at time t.

Agents are located at different points in space. We represent their position on a complete, simple digraph, with each agent occupying a different node. There is a special node, 0, that we call the *source*. For all $S \subseteq N$, $S_0 := S \cup \{0\}$. Let $E := \{(i,j) | i, j \in N_0 \text{ and } i \neq j\}$ denote the set of directed edges. To obtain their demand, agents need to build paths to the source. In addition, the capacity of these paths must be large enough to carry their demands in every period.

For each pair $(i,j) \in E$, $c_{ij} \geq 0$ represents the cost to install a capacity of one unit from i to j. Cost is linear with capacity, so for any $k \geq 0$, the cost of installing a capacity of k on edge (i,j) is $k \cdot c_{ij}$. Let $c = (c_{ij})_{(i,j)\in E}$ be the collection of costs of all edges. We abuse language slightly by calling it a *cost matrix*. We sometimes make assumptions on c. We say that c is symmetric if $c_{ij} = c_{ji}$ for all $i, j \in N_0$. We say that c satisfies the triangle inequality if for any $i, j, k \in N_0$, $c_{ik} \leq c_{ij} + c_{jk}$.

The tuple (N, M, Q, c) is called a *multi-period shortest path problem* (MSP problem for short).

2.1 Optimal networks

The first objective is to build a network connecting all agents to the source that contains enough capacity to simultaneously carry all demands at each period. For any $(i,j) \in E$, let $z_{ij} \geq 0$ be the *capacity* installed from i to j. Let $z = (z_{ij})_{(i,j)\in E}$ be the collection of capacities on all edges. We call z a *network*.

For $i \in N$, a *path from i to the source* P_i is a sequence of edges $((i_s, i_{s+1}))_{s=1}^{r}$ with $(i_s, i_{s+1}) \in E$ for all $s \in \{1, ..., r\}$ such that $i_1 = i$, $i_{r+1} = 0$ and for $l, m \in \{1, ..., r\}$, $i_l = i_m$ implies $l = m$. We use $(i,j) \in P_i$ to denote that (i,j) is an element of P_i. We say that r is the length of path P_i. The set of paths for agent i to the source is denoted by $\mathcal{P}_i$. Note that all $\mathcal{P}_i$ have the same finite cardinality. A connection plan for $i \in N$ is a tuple $f_i = (f_{i,t})_{t\in M}$, denoting the flows for agent i for all paths and all periods, where for all $t \in M$, $f_{i,t} = (f_{P_i,t})_{P_i \in \mathcal{P}_i}$ such that $f_{P_i,t} \geq 0$ for all $P_i \in \mathcal{P}_i$. The value $f_{P_i,t}$ denotes the flow that agent i sends through path P_i in period t. Let $f = (f_i)_{i\in N}$ be a profile of connection plans for all agents. The connection plan f is *feasible* for demand Q if for all $i \in N$ and all $t \in M$, $\sum_{P_i \in \mathcal{P}_i} f_{P_i,t} = q_{it}$. Let $F(N, Q)$ be the set of feasible connection plans for demand Q.

A network z *realizes* f if

$$z_{jk} \geq \max_{t\in M} \sum_{i\in N} \sum_{\substack{P_i \in \mathcal{P}_i \\ s.t.\ (j,k)\in P_i}} f_{P_i,t}$$

for all $(j,k) \in E$, i.e., if the capacity installed is sufficient to carry all flows, on all edges and in all periods. We call z^f the network in which all weak inequalities hold with equality.

For any z, its cost is $\gamma(z, c) = \sum_{(j,k)\in E} z_{jk} c_{jk}$. Let $C(N, Q) = \min_{f\in F(N,Q)} \gamma(z^f, c)$ be the cost of the cheapest feasible connection plan for the grand coalition N. In the same manner, for all

$S \subseteq N$, we define $C(S,Q) = \min_{f \in F(N,Q^S)} \gamma(z^f, c)$ where Q^S is the restriction of Q to S, i.e. for all $t \in M$, $q_{it}^S = q_{it}$ if $i \in S$ and $q_{it}^S = 0$ otherwise. When there is no confusion we write $C(N)$ instead of $C(N,Q)$ and $C(S)$ instead of $C(S,Q)$

Example 1 *Throughout the paper we will illustrate our results with a 4-player example, with the cost matrix described in Figure 1. Circled numbers represent the agents and the source. We suppose that the cost matrix is symmetric and the number on an edge $\{i,j\}$ represents the cost from i to j and j to i. It is easy to see that the example satisfies the triangle inequality.*

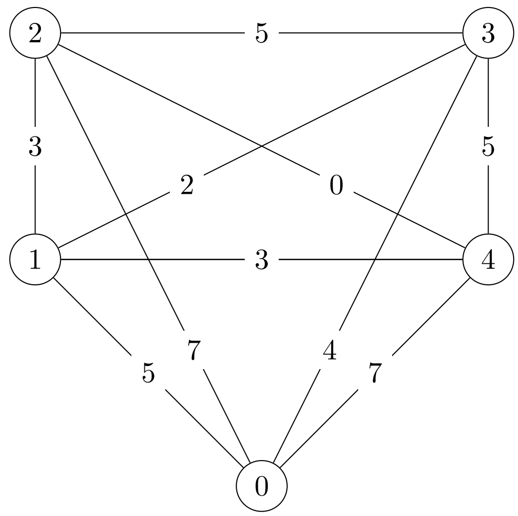

Figure 1: Example of a cost matrix with 4 agents.

Suppose that $m = 2$ and that $Q = ((1,0),(2,1),(0,1),(2,3))$. While there are 10 units being consumed, they are evenly split between the two periods. We consider paths of length 1 and 2 to the source. Let $P_i^k = ((i,k),(k,0))$ if $i \neq k$ and $P_i^k = ((i,0))$ otherwise. Consider the following connection plans f and the cost of the corresponding network z^f.

	Agent 1		*Agent 2*		*Agent 3*		*Agent 4*		
	Period 1	*Period 2*	*Period 1*	*Period 2*	*Period 1*	*Period 2*	*Period 1*	*Period 2*	$\gamma(z^f, c)$
f^1	P_1^1	$\emptyset$	P_2^2	P_2^2	$\emptyset$	P_3^3	P_4^4	P_4^4	*44*
f^2	P_1^4	$\emptyset$	P_2^2	P_2^2	$\emptyset$	P_3^2	P_4^4	P_4^4	*43*
f^3	P_1^3	$\emptyset$	P_2^4	P_2^4	$\emptyset$	P_3^3	P_4^4	P_4^4	*34*

Connection plan f^1 connects all agents directly to the source, with a capacity equal to their largest demand in any period. f^2 takes advantage of the fact that agents 1 *and* 4 *jointly demand 3 units in each period. Thus, while agent 4 still installs* 3 *units of capacity on* $(4,0)$, *now agent 1 consumes the unused unit in period 1, connecting through* $(1,4)$. *We do the same for agents* 2 *and* 3, *who together demand* 2 *units in each period. f^3 follows the same idea, but instead has agents* 1 *and* 3 *and agents* 2 *and* 4 *partner up. There are many other feasible connection plans, but it can* *be verified that none are cheaper than* f^3*. Thus,* $C(N) = 34$.

2.2 Cooperative game

Let (N, M, Q, c) be a MSP problem. Note that $C(\cdot)$ can be regarded as the characteristic function of a cooperative cost game (N, C), describing the cheapest way for any $S \subseteq N$ to obtain its demand. By definition $C(\emptyset) = 0$ and $S \subseteq T$ implies that $C(S) \leq C(T)$. For each $S \subseteq N$ we define the subgame C_S such that for all $T \in 2^S$, $C_S(T) \equiv C(T)$.

An *allocation* is a vector $y \in \mathbb{R}^N$ such that $\sum_{i \in N} y_i = C(N)$. For any $S \subseteq N$, we define $y(S) \equiv \sum_{i \in S} y_i$. We say that an allocation y is a *core allocation* if $y(S) \leq C(S)$ for all $S \subset N$. For a cooperative game with characteristic function C, the set of all core allocations is denoted with $Core(C)$.

A map $\lambda : 2^N \setminus \{\emptyset\} \to [0, 1]$ is said to be *balanced* if for all $i \in N$, $\sum_{\substack{S \in 2^N \\ i \in S}} \lambda_S = 1$. Bondareva (1963) and Shapley (1967) have shown that the core of the cooperative (cost) game (N, C) is non-empty if and only if for any balanced map λ, $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S C(S) \geq C(N)$, i.e. if (N, C) is *balanced*. We say that a game is *totally balanced* if all subgames have a non-empty core.

We often convert (N, C) into a *savings game* (N, V) as follows: for any $C(\cdot)$ and $S \subseteq N$, let $V(S) = \sum_{i \in S} C(\{i\}) - C(S)$. $V(S)$ is the savings generated by the cooperation of agents in S, compared to each acting on their own. The allocation y^v is the savings game-equivalent of y and is defined as follows: for all $i \in N$, $y_i^v = C(\{i\}) - y_i$. The allocation y^v is a core allocation if $y^v(S) \geq V(S)$ for all $S \subset N$. By definition, $y^v \in Core(V)$ if and only if $y \in Core(C)$.

3 The core of MSP problems

3.1 MSP problems generate balanced games

We show in this section that the core of a MSP game is always non-empty. This is done by showing that the game is balanced, with a proof inspired by Quant et al. (2006). We take a balanced map, and consider the optimal connection plan and network for each coalition. Multiplying by the balanced map and then summing up, we show that we obtain a connection plan that is feasible for the grand coalition and a network that realizes it. Since the cost is linear on each edge, the cost of this network is the weighted sum of the costs of the networks for all separate coalitions. As this network realizes a feasible connection plan for the grand coalition, the cost to connect the grand coalition can thus not be higher and the game is thus balanced.

Theorem 1 *For all MSP problems* (N, M, Q, c)*,* $Core(C)$ *is non-empty.*

Proof.

Let λ be a balanced map and $B = \left\{S \in 2^N \setminus \{\emptyset\} \mid \lambda_S > 0\right\}$. For all $S \in B$, let f^S be (one of) its optimal connection plan(s) and z^S the corresponding network. Let $z^* = \sum_{S\in B} \lambda_S z^S$. Let f^* be the connection plan such that $f^*_i = \sum_{S\in B} \lambda_S f^S_i$ for all $i \in N$. Note that $f^S_i = 0$ for all $i \notin S$.

We verify that $f^* \in F(N, Q)$.

Since $f^S \in F(N, Q^S)$, we have that for all $i \in S$ and all $t \in M$, $\sum_{P_i \in \mathcal{P}_i} f^S_{P_i,t} = q_{it}$. Multiplying by λ_S and summing over $S \in B, i \in S$ on both sides, we obtain:

$$\begin{aligned}
\sum_{\substack{S\in B\\ i\in S}} \lambda_S \sum_{P_i\in\mathcal{P}_i} f^S_{P_i,t} &= \sum_{\substack{S\in B\\ i\in S}} \lambda_S q_{it} \\
\sum_{P_i\in\mathcal{P}_i} \sum_{\substack{S\in B\\ i\in S}} \lambda_S f^S_{P_i,t} &= \sum_{\substack{S\in B\\ i\in S}} \lambda_S q_{it} \\
\sum_{P_i\in\mathcal{P}_i} f^*_{P_i,t} &= q_{it}
\end{aligned}$$

and thus $f^* \in F(N, Q)$.

We next verify that z^* realizes f^*.

Since z^S realizes f^S, we have that

$$z^S_{jk} = \max_{t\in M} \sum_{i\in N} \sum_{\substack{P_i\in\mathcal{P}_i\\ s.t.\ (j,k)\in P_i}} f^S_{P_i,t}$$

for all $(j, k) \in E$. Multiplying by λ_S and summing over $S \in B$ on both sides, we obtain:

$$\begin{aligned}
\sum_{S\in B} \lambda_S z^S_{jk} &= z^*_{jk} = \sum_{S\in B} \lambda_S \max_{t\in M} \sum_{i\in N} \sum_{\substack{P_i\in\mathcal{P}_i\\ s.t.\ (j,k)\in P_i}} f^S_{P_i,t} \\
&\geq \max_{t\in M} \sum_{i\in N} \sum_{\substack{P_i\in\mathcal{P}_i\\ s.t.\ (j,k)\in P_i}} \sum_{S\in B} \lambda_S f^S_{P_i,t} \\
&= \max_{t\in M} \sum_{i\in N} \sum_{\substack{P_i\in\mathcal{P}_i\\ s.t.\ (j,k)\in P_i}} f^*_{P_i,t}
\end{aligned}$$

for all $(j, k) \in E$ and thus z^* realizes f^*.

Therefore, we have that

$$\begin{aligned}
\sum_{S\in B} \lambda_S C(S) &= \sum_{S\in B} \lambda_S \gamma(z^S, c) \\
&= \gamma(z^*, c) \\
&\geq C(N)
\end{aligned}$$

where the last inequality comes from the fact that z^* realizes $f^* \in F(N, Q)$.

Thus, C is balanced. ■

MSP problems therefore have the interesting feature that we can always share the cost in a stable manner, guaranteeing that no group has incentives to secede and do the project by itself. While this is known to be true for classic, one-period shortest path problems, it is interesting to see that it also holds in our more general setting, with no conditions on the number of players, number of periods, demands or cost structure.

It is easy to see that for a MSP problem (N, M, Q, c) the above proof extends to all subgames, yielding the following corollary.

Corollary 1 *For all MSP problems* (N, M, Q, c), $C(\cdot)$ *is totally balanced.*

3.2 Extension to the private property game

So far, we have made the assumption that agents in S can use the nodes of players in $N \backslash S$ to construct their paths to the source. We can amend this assumption by supposing that a coalition can only use its own nodes (and the source). In many source connection problems, like minimum cost spanning tree problems, these distinct games are called *private* or *public* games, depending on whether the nodes are privately or publicly owned. Stated differently, in a private game, an agent can refuse to let others use his node when he is not cooperating with them. Among others, see Trudeau and Vidal-Puga (2019) for a discussion of the two approaches.

For a MSP problem (N, M, Q, c) the associated private game is denoted $C^{PRV}(\cdot, Q)$, or, once again when there is no confusion, $C^{PRV}(\cdot)$. Recalling that $F(S, Q^S)$ is the set of feasible connection plans when agents have access to nodes in S_0 and the agents in S have demand profile Q^S, we have that $C^{PRV}(S) = \min_{f \in F(S,Q^S)} \gamma(z^f, c)$.

We can immediately notice that $C^{PRV}(S) \geq C(S)$ for all $S \subset N$ and $C^{PRV}(N) = C(N)$. It is thus immediate that if $y \in Core(C)$, then $y \in Core(C^{PRV})$. It is obvious that the opposite is not true. We therefore have the following results.

Lemma 1 *For all MSP problems* (N, M, Q, c), $Core(C) \subseteq Core(C^{PRV})$.

Corollary 2 *For all MSP problems* (N, M, Q, c), $Core(C^{PRV})$ *is non-empty.*

3.3 Returns to scale

Another assumption of our model is that we have a linear cost function for each edge. Alternatively, we could have concave or convex cost functions. Assuming concave cost functions on all edges would lead to increasing returns to scale, as edges become cheaper the more they are used. If we take all cost functions to be convex, we get decreasing returns to scale and spreading flow over several paths might be beneficial.

Perhaps surprisingly, if we combine concave cost functions and cooperative gains, we might not be able to find core allocations. Specifically, if the bulk of the gains are generated by small coalitions, the core might be empty. In Trudeau (2009), an example of a (single-period) network flow problem with concave cost functions that has an empty core is provided. The result extends to our multiple period framework.

Though convex cost functions give rise to decreasing returns, this can, however, be partially offset by gains from other factors. In our setup, the ability of an agent to share his unused capacity in a given period is such a gain, and the access to new edges in the private game is another. As the coalition grows, the possibilities to minimize cost increase. We call these technological gains.

Suppose first that we have a single period and a public game. In this case, there are no technological gains to compensate the decreasing returns to scale. The core as defined is empty, as coalitions would prefer to act on their own.[1]

If, however, we do have positive technological gains, they can offset the losses caused by the convex cost functions. Quant et al. (2006) consider a (single-period) private network flow game with convex cost functions and show that the core, as defined in this paper, is always non-empty. Although the returns to scale are negative, as a coalition grows, it also gains access to new edges and can now spread its flows to avoid the increasing marginal costs. This is enough to generate a non-empty core.

We can extend the result of Quant et al. (2006) to multiple periods. Suppose that for each edge (i,j) we have a convex and increasing cost function $\theta_{ij}(k_{ij})$ with $\theta_{ij}(0)=0$. Let $\Theta = (\theta_{ij})_{(i,j)\in E}$ and (N, M, Q, Θ) be the resulting network-flow problem with demand over multiple periods. In all our definitions we replace c by Θ. (As we have non-constant returns to scale, it is less obvious to identify flows with particular paths between an agent and the source. We talk of network flows instead.)

Theorem 2 *If Θ contains only convex cost functions, then $Core(C^{PRV})$ is non-empty.*

Proof. The proof follows Quant et al. (2006) and our earlier proof for the case with linear cost functions.

Let λ be a balanced map and $B = \left\{S \in 2^N \setminus \{\emptyset\} \mid \lambda_S > 0\right\}$. For all $S \in B$, let f^S be (one of) its optimal connection plan(s) and z^S the corresponding network. Let $z^* = \sum_{S\in B} \lambda_S z^S$. Let f^* be the connection plan such that $f_i^* = \sum_{S\in B} \lambda_S f_i^S$ for all $i \in N$.

[1] The interpretation of the core collapses in this context, as the threat to act on your own to avoid negative externalities is an empty one. It is in fact more natural to reverse the sign of the inequalities in the definition of the core, so that each coalition bears a part of the decreasing returns to scale.

All we need to do is show that $\gamma(f^*,\Theta) \leq \sum_{S\in B} \lambda_S \gamma(f^S,\Theta)$. We have

$$\begin{aligned}
\gamma(f^*,\Theta) &= \sum_{(i,j)} \theta_{ij}(z^*_{ij}) \\
&= \sum_{(i,j)} \theta_{ij}(\sum_{S\in B} \lambda_S z^S_{ij}) \\
&\leq \sum_{(i,j)} \sum_{S\in B} \lambda_S \theta_{ij}(z^S_{ij}) \\
&= \sum_{S\in B} \sum_{(i,j)} \lambda_S \theta_{ij}(z^S_{ij}) \\
&= \sum_{S\in B} \lambda_S \gamma(f^S,\Theta) \\
&= \sum_{S\in B} \lambda_S C^{PRV}(S)
\end{aligned}$$

where the inequality follows from the fact that in the private game, only coalitions containing $\{i,j\}$ can use edge (i,j). This implies that $\sum_{\substack{S\in B \\ z^S_{ij}>0}} \lambda_S \leq 1$ and by the properties of a convex function, the inequality follows. ■

4 Applications

The main attraction of multi-period shortest path problems is that they encompass a large number of well-studied problems. We describe some of those in this section.

4.1 Source connection problems

The most obvious application is to the various source-connection problems. We discuss in this subsection (classic) shortest path problems, minimum cost spanning tree problems and minimum cost arborescence problems.

4.1.1 Shortest-path problems

As discussed in the introduction, MSP problems are extensions of the classic shortest path problems. By setting $m = 1$, we recover the full set of (classic) shortest path problems. Note that the public game is not particularly interesting, as each agent paying the cost of his path(s) is the only allocation in the core. More interesting allocations are found in the core of the private game, which include the possibility of subsidies for well-located agents. See Rosenthal (2013) and Bahel and Trudeau (2014).

4.1.2 Minimum cost spanning tree and minimum cost arborescence problems

Less obviously, MSP problems also encompass the well-studied *minimum cost spanning tree* (mcst) problems, in which the cost function on each edge is a fixed cost that has to be paid if the link is used (in any direction), with the cost not depending on the capacity. It is well established that the cores of the public and private versions of that problem are non-empty (Bird (1976)).

We can obtain mcst problems as MSP problems by letting $n = m$ and by having $Q = D^n$, where D^n is a "diagonal matrix" such that $D^n_{it} = 1$ if $i = t$ and $D^n_{it} = 0$ otherwise. In words, we have the same number of periods as agents, and each agent demands in a different period. It now becomes possible to construct a single tree (with capacity 1) connecting all agents to the source as they will each use it in a different period. Because all demands are of one unit, the per-unit cost in MSP problems behaves like the fixed cost of mcst problems.

Lemma 2 *If the MSP problem (N, M, Q, c) is such that $|N| = |M|$, $Q = D^n$ and c is symmetric, then (N, M, Q, c) is equivalent to a mcst problem. In addition, all mcst problems can be written as MSP problems.*

The second statement in the lemma is obvious: starting from the mcst problem, composed of N and a cost matrix c, the equivalent MSP problem is (N, N, D^n, c).

Example 2 *We revisit Example 1. Suppose that to N and c we add $M = \{1, ..., 4\}$ and $Q = D^4$, i.e. agent i demands* 1 *unit in period i and none in the other periods. We can then build a minimum cost spanning tree composed of edges (4,2), (2,1), (1,3) and (3,0), for a total cost of* 9*. We build a capacity of 1 on each of those edges, and in period i, agent i uses the path on that tree from i to the source.*

Minimum cost arborescence (mca) problems are extensions of mcst problems where the cost matrix might be asymmetric. The core of both the private and public games are non-empty, and the cooperative games generated by these problems were studied in Dutta and Mishra (2012) and Bahel and Trudeau (2017). The link with MSP problems is the same as for mcst problems, except that we now allow for asymmetric cost functions.

Lemma 3 *If the MSP problem (N, M, Q, c) is such that $|N| = |M|$ and $Q = D^n$, then (N, M, Q, c) is equivalent to a mca problem. In addition, all mca problems can be written as MSP problems.*

4.1.3 Steiner trees and other source connection problems

Steiner tree problems are identical to mcst problems, with the exception that we have public nodes not occupied by any agent. While it might seem that Steiner tree problems can be written as MSP problems, it cannot be, as we have examples of Steiner tree problems generating empty cores (Skorin-Kapov, 1995).

Consider a problem with 3 agents (1, 2 and 3), and 3 Steiner nodes (a, b and c). Costs are represented in Figure 2, with drawn edges having a cost of 1 and others having a cost of 10.

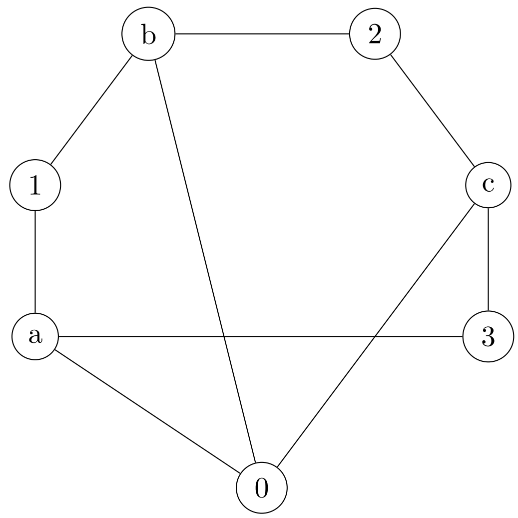

Figure 2: Steiner tree problem

In the Steiner tree problem, any pair of agents can connect at a total cost of 3 (with both connecting to the Steiner node they share a cheap connection to) while the three agents together connect at a total cost of 5 (with one pair connecting as before, and the third agent connecting through another Steiner node). The core of this game is empty.

In the corresponding MSP problem with the same graph, and with agent i demanding one unit in period i and none in other periods (we can view the Steiner nodes as being occupied by agents with null demands), the grand coalition can do better by putting a weight of 1/2 on the connection plan of each pair: build capacities of 1/2 on each of the represented edges, for a total cost of 4.5. The corresponding game is then balanced.

Thus, we can view Steiner tree problems as a MSP problem with the additional constraint that installed capacities must be 0 or 1. The additional constraint is what compromises the balancedness of the game. The presence of Steiner nodes is what can make non 0-1 capacities optimal, which explains why we can represent mcst problems as MSP problems.

As discussed previously, if we considered the extension of (private) MSP problems to cases with convex functions on every edge, we would obtain the network flow problems of Quant et al. (2006) as a special case with a single period.

4.2 Assignment and other matching problems

The next set of problems covered by MSP problems are assignment problems (Shapley and Shubik (1971)), in which we have two sets of agents and there is a value to match agents from different sets. In an assignment, an agent can be assigned to at most one agent from the other side, and the value created is pair-specific. An application is a market for houses. The well-studied model always has a non-empty core. We show in this section that all assignment problems can be written as a MSP problem. This result is much less obvious than the previous subsection, as it involves defining c in a precise way to reflect the pair-specific benefits of matching, before reinterpreting the cost game as a savings game.

There are many extensions of (2-sided) assignment games to m-sided assignment games, and we lose the certain non-emptiness of the core. We discuss two different extensions and by writing them as MSP problems, we obtain new sets of such problems with non-empty cores.

4.2.1 Classic assignment problems

An *assignment problem* is a tuple (N^1, N^2, v), where N^i is the set of agents on side i of the market and $v = (v_{ij})_{i\in N^1, j\in N^2}$ gives the value created by each pair of agents from different sides. We have that $N = \cup_i N^i$ and for all $S \subset N$, $S^i = S \cap N^i$,

An *eligible assignment* is a set of pairs $a \subset N^1 \times N^2$ such that if $(i,j) \in a$, there is no $k \neq j$ such that $(i,k) \in a$ or $l \neq i$ such that $(l,j) \in a$. We slightly abuse notation and use $k \notin a$ to denote that there is no $(i,j) \in a$ such that $k = i$ or $k = j$.

Let $\Omega(N^1, N^2)$ be the set of eligible assignments. To find the *optimal assignment* we need to find a^* such that

$$a^* \in \arg\max_{a\in\Omega(N^1,N^2)} \sum_{(i,j)\in a} v_{ij}.$$

Let $V^A(S) = \max_{a\in\Omega(S^1,S^2)} \sum_{(i,j)\in a} v_{ij}$ for all $S \subseteq N$.

As can be expected, the way to model assignment problems as MSP problems is to have two periods, with agents demanding in only one of the two periods.

Example 3 *We reconsider Example 1. Suppose that to* N *and* c *we add* $M = \{1,2\}$ *and* $Q = ((1,0),(1,0),(0,1),(0,1))$, *i.e. agents* 1 *and* 2 *are on one side of the market and* 3 *and* 4 *are on the other.*

Agents 1 and 3 can join forces by building the path $((1,3),(3,0))$ *at a cost of* 6; *a saving of* 3 *compared to both of them connecting directly to the source. In the same way, agents* 1 *and* 4 *would obtain a saving of* 4 *by building the path* $((4,1),(1,0))$, *agents* 2 *and* 3 *generate a saving of* 2 *by building the path* $((2,3),(3,0))$ *and agents* 2 *and* 4 *generate a saving of* 7 *by building the path* $((4,2),(2,0))$.

Thus, the optimal assignment is to match agent 1 *with agent* 3 *and agent* 2 *with agent* 4, *for total savings of* 10.

We show that all MSP problems of this form and such that the cost matrix satisfies the triangle inequality are assignment games.

Lemma 4 *Let $m = 2$, $Q \in \{0,1\}^{N\times M}$ and $q_{i1} + q_{i2} = 1$ for all $i \in N$ and let c satisfy the triangle inequality. The MSP problem (N, M, Q, c) can then be written as an assignment game.*

Proof. Let $N^1 \cup N^2 = N$ be such that $N^1 = N^1(Q)$ and $N^2 = N^2(Q)$. Note that by definition of Q, $N^1 \cap N^2 = \emptyset$ and recall that $V(S) = \sum_{i\in S} C(\{i\}) - C(S)$.

Let $v = (v_{ij})_{i\in N^1, j\in N^2}$ be such that $v_{ij} = V(\{i,j\})$. We show that $V(S) = \max_{a\in\Omega(S^1,S^2)} \sum_{(i,j)\in a} v_{ij}$ for all $S \subseteq N$.

It is easy to see that for all S such that $S \subseteq N^1$ or $S \subseteq N^2$, $C(S) = \sum_{i\in S} C(\{i\})$ and thus $V(S) = 0$.

Now let $S \subseteq N$ be such $S^1 \neq \emptyset$ and $S^2 \neq \emptyset$. As c satisfies the triangle inequality we only need to consider two types of paths for any $i \in S$: $((i,0))$ and $((i,k),(k,0))$ for $(i,k) \in S^1 \times S^2$ or $(k,i) \in S^1 \times S^2$. Clearly, if in the optimal network for S all agents send their flow over one path only, we can find an $a \in \Omega(S^1,S^2)$ such that $C(S,Q) = \sum_{(i,j)\in a} C(\{i,j\}) + \sum_{k\notin a} C(\{k\})$.

Now suppose that the optimal network z^f for S realizes a connection plan in which some agents send their flow over more than one path. Let f_i^k denote the flow agent i sends over path $((i,k),(k,0))$ and f_i^i the flow of agent i over path $((i,0))$. We omit t for simplicity, as for all $i \in N$ either $q_{i1} = 0$ or $q_{i2} = 0$. As c satisfies the triangle inequality, we can moreover assume that $\sum_{i\neq k} f_i^k \leq f_k^k \leq 1$ for all $i,k \in S$. This follows from the fact that it is only optimal for i to connect through k if it can use ks unused capacity. If the first inequality is violated, not all agents connecting through k can simultaneously use that unused capacity. Then

$$C(S) = \sum_{i,j\in S} c_{ij} \cdot z_{ij} = \sum_{\{i,j\};f_i^j>0} C(\{i,j\}) \cdot f_i^j + \sum_{k;\sum_{i\neq k} f_i^k < f_k^k} C(\{k\}) \cdot (f_k^k - \sum_{i\neq k} f_i^k).$$

It then follows that there must be a set of eligible assignments $\{a_1,...,a_n\} \subseteq \Omega(S^1,S^2)$ and a set of weights $(w_1,...,w_n) \in (0,1)^n$, $\sum_{i=1}^n w_i = 1$ s.t.

$$C(S) = \sum_{l=1}^{n} \sum_{(i,j)\in a_l} C(\{i,j\}) \cdot w_l + \sum_{k;\sum_{i\neq k} f_i^k < f_k^k} C(\{k\}) \cdot (f_k^k - \sum_{i\neq k} f_i^k).$$

Connection plan f is thus a convex combination of eligible assignments and therefore there must exist an $a_m \in \Omega(S^1,S^2)$ such that

$$\begin{aligned} C(S) &= \sum_{l=1}^{n} \sum_{(i,j)\in a_l} C(\{i,j\}) \cdot w_l + \sum_{k;\sum_{i\neq k} f_i^k < f_k^k} C(\{k\}) \cdot (f_k^k - \sum_{i\neq k} f_i^k) \\ &\geq \sum_{(i,j)\in a_m} C(\{i,j\}) + \sum_{\substack{k\in S \\ k\notin a_m}} C(\{k\}) \end{aligned}$$

We can therefore conclude that if a network is optimal, it must realize a connection plan in which for all $i \in S$, $f_i^k = 1$ for some $k \in S$ and $f_i^j = 0$ for all $j \in N\backslash\{k\}$. Thus

$$C(S) = \min_{a \in \Omega(S^1, S^2)} \sum_{(i,j) \in a} C(\{i,j\}) + \sum_{\substack{k \in S \\ k \notin a}} C(\{k\})$$

and by definition of $V(S)$ it follows that $V(S) = V^A(S)$.

■

The following example shows that it is necessary for our result that c satisfies the triangle inequality.

Example 4 *Let $N^1 = \{1, 2, 3\}$ and $N^2 = \{4, 5, 6\}$. The cost structure is as in the figure below, where all edges not drawn have a cost of 10.*

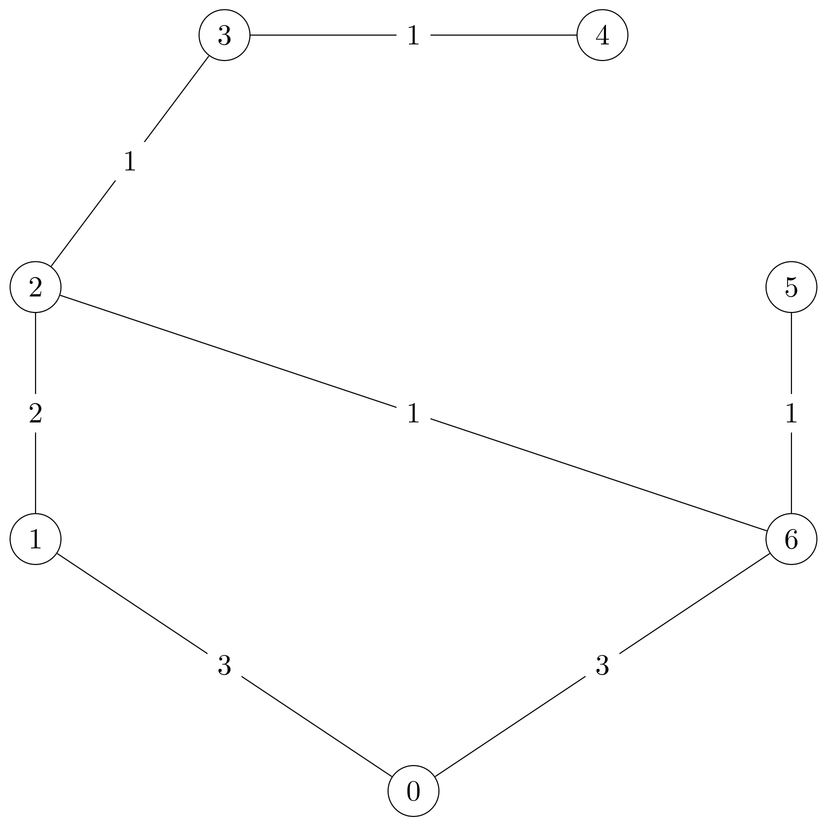

Figure 3: Example of a cost matrix violating the triangle inequality.

This cost matrix gives rise to the following cost savings for each pair:

	1	2	3
4	*2*	*4*	*5*
5	*0*	*3*	*3*
6	*0*	*3*	*3*

We can see that $V^A(N) = 8$. However, the cheapest network satisfying all demands has a total cost of 16, while the sum of the costs of the individual agents is 25, giving $V(N) = 25 - 16 = 9$. The difference between $V^A(N)$ and $V(N)$ arises due to the fact that in the grand coalition, agent 4 can both share edge $(3,2)$ with agent 3 and edge $(1,0)$ with agent 1.

We show that the full set of assignment problems can be written as MSP problems.

Lemma 5 *All assignment problems (N^1, N^2, v) can be written as a MSP problem.*

Proof. Let $N = N^1 \cup N^2$, $m = 2$ and Q such that $q_i = (1,0)$ if $i \in N^1$ and $q_i = (0,1)$ if $i \in N^2$.

Let $v^{\max} = \max_{i \in N^1, j \in N^2} v_{ij}$. Let $c_{i0} = 2v^{\max}$, $c_{ij} = c_{ji} = 2v^{\max} - v_{ij}$ if $i \in N^1$ and $j \in N^2$ and $c_{ij} = c_{ji} = 2v^{\max}$ otherwise.

It is easy to see that our cost matrix satisfies the triangle inequality: all edges have a cost of at least $v^{\max}$, and thus a path of 2 edges costs at least $2v^{\max}$. This means that the cost of a path of 2 edges is at least as large as the cost of a direct connection, which has a cost of at most $2v^{\max}$. Therefore, we have a limited number of paths to consider.

Every singleton $\{i\}$ connects through the path $((i,0))$ at a cost of $2v^{\max}$, and thus $C(\{i\}) = 2v^{\max}$.

Consider pairs $\{i,j\}$. If they belong to the same group N^1 or N^2, then the best they can do is both connect directly to the source, and $C(\{i,j\}) = 4v^{\max}$. If they don't belong to the same group, then they can share (part of) the same path. Since they are completely symmetric, we connect j to i and i to the source, at a cost of $4v^{\max} - v_{ij}$. Thus, $C(\{i,j\}) = 4v^{\max} - v_{ij}$.

By the previous lemma, if $|S| > 2$, then

$$\begin{aligned}
C(S) &= \min_{a \in \Omega(S^1,S^2)} \sum_{(i,j) \in a} C(\{i,j\}) + \sum_{\substack{k \in S \\ k \notin a}} C(\{i\}) \\
&= \min_{a \in \Omega(S^1,S^2)} \sum_{(i,j) \in a} (4v^{\max} - v_{ij}) + \sum_{\substack{k \in S \\ k \notin a}} 2v^{\max} \\
&= 2\,|S|\,v^{\max} - \max_{a \in \Omega(S^1,S^2)} \sum_{(i,j) \in a} v_{ij}.
\end{aligned}$$

We then obtain

$$\begin{aligned}
V(S) &= \sum_{i \in S} C(\{i\}) - C(S) \\
&= 2\,|S|\,v^{\max} - \left(2\,|S|\,v^{\max} - \max_{a \in \Omega(S^1,S^2)} \sum_{(i,j) \in a} v_{ij} \right) \\
&= \max_{a \in \Omega(S^1,S^2)} \sum_{(i,j) \in a} v_{ij} \\
&= V^A(S)
\end{aligned}$$

for all $S \subseteq N$. ■

4.2.2 m-sided assignment problems

Many extensions of the assignment problem to $m > 2$ sides have been proposed, and we focus here on the extensions proposed by Quint (1991) and Atay et al. (2016). To simplify notations, we focus here on 3-sided assignment problems, but all results can be extended to m-sided, for any positive integer m. Both extensions consider cases in which we have 3 groups, and value is created by triplets consisting of one member of each group. While Quint (1991) supposes that these are the only groups creating value, Atay et al. (2016) also allow for pairs to create value. We call the model proposed by Quint (1991) the *strict m-sided assignment problem* and the one by Atay et al. (2016) the *generalized m-sided assignment problem.*

Our set of agents N is now partitioned in three groups, N^1, N^2, N^3. For $k = 2, 3$, let $\mathcal{A}^k = \{S \subseteq N|\ |S| = k \text{ and } |S^l| \leq 1 \text{ for } l = 1, 2, 3\}$ be the set of groups containing k agents from different markets. Let $\mathcal{A}^{2,3} = \mathcal{A}^2 \cup \mathcal{A}^3$.

The strict 3-sided assignment problem is given by $((N^1, N^2, N^3), w)$, with $w \in \mathbb{R}_+^{\mathcal{A}^3}$ giving the value created by any triplet of agents from different groups. The generalized 3-sided assignment problem is given by $((N^1, N^2, N^3), w)$, with $w \in \mathbb{R}_+^{\mathcal{A}^{2,3}}$.

For the strict 3-sided assignment problem an eligible assignment is a set of triplets a such that if $S \in a$, there is no other S' in a such that $S' \cap S \neq \emptyset$. Let $\Omega(N^1, N^2, N^3)$ be the set of eligible assignments. For the generalized 3-sided assignment problem an eligible assignment is a set of pairs and triplets a such that if $S \in a$, there is no other S' in a such that $S' \cap S \neq \emptyset$. Let $\Omega^G(N^1, N^2, N^3)$ be the set of eligible assignments.

To find the optimal assignment for the strict 3-sided assignment problem we need to find a^* such that

$$a^* \in \arg \max_{a \in \Omega(N^1, N^2, N^3)} \sum_{S \in a} w_S$$

while for the generalized 3-sided assignment problem we are looking for a^* such that

$$a^* \in \arg \max_{a \in \Omega^G(N^1, N^2, N^3)} \sum_{S \in a} w_S.$$

Let $V^S(S) = \max_{a \in \Omega(S^1, S^2, S^3)} \sum_{T \in a} w_T$ and $V^G(S) = \max_{a \in \Omega^G(S^1, S^2, S^3)} \sum_{T \in a} w_T$ for all $S \subseteq N$.

To express these problems as MSP problems, a natural way to proceed is to extend what we've done for the classic assignment game, by now creating three groups of agents, with each group demanding a single unit in a different period. This leads us to generalized 3-sided problems.

Example 5 *We revisit once again Example 1. Suppose that to N and c we add $M = \{1, 2, 3\}$ and $Q = ((1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 1))$, i.e. $N^1 = \{1\}$, $N^2 = \{2\}$ and $N^3 = \{3, 4\}$. We now obtain a game in which there are benefits in combining pairs of agents from different groups, and triplets containing agents of different groups create even more savings. We obtain the same* *savings as above, in the classic assignment game, for coalitions* $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ *and* $\{2,4\}$. *In addition, coalition* $\{1,2\}$ *now creates a saving of* 4 *(using the path* $((2,1),(1,0))$ *instead of individual connections to the source). The triplets* $\{1,2,3\}$ *and* $\{1,2,4\}$ *also create savings. Since they consume in different periods, a single path of capacity* 1 *is sufficient, and for* $\{1,2,3\}$ *building the path* $((2,1),(1,3),(3,0))$ *gives rise to savings of* 7*. In the same manner, coalition* $\{1,2,4\}$ *generates savings of* 11 *with the path* $((4,2),(2,1),(1,0))$.

Strict 3-sided assignment problems are trickier to express as MSP problems, as pairs should not create any surplus. We do so in the following way. We still define three groups of agents, but now they all demand a single unit in all periods except one, with the inactive period varying depending on the group. A triplet composed of agents of different groups can now generate savings, requiring only a network with a capacity of 2 instead of 3.

Example 6 *We modify Example 5, keeping everything but the demand vector, which we change to* $Q = ((1,1,0),(1,0,1),(0,1,1),(0,1,1))$. *Notice that pairs do not create any savings: we still need to build a separate path for each agent. But triplets containing a member of each group can generate savings. Coalition* $\{1,2,3\}$ *can build a network with 2 units of capacity on* $(1,0)$ *and one unit each on* $(2,1)$ *and* $(3,1)$. *In each period only two agents are demanding and the network has enough capacity. The cost is* 15*, instead of* 16 *if they all connect directly to the source.*

Similarly, coalition $\{1,2,4\}$ *can build a network with one unit of capacity each on edges* $(1,0)$, $(2,0)$, $(4,1)$ *and* $(4,2)$. *This generates a saving of* 4 *compared to independent connections.*

There is nothing special with our examples, and we can always build $3-$sided assignment games through MSP problems in these manners. Since they are MSP problems, the resulting games are stable.

Lemma 6 *Suppose a MSP problem* (N,M,Q,c) *with* $m=3$ *and let* c *satisfy the triangle inequality. If we can partition* N *into* $\left(N^1,N^2,N^3\right)$ *such that:*

i) Q *is such that* $q_{it}=1$ *if* $i \in N^t$ *and* $q_{it}=0$ *otherwise, then* (N,M,Q,c) *can be written as a stable generalized* 3*-sided assignment problem.*

ii) Q *is such that* $q_{it}=0$ *if* $i \in N^t$ *and* $q_{it}=1$ *otherwise, then* (N,M,Q,c) *can be written as a stable strict* 3*-sided assignment problem.*

Quint (1991) and Atay et al. (2016) report that both the strict and the generalized 3-sided assignment problems can have an empty core, and they offer subsets that are always stable. We show through examples in what follows that the subset of 3-sided assignment games which is representable as MSP problems, and which therefore has a non-empty core, neither contains nor is contained by the sets they propose.

We start with the generalized 3-sided assignment problem (Atay et al. (2016)). Suppose that $\{i,j,k\}$ is a coalition of agents from different groups. They show that the game is stable if $w_{\{i,j,k\}} = w_{\{i,j\}} + w_{\{i,k\}} + w_{\{j,k\}}$ and if, when i and j are assigned to each other when everybody cooperates (possibly with a member of the third group), they stay assigned to each other if the third group stops cooperating.

In Example 5, we have that $7 = w_{\{1,2,3\}} \neq w_{\{1,2\}} + w_{\{1,3\}} + w_{\{2,3\}} = 4 + 3 + 2 = 9$. In MSP problems, we actually have that $w_{\{i,j,k\}} = w_{\{i,j\}} + w_{\{i,k\}} + w_{\{j,k\}} - \min\left\{w_{\{i,j\}}, w_{\{i,k\}}, w_{\{j,k\}}\right\}$, i.e. we sum up the two largest surpluses of the pairs. Thus, aside from when $\min\left\{w_{\{i,j\}}, w_{\{i,k\}}, w_{\{j,k\}}\right\} = 0$, the two sets of problems are disjoint.

For strict 3-sided assignment problems, the sufficient condition of Quint (1991) is as follows. Suppose first that $\left|N^1\right| = \left|N^2\right| = \left|N^3\right| = K$ and relabel agents such that agent $j - i$ is the i^{th} player in N^j. Suppose next that an optimal assignment is to assign, for $k = 1, ...K$, the k^{th} player of all groups to each other. We then need the following two conditions: 1) there exists d_{ij} for all i, j in different groups such that $w_{\{i,j,k\}} = d_{ij} + d_{ik} + d_{jk}$ for all i, j, k in distinct groups, and 2), for any $\alpha \in [0, 1]$, $d_{i-k,j-l} \leq \alpha\, d_{i-k,j-k} + (1 - \alpha) d_{i-l,j-l}$. The conditions are similar in spirit to those of Atay et al. (2016): condition 1) is the same except that now there are no known values for pairs. Condition 2) describes the strength of the pairwise values of matched agents, compared to unmatched agents.

As above, it is easy to see that some assignment problems satisfying these conditions will not be representable as a MSP problem. We need to build a bigger example to show that some strict assignments problems representable as a MSP problem do not satisfy these conditions.

Example 7 *We suppose that $n = 6$ and $m = 3$, with two agents in each groups. Agents 1 and 2 have demands $(0, 1, 1)$, agents 3 and 4 have demands $(1, 0, 1)$ and agents 5 and 6 have demands $(1, 1, 0)$. The cost structure is as in the figure below.*

We obtain the following savings for each triplet of different groups:

S	$V(S)$	S	$V(S)$
$\{1,3,5\}$	*2*	$\{2,3,5\}$	*2*
$\{1,3,6\}$	*3*	$\{2,3,6\}$	*4*
$\{1,4,5\}$	*2*	$\{2,4,5\}$	*2*
$\{1,4,6\}$	*1*	$\{2,4,6\}$	*0*

It can be shown that there are no sets of d_{ij} that satisfy condition 1. Thus, the example is a stable strict 3-sided assignment game that does not satisfy the conditions of Quint (1991).

We conclude this subsection by describing new sets of generalized and strict 3-sided assignment problems that are always stable, using their representability as MSP problems.

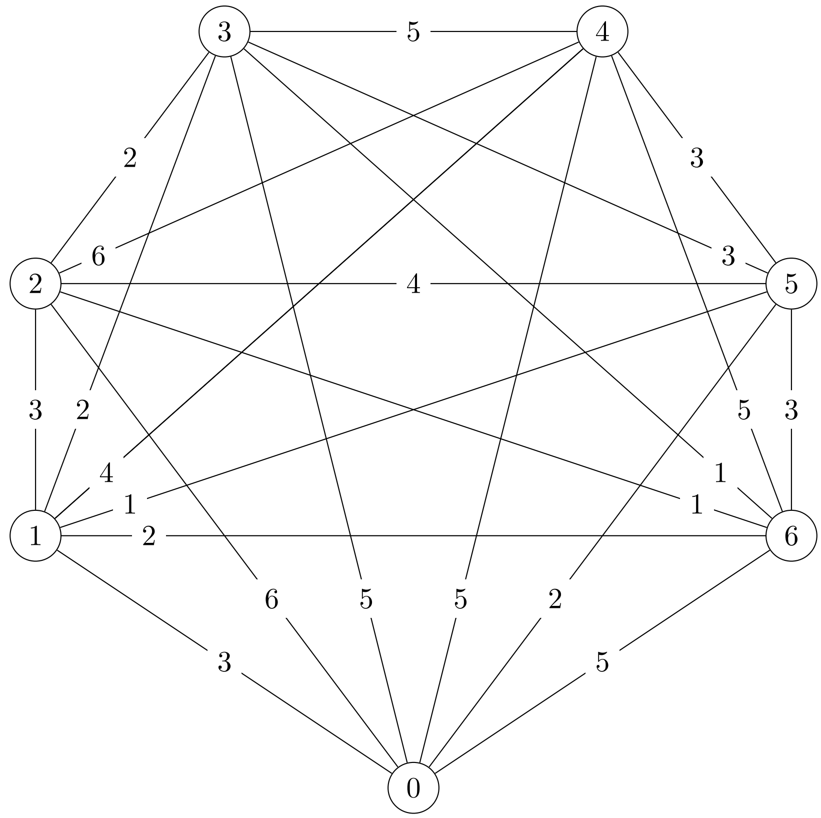

Figure 4: Example of a cost matrix with 6 agents.

Theorem 3 *i) Let* $\left(\left(N^1, N^2, N^3\right), w\right)$ *be a generalized* 3*-sided assignment problem such that* $w_{\{i,j,k\}} = w_{\{i,j\}} + w_{\{i,k\}} + w_{\{j,k\}} - \min\left\{w_{\{i,j\}}, w_{\{i,k\}}, w_{\{j,k\}}\right\}$ *for all* $i \in N^1, j \in N^2$ *and* $k \in N^3$*. Then* $Core(V^G)$ *is non-empty.*

ii) Let $\left(\left(N^1, N^2, N^3\right), w\right)$ *be a strict* 3*-sided assignment problem such that there exists a* d_{ij} *for all* i, j *in different groups such that* $w_{\{i,j,k\}} = d_{ij}+d_{ik}+d_{jk}-\min\{d_{ij}, d_{ik}, d_{jk}\}$ *for all* $i \in N^1, j \in N^2$ *and* $k \in N^3$*. Then* $Core(V^S)$ *is non-empty.*

Proof. For both cases, we build a MSP problem that represents the problem. Fix $\left(N^1, N^2, N^3\right)$ and let $N = N^1 \cup N^2 \cup N^3$. Let $m = 3$.

i) Fix w such that $w_{\{i,j,k\}} = w_{\{i,j\}}+w_{\{i,k\}}+w_{\{j,k\}}-\min\left\{w_{\{i,j\}}, w_{\{i,k\}}, w_{\{j,k\}}\right\}$ for all $i \in N^1, j \in N^2$ and $k \in N^3$.

Let Q be such that $q_{it} = 1$ if $i \in N^t$ and $q_{it} = 0$ otherwise.

Let $v^{\max} = \max_{i\in N^1, j\in N^2, k\in N^3} w_{\{i,j,k\}}$. Let $c_{i0} = 2v^{\max}$ for all $i \in N$, $c_{ij} = c_{ji} = 2v^{\max} - w_{\{i,j\}}$ if $i \in N^k$ and $j \in N^l$, with $k \neq l$ and $c_{ij} = c_{ji} = 2v^{\max}$ otherwise. It is easy to see that c is symmetric and satisfies the triangle inequality.

We discuss $C(S)$. If $|S| = 1$, then, $C(S) = 2v^{\max}$. If $S = \{i, j\}$ and $i, j \in N^k$, then $C(\{i, j\}) = C(\{i\}) + C(\{j\}) = 4v^{\max}$. If $S = \{i, j\}$ and $i \in N^k$ and $j \in N^l$ for $k \neq l$, then we can connect them both through the same path, say $((j,i),(i,0))$. The cost is $4v^{\max} - w_{\{i,j\}}$. As we cannot do better, we have $C(\{i,j\}) = 4v^{\max} - w_{\{i,j\}}$. It is then obvious that $V(\{i,j\}) = w_{\{i,j\}}$.

If $S = \{i,j,k\}$ and there is N^l such that $N^l \cap \{i,j,k\} = \emptyset$, then we have (at least) two agents from the same side. We cannot do better than assign one pair of agents from different sides to each other, if such exist. Otherwise, the three agents are from different sides. We can connect them through the same path, connecting one agent to the source and the two others through him. Then, the cost is

$$\begin{aligned} C(\{i,j,k\}) &= \min_{l \in \{i,j,k\}} \left\{ c_{l0} + \sum_{l' \in \{i,j,k\} \setminus \{l\}} c_{ll'} \right\} \\ &= \min_{l \in \{i,j,k\}} \left\{ 2v^{\max} + \sum_{l' \in \{i,j,k\} \setminus \{l\}} \left(2v^{\max} - w_{\{l,l'\}}\right) \right\} \\ &= 6v^{\max} - \max_{l \in \{i,j,k\}} \sum_{l' \in \{i,j,k\} \setminus \{l\}} w_{\{l,l'\}} \\ &= 6v^{\max} - w_{\{i,j\}} - w_{\{i,k\}} - w_{\{j,k\}} + \min \left\{ w_{\{i,j\}}, w_{\{i,k\}}, w_{\{j,k\}} \right\}. \end{aligned}$$

It is easy to see that any other structure (for example a path from k to j to i to 0) doesn't have a lower cost. We then have that $V(\{i,j,k\}) = w_{\{i,j\}} + w_{\{i,k\}} + w_{\{j,k\}} - \min\left\{w_{\{i,j\}}, w_{\{i,k\}}, w_{\{j,k\}}\right\} = w_{\{i,j,k\}}$.

For any S such that $|S| > 3$, we partition (connect independently to the source) the agents into triplets, pairs and singletons, summing up their cost. Thus, $C(S) = \min_{a \in \Omega^G(S^1,S^2,S^3)} \sum_{T \in a} C(T) + \sum_{\substack{i \in S \\ i \notin a}} C(\{i\})$.

Thus, for all $S \subseteq N$ we have $V(S) = V^G(S)$.

ii) Fix w such that there exists d_{ij} for all i,j in different groups such that $w_{\{i,j,k\}} = d_{ij} + d_{ik} + d_{jk} - \min\{d_{ij}, d_{ik}, d_{jk}\}$ for all $i \in N^1, j \in N^2$ and $k \in N^3$.

Let Q be such that $q_{it} = 0$ if $i \in N^t$ and $q_{it} = 1$ otherwise.

Let $v^{\max} = \max_{i \in N^1, j \in N^2, k \in N^3} w_{\{i,j,k\}}$. Let $c_{i0} = 2v^{\max}$ for all $i \in N$, $c_{ij} = c_{ji} = v^{\max} - d_{ij}$ if $i \in N^k$ and $j \in N^l$, with $k \neq l$ and $c_{ij} = c_{ji} = v^{\max}$ otherwise. It is easy to see that c is symmetric and satisfies the triangle inequality.

We discuss $C(S)$. If $|S| < 3$ or $|S| = 3$ and there is N^l such that $N^l \cap S = \emptyset$, then there is no gain to cooperation and $C(S) = 2|S|v^{\max}$. Thus, $V(S) = 0$.

Let $S = \{i,j,k\}$ with $i \in N^1$, $j \in N^2$ and $k \in N^3$. Then, we can connect them through 2 paths to the source instead of 3. We consider the structure where one agent is connected to the source (capacity of 2) and the 2 other agents directly to him. It dominates the structure where we build one path (which requires capacity of 2 on two of the three edges). The structure where 2 agents are connected to the source and the third to both of them is also feasible but not cheaper. We obtain:

$$\begin{aligned}
C(\{i,j,k\}) &= \min_{l\in\{i,j,k\}} \left\{ 2c_{l0} + \sum_{l'\in\{i,j,k\}\setminus\{l\}} c_{ll'} \right\} \\
&= \min_{l\in\{i,j,k\}} \left\{ 4v^{\max} + \sum_{l'\in\{i,j,k\}\setminus\{l\}} (v^{\max} - d_{ll'}) \right\} \\
&= 6v^{\max} - \max_{l\in\{i,j,k\}} \sum_{l'\in\{i,j,k\}\setminus\{l\}} d_{ll'} \\
&= 6v^{\max} - d_{ij} - d_{ik} - d_{jk} + \min\{d_{ij}, d_{ik}, d_{jk}\}.
\end{aligned}$$

We then have that $V(\{i,j,k\}) = d_{ij} + d_{ik} + d_{jk} - \min\{d_{ij}, d_{ik}, d_{jk}\} = w_{\{i,j,k\}}$.

For any S such that $|S| > 3$, we partition (connect independently to the source) the agents into triplets and singletons, summing up their cost. Thus, $C((S^1, S^2, S^3)) = \min_{a\in\Omega(S^1,S^2,S^3)} \sum_{T\in a} C(T) + \sum_{\substack{i\in S\\ i\notin a}} C(\{i\})$.

Thus, for all $S \subseteq N$ we have $V(S) = V^S(S)$. ■

However, these conditions are merely sufficient for representability as a MSP problem. In particular, for the problem in Example 7, there are no sets of d_{ij} that satisfy the condition of Theorem 3 ii).

4.3 Compatibility problems

The minimum coloring problem is a classic operations research problem in which we need to partition a group into elements that have no conflicts with each other - we can think of scheduling for instance. Conflicts, or incompatibilities, are expressed in a graph composed of undirected edges, that we express as $\{i,j\}$. We have that i and j are incompatible if the undirected edge $\{i,j\}$ belongs to the graph. Okamoto (2008) studies the problem in which we want to minimize the cost of providing services to all agents, with each element of the partition having the same cost. For example we pay k if we need to schedule k different time slots.

Formally, a *minimum coloring problem* is (N,G), where G is an undirected graph. For all $S \subset N$, let $G[S]$ be the subgraph induced by S. Let $\chi(G)$ be the *chromatic number of* G, i.e. the minimum number of elements in a partition of N so that if T is an element of the partition, $G[T] = \emptyset$. The coalitional cost function associated with the minimum coloring problem (N,G) is $C^M(S,G) = \chi(G[S])$.

For a graph G, $S \subseteq N$ is a *clique* of G if $\{i,j\} \in G$ for all $i,j \in S$. Let $\mathcal{W}(G)$ be the set of cliques. A clique is *maximal* if there is no other clique that contains it. Let $\bar{\mathcal{W}}(G)$ be the set of maximal cliques. Let $\omega(G) = \max_{S\in\mathcal{W}(G)} |S|$ be the size of the largest clique. Note that $\chi(G) \geq \omega(G)$. We say that a graph G is a *weakly perfect graph* if $\chi(G) = \omega(G)$. We say that a graph G is a *perfect graph* if $\chi(G[S]) = \omega(G[S])$ for all $S \subseteq N$.

Okamoto (2008) shows that the core of a minimum coloring problem is non-empty when G is a perfect graph. These can all be represented as MSP problems. For a graph G, we build Q^G as follows: order cliques in $\bar{\mathcal{W}}(G) = \left\{W_1, ..., W_{|\bar{\mathcal{W}}(G)|}\right\}$. For all $t = 1, ..., \left|\bar{\mathcal{W}}(G)\right|$ and all $i \in N$, let $q_{it}^G = 1$ if $i \in W_t$ and $q_{it}^G = 0$ otherwise.

Lemma 7 *Let (N, G) be a minimum coloring problem, with G a perfect graph. Let (N, M, Q^G, c) be such that $m = \left|\bar{\mathcal{W}}(G)\right|$, $c_{i0} = 1$ for all $i \in N$ and $c_{ij} = 0$ otherwise. Then, $C^M(\cdot, G) = C(\cdot, Q^G)$.*

Proof. First, notice that by definition of c, we have that for any connection plan f, $\gamma(z^f, c) = \sum_{i \in N} z_{i0}^f$, i.e. the cost is the sum of the capacities to the source.

Fix $S \subseteq N$. Let $\bar{\mathcal{W}}^+(G) = \left\{T \in \bar{\mathcal{W}}(G) \mid |T| = \omega(G)\right\}$ and let G^{S+} be such that $\{i, j\} \in G^{S+}$ if and only if $i, j \in R \in \bar{\mathcal{W}}^+(G[S])$. For any $G' \subset G$, we construct $\mathsf{Q}^{\mathsf{G}'}$ from Q^G as follows: for all $t = 1, ..., \left|\bar{\mathcal{W}}(G)\right|$, if there exists a $T \in \bar{\mathcal{W}}(G')$ such that $T \subseteq W_t$, we set $\mathsf{q}_{\mathsf{it}}^{\mathsf{G}'} = q_{it}^G$ for all $i \in T$. Otherwise $\mathsf{q}_{\mathsf{it}}^{\mathsf{G}'} = 0$.

We consider the problem $(N, M, \mathsf{Q}^{\mathsf{G}^{\mathsf{S}+}}, c)$ and in particular $C(S, \mathsf{Q}^{\mathsf{G}^{\mathsf{S}+}})$. Take any $R \in \bar{\mathcal{W}}^+(G[S])$. Then, $\{i, j\} \in G^{S+}$ for all $i, j \in R$ and by construction of $\mathsf{Q}^{\mathsf{G}^{\mathsf{S}+}}$, there exists $t \in M$ such that $\mathsf{q}_{\mathsf{it}}^{\mathsf{G}^{\mathsf{S}+}} = 1$ for all $i \in R$. To accommodate this demand in period t, we will need (at least) $|R| = \omega(G[S])$ units of capacity to the source, i.e. we need f such that $\sum_{i \in N} z_{i0}^f \geq |R| = \omega(G[S])$. Thus, we have $C(S, \mathsf{Q}^{\mathsf{G}^{\mathsf{S}+}}) \geq \omega(G[S]) = \chi(G[S])$, where the last equality comes from G being a perfect graph.

Next, observe that $\left(Q^G\right)^S \geq \mathsf{Q}^{\mathsf{G[S]}} \geq \mathsf{Q}^{\mathsf{G}^{\mathsf{S}+}}$. Thus, it follows that $C(S,\, Q^G) \geq C(S, \mathsf{Q}^{\mathsf{G}^{\mathsf{S}+}})$. Combining with the previous result, we obtain $C(S,\, Q^G) \geq \omega(G[S]) = \chi(G[S])$.

Let $\left\{S_1, ..., S_{\omega(G[S])}\right\}$ be a partition of S that solves the minimum coloring problem for S. By definition of Q^G, if $i, j \in S_r$ for some $r = 1, ..., \omega(G[S])$, then $\{i, j\} \notin G$ and there is no $t \in M$ such that $q_{it}^G = q_{jt}^G = 1$. Thus, consider the connection plan f^S such that for all r, we randomly pick an agent i_r in S_r, and build a capacity of 1 from i_r to the source as well as from all other members of S_r to i_r. This is a feasible connection plan, and it is obvious that $\gamma(z^{f^S}, c) = \sum_{i \in N} z_{i0}^{f^S} = \omega(G[S])$. Thus, $C(S, Q^G) \leq \omega(G[S])$.

Combining, we obtain that $C(S, Q^G) = \omega(G[S]) = C^M(S, G)$ for all $S \subseteq N$. ■

We can provide an additional result: minimum coloring problems for which the graph is weakly perfect still have non-empty cores.

Lemma 8 *Let (N, G) be a minimum coloring problem, with G a weakly perfect graph. Then, $Core(C^M)$ is non-empty.*[2]

[2]The result can be obtained without MSP representation, by identifying a clique S with the largest size, and

Proof. Let (N, M, Q^G, c) be as above and consider $C(N, Q^G)$. Since G is a weakly perfect graph, $\chi(G) = \omega(G)$, and by the same logic as in the previous lemma, $C(N, Q^G) = \omega(G) = C^M(N, G)$.

Fix $S \subset N$ and let $\{S_1, ..., S_{\chi(G[S])}\}$ be a partition of S that solves the minimum coloring problem for S. By definition of Q^G, if $i, j \in S_r$ for some $r = 1, ..., \chi(G[S])$, then $\{i, j\} \notin G$ and there is no $t \in M$ such that $q^G_{it} = q^G_{jt} = 1$. Thus, we can build a path of capacity 1 connecting all members of S_r to each other, with one agent connected to the source. By the definition of c, the cost of such a path is 1. Thus, $C(S, Q^G) \leq \chi(G[S]) = C^M(S, G)$.

Since (N, M, Q^G, c) is a MSP problem, $Core(C)$ is non-empty. Since $C(S, Q^G) \leq C^M(S, G)$ for all $S \subset N$ and $C(N, Q^G) = C^M(N, G)$, $Core(C) \subseteq Core(C^M)$. Thus, $Core(C^M)$ is non-empty. ■

A particular subset of minimum coloring problems is the set of job scheduling problems (Bahel and Trudeau (2019)). In those problems, each agent has a single job that has a starting and finishing time, and must be processed on a machine without interruption from the starting to the finishing time. Evidently, a group of agents is incompatible if their jobs intersect. This gives a lot of structure to the incompatibility graph, which will always be a perfect graph. Thus the core is always non-empty. We can thus represent those problems as MSP problems. To make the representation even closer to the original problem, we can reorder the periods in M such that for any $i \in N$, if $q_{ir} = q_{it} = 1$, then $q_{is} = 1$ for all $r < s < t$.

5 Concluding remarks

We have shown that MSP problems do not just generate balanced games, but in fact totally balanced games. The total balancedness of MSP games extends to many, but not all games that we have discussed. For total balancedness to extend, we need that the subgames C_S remain games of the same type as the original game. This is not necessarily true for source connection problems. For instance, the subgame of a public mcst problem is a Steiner tree problem, which can have an empty core, while the subgame of its MSP representation has a non-empty core, but does not correspond to a mcst problem. It is in fact known that the public mcst problem does not always generate totally balanced games (Norde et al., 2001).

For assignment (2 or m-sided) and minimum coloring problems, the subset of games that are representable as MSP problems are not only balanced, but also totally balanced. It is an open question if all totally balanced m-sided assignment games are representable as a MSP problem. Deng et al. (2000) show that a minimum coloring problem is totally balanced if and only if its

building a graph that contains only edges between members of S. This graph is obviously perfect, and thus it has core allocations. Since the grand coalition has the same cost of $|S|$ as in the original graph, with all other coalitions having no larger cost than in the original graph, these core allocations are also in the core of the original game.

We present it with the MSP representation to show how it can be used in various contexts to extend stability results.

graph is perfect.

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