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Characterization of TU games with stable cores by nested balancedness*

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Abstract

A balanced transferable utility game (N, v) has a stable core if its core is externally stable, that is, if each imputation that is not in the core is dominated by some core element. Given two payoff allocations x and y, we say that x *outvotes* y via some coalition S of a feasible set if x dominates y via S and x allocates at least $v(T)$ to any feasible T that is not contained in S. It turns out that outvoting is transitive and the set M of maximal elements with respect to outvoting coincides with the core if and only if the game has a stable core. By applying the duality theorem of linear programming twice, it is shown that M coincides with the core if and only if a certain nested balancedness condition holds. Thus, it can be checked in finitely many steps whether a balanced game has a stable core. We say that the game has a *super-stable* core if each payoff vector that allocates less than $v(S)$ to some coalition S is dominated by some core element and prove that core super-stability is equivalent to vital extendability, requiring that each vital coalition is extendable.

Keywords: Domination, stable set, core, TU game

JEL Classification: C71

1 Introduction

In their seminal work on game theory, von Neumann and Morgenstern (1953) introduced the notion of stable sets as the main and most natural solution concept for cooperative games. Stable sets are based on the notion of dominance: A payoff vector x dominates another one y if there is a coalition S for which x is strictly better than y for every member of S, but still remains affordable for them in the game under consideration. A set X of payoff vectors is said to be stable if no vector in X dominates another one of X (internal stability), and every payoff vector outside X is dominated by a vector of X (external stability).

Despite the attractive character of this definition, which has been disseminated to other domains like decision theory, the notion of stable set revealed to be of difficult use. Stable sets are in general not

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unique, there may even be uncountably many of them, and as shown by Lucas (1968), there exist games with no stable sets. In addition, they are very difficult to find and there is no known algorithm to find them. Deng and Papadimitriou (1994) remarked that the problem of the existence of a stable set for a given game is undecidable. These unpleasant features made Aumann write in 1987: *"Finding stable sets involves a new tour de force of mathematical reasoning for each game or class of games that is considered. Other than a small number of elementary truisms (e.g., that the core is contained in every stable set) there is no theory, no tools, certainly no algorithms."* This is why the main solution concept of cooperative game theory turned to be the core.

There is, however, a close relationship between the core and stable sets. As mentioned above by Aumann, the core is contained in every stable set, and moreover, if the core is itself stable, then it becomes the unique stable set. Consequently, games with a stable core appear to be especially attractive, as they have a solution possessing the appealing properties of stable sets without having its drawbacks. The central question is then: *For which games is the core a stable set?* Are there simple sufficient conditions for core stability? Are there simple necessary conditions? The literature gives some answers to these questions, but not in full generality. An important notion here is the notion of extendability, introduced by Kikuta and Shapley (1986). A game v is extendable if, for every coalition S, every core element of the subgame $v_{|S}$ (the restriction of v to the coalitions in S) can be extended to a core element of v. It turns out that extendable games have a stable core. Extendability is related to other well-known concepts in cooperative game theory. For example, subconvexity introduced by Sharkey (1982), which is weaker than convexity, implies that the core is large, which in turn implies extendability, as proved by van Gellekom, Potters, and Reijnierse (1999). This permits to conclude that some well-known classes of games (convex, with a large core) have a stable core. However, extendability is far from being necessary, as shown by Shellshear and Sudhölter (2009). Indeed, it is enough to restrict extendability to those coalitions S which are strongly vital-exact, which the authors call *vital-exact extendability*. It turns out that vital-exact extendability is a sufficient and necessary condition for core stability for some classes of games, like matching and assignment games, simple flow games, and minimum coloring games, but fails to be necessary in general. Moreover, it is not known whether a further weakening of extendability could lead to a necessary and sufficient condition of core stability. On the other side, very few necessary conditions for core stability are known. It has been remarked by Gillies (1959) that core stability implies that all singletons are exact.

In 1971 T. E. Kulakovskaya published a note, giving a sufficient and necessary condition for core stability. This note was a short summary in English from her PhD dissertation in Russian from Leningrad State University (finished 1973). Unfortunately, T. E. Kulakovskaya died in 1996, and her work remained almost unnoticed. The second author of the present paper found a simple counterexample that disproves the published result. However, after inquiry, through the kind help of Natalia Naumova, he could obtain an English translation by Professor Joseph Romanovsky of an extract of around 35 pages of the thesis. We started our investigation from her ideas and were able to build a new sufficient and necessary condition for core stability, which is the main result of the present paper.

The paper is organized as follows. Section 2 introduces the main definitions and sets the framework. Section 3 generalizes the notion of balancedness, while Section 4 introduces a strengthening of the dominance relation, called the outvoting relation. We investigate in Section 5 the set of maximal elements w.r.t. the outvoting relation, and conclude from its properties that it does not constitute a valuable solution concept. Section 6 gives the general scheme of the construction as well as preparatory results, while the main result is presented in Section 7. In the next section, we show that vital extendability characterizes a stronger form of stability of the core that we call super-stability. Finally, Section 9 is a discussion, giving additional results related to the complexity of using the condition of core stability.

2 Preliminaries

A (cooperative TU) *game* is a pair (N, v) such that $N \neq \emptyset$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. We often identify (N, v) with its coalition function v. For $S \subseteq N$, we denote by $\mathbb{R}^S$ the $|S|$-dimensional Euclidean space of real functions on S. For $x, y \in \mathbb{R}^S$ we write $x \geqslant y$ if $x_i \geqslant y_i$ for all $i \in S$. Moreover we use $x > y$ for $x \geqslant y$ and $x \neq y$, and we write $x \gg y$ if $x_i > y_i$ for all $i \in S$. Let $X(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ and $I(v) = \{x \in X(v) \mid x_i \geqslant v(\{i\})$ for all $i \in N\}$ where $x(S) = \sum_{i \in S} x_i$ for each $x \in \mathbb{R}^N$ and $S \subseteq N$, i.e., $X(v)$ is the set of *Pareto optimal allocations* (*preimputations*) and $I(v)$ is the set of *imputations* (*individually rational* preimputations). We also use $x_S = (x_i)_{i \in S}$ and denote, if $S \neq \emptyset$, the subgame of (N, v) restricted to the subsets of S, slightly abusing notation, by $\left(S, v_{|S}\right)$. The *core* of v is the set $C(v) = \{x \in X(v) \mid x(S) \geqslant v(S)$ for all $S \subseteq N\}$. A *coalition* in N is a nonempty subset of N. Let $x, y \in \mathbb{R}^N$ and S be a coalition in N. We say that x *dominates* y *via* S (at v), written $x \operatorname{dom}_S y$, if $x_S \gg y_S$ and $x(S) \leqslant v(S)$. We say that x *dominates* y ($x \operatorname{dom} y$) if $x \operatorname{dom}_S y$ for some coalition S. We say that $X \subseteq \mathbb{R}^N$ is *internally stable* (at v) if $x \in X$, $y \in \mathbb{R}^N$, $x \operatorname{dom} y$ imply $y \notin X$. Moreover, X is *externally stable* (at v) if for all $y \in I(v) \setminus X$ there exists $x \in X$ such that $x \operatorname{dom} y$. Finally, a set X is *stable* if it is internally and externally stable.

Note that $C(v)$ is internally stable and that each externally stable set contains $C(v)$. If $I(v) = \emptyset$, then $\emptyset = C(v)$ is a stable set. Hence we shall not further consider the case that $\sum_{i \in N} v(\{i\}) > v(N)$.

The following remark formulates Proposition 2.1 and Corollary 2.1 of Shellshear and Sudhölter (2009) (see also Gillies 1959).

Remark 2.1. Let (N, v) be a game with a stable core such that $I(v) \neq \emptyset$.

(1) Then, for all $i \in N$, there exists $x \in C(v)$ such that $x_i = v(\{i\})$.

(2) Hence, each preimputation that is not an imputation is dominated (via some singleton) by a core element.

A coalition S in N is *vital* (at $v_{|S}$) if there exists $x_S \in C(v_{|S})$ such that $x(T) > v(T)$ for all $T \in 2^S \setminus \{S, \emptyset\}$, i.e., if the dimension of $C(v_{|S})$ is full ($|S| - 1$); S is *exact* (at v) it there exists $y \in C(v)$ such that $y(S) = v(S)$, and it is *strictly vital-exact* (at v) if S is vital and there exists $x \in C(v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Let $\mathcal{E}(v)$ and $\mathcal{VE}^{\text{strict}}(v)$ denote the set of exact and strictly vital-exact proper coalitions of N, respectively (i.e., $\mathcal{E}(v) = \{S \in 2^N \setminus \{\emptyset, N\} \mid S \text{ is exact at } v\}, \mathcal{VE}^{\text{strict}}(v) = \{S \in 2^N \setminus \{\emptyset, N\} \mid S \text{ is strictly vital-exact at } v\}$).

For any game (N, v) let $\mathcal{F}(v) = \mathcal{F}$ be a set of coalitions such that

$$\mathcal{F} \subseteq \mathcal{E}(v), \tag{2.1}$$

$$\mathcal{F} \supseteq \mathcal{VE}^{\text{strict}}(v), \text{ and} \tag{2.2}$$

$$C(v) = \{x \in X(v) \mid x(S) \geqslant v(S) \forall S \in \mathcal{F}\}. \tag{2.3}$$

Remark 2.2. Let (N, v) be a game with a stable core such that $I(v) \neq \emptyset$. Then $\mathcal{VE}^{\text{strict}}(v)$ satisfies (2.3), hence it is necessary for core stability that the strictly vital-exact coalitions determine the core (i.e., that $\mathcal{VE}^{\text{strict}}(v)$ satisfies (2.3)). Indeed, let $x \in X(v) \setminus C(v)$. It remains to find a strictly vital-exact coalition S with $x(S) < v(S)$. To this end choose a core element y that dominates x (its existence is guaranteed because the core is stable) and a minimal coalition S with respect to (w.r.t.) inclusion such that y dominates x via S. Then $y(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$ so that S is strictly vital-exact and $x(S) < v(S)$.

Let $C(v) \neq \emptyset$. Then the set of exact coalitions, $\mathcal{F}(v) = \mathcal{E}(v)$, satisfies (2.1), (2.2), and (2.3). We provide another interesting example of an often much smaller set $\mathcal{F}(v)$ that also satisfies (2.1), (2.2), and (2.3).

Say that a coalition S of N is *strongly vital-exact* at v, if S is vital and there exists $x \in C(v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{S, \emptyset\}$ for which there exists $y \in C(v)$ with $y(T) > v(T)$. Let $\mathcal{VE}^{\text{strong}}$ denote the set of strongly vital-exact proper coalitions of (N, v), i.e., $\mathcal{VE}^{\text{strong}}(v) = \{S \in 2^N \setminus \{\emptyset, N\} \mid S \text{ is strongly vital-exact at } v\}$.

Example 2.3. Let (N, v) be a game such that $C(v) \neq \emptyset$. Then $\mathcal{VE}^{\text{strong}}(v)$ satisfies (2.1), (2.2), and (2.3) by Lemma 3.7 of Shellshear and Sudhölter (2009).

3 Balancedness

Let N be a finite nonempty set and S be a coalition in N. The vector $\mathbb{1}^S \in \mathbb{R}^N$ denotes the *indicator* vector of S, i.e., $\mathbb{1}^S_j = \begin{cases} 1, \text{ if } j \in S, \\ 0, \text{ if } j \in N \setminus S. \end{cases}$

Let $Z \subseteq \mathbb{R}^N_+ \setminus \{0\}$ be a finite set. Define $F(Z) = \{\delta \in \mathbb{R}^Z_+ \mid \sum_{z \in Z} \delta_z z = \mathbb{1}^N\}$. Then $F(Z)$ is a convex polyhedral set. By our assumption $z > 0$ for all $z \in Z$, $F(Z)$ is bounded, hence the convex hull of its extreme points (by the Krein-Milman theorem). Moreover, $\delta = (\delta_z)_{z \in Z} \in F(Z)$ is a vertex of $F(Z)$ if and only if supp $\delta = \{z \in Z \mid \delta_z > 0\}$ is linearly independent. We say that $Z' \subseteq Z$ is *balanced* (in Z) if there is a system $(\delta_z)_{z \in Z'}$ of positive weights (called *balancing weights*) such that $\sum_{z \in Z'} \delta_z z = \mathbb{1}^N$. A balanced set is minimal if and only if its system of balancing weights is unique. Hence, the vertices in $F(Z)$ correspond to the minimal balanced sets. Each minimal balanced set is the support of a unique vertex and vice versa, an element of $F(Z)$ is a vertex if its support is a minimal balanced set in Z.

Recall that a collection $\mathcal{B}$ of coalitions (i.e., $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$) is called *balanced* (in N) if $\{\mathbb{1}^S \mid S \in \mathcal{B}\}$

is balanced. If $\mathcal{B}$ is a minimal balanced collection of coalitions, then we denote the unique system of balancing weights by $\left(\lambda_S^{\mathcal{B}}\right)_{S\in\mathcal{B}}$ in this case.

Remark 3.1. Let (N, v) be a game.

(1) $C(v) \neq \emptyset$ if and only if (N, v) is balanced, i.e., $v(N) \geqslant \sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S)$ for all minimal balanced collections $\mathcal{B}$ in N (Bondareva 1963, Shapley 1967). The result is "sharp" in the sense that, for each minimal balanced collection $\mathcal{B} \neq \{N\}$ in N there exists a game (N, v), which exclusively violates the foregoing inequality for this $\mathcal{B}$.

(2) Let (N, v) be balanced and let $\mathcal{F} \subseteq 2^N \setminus \{N, \emptyset\}$ satisfy (2.3). If $|N| \geqslant 2$, then $\mathcal{F}$ is balanced and the linear span of $\{\mathbb{1}^S \mid S \in \mathcal{F}\}$ is $\mathbb{R}^N$.

This statement may be deduced from Corollary 6 of Derks and Reijnierse (1998), but for the sake of completeness, we recall the proof. Let $x \in C(v)$ and $y \in \mathbb{R}^N$ satisfy $y(N) = 0$ and $y(S) \geqslant 0$ for all $S \in \mathcal{F}$. Then $x + ty \in C(v)$ for all $t \geqslant 0$ so that $y = 0 \in \mathbb{R}^N$ by compactness of $C(v)$. As $|N| \geqslant 2$, we also conclude that $\mathcal{F} \neq \emptyset$. Hence, the linear programming problem "max $\sum_{S\in\mathcal{F}} y(S)$ subject to $y(S) \geqslant 0$ for all $S \in \mathcal{F}$ and $y(N) = 0$" is feasible and its value is 0. By the duality theorem of linear programming (Franklin 1980, p. 62) there exist $\beta_S \geqslant 0$, $S \in \mathcal{F}$, and $\beta_N \in \mathbb{R}$ such that $\beta_N \mathbb{1}^N = \sum_{S\in\mathcal{F}} (1 + \beta_S)\mathbb{1}^S$. Hence, $\beta_N > 0$ and $\mathcal{F}$ is balanced. We conclude that y is determined by $y \in \mathbb{R}^N$ and $y(S) = 0$ for all $S \in \mathcal{F}$. Therefore, $\{\mathbb{1}^S \mid S \in \mathcal{F}\}$ contains a vector space basis of $\mathbb{R}^N$.

4 The Outvoting Relation

Let (N, v) be a game and let $\mathcal{F} \subseteq 2^N \setminus \{N, \emptyset\}$. We now define a sub-relation of domination that is transitive.

Let $x, y \in \mathbb{R}^N$ and $P \in \mathcal{F}$. We say that x *outvotes* y via P (in $\mathcal{F}$), written $x \succ_P y$, if $x \operatorname{dom}_P y$ (i.e., $x_P \gg y_P$ and $x(P) \leqslant v(P)$), and $x(S) \geqslant v(S)$ for all $S \in \mathcal{F} \setminus 2^P$. Also, we say that x *outvotes* y ($x \succ y$) if there is $P \in \mathcal{F}$ such that $x \succ_P y$.

Lemma 4.1. *Let (N, v) be a game, $x, y, z \in \mathbb{R}^N$, and $P, Q \in \mathcal{F}$. If $x \succ_P y \succ_Q z$, then $P \subseteq Q$ and $x \succ_P z$.*

Proof. As $y(S) \geqslant v(S)$ for all $S \in \mathcal{F}\setminus 2^Q$ and $y(P) < x(P) \leqslant v(P)$, $P \subseteq Q$. We conclude that $x \succ_P z$. □

Note that by Lemma 4.1 the binary relation $\succ$ is transitive.

Let $M(v) = \{x \in X(v) \mid y \not\succ x \forall y \in X(v)\}$. Hence, $M(v)$ is the set of preimputations that are maximal w.r.t. outvoting.

Lemma 4.2. *Let (N, v) be a balanced game and assume that $\mathcal{F}$ satisfies* (2.3). *For all $x \in X(v) \setminus M(v)$ there exists $y \in M(v)$ such that $y \succ x$.*

Proof. We may assume $|N| \geqslant 2$ (for $|N| = 1$, $M(v) = X(v)$). For $Q \in \mathcal{F}$ let $Y(Q) = \{y \in X(v) \mid y \succ_Q x\}$. As $x \notin M(v)$, there exists $Q \in \mathcal{F}$ such that $Y(Q) \neq \emptyset$. Now, among those coalitions $Q \in \mathcal{F}$ satisfying $Y(Q) \neq \emptyset$, let P be a minimal (w.r.t. inclusion) one, and choose $y \in Y(P)$. Define $X = \{z \in X(v) \mid z_P \geqslant y_P, z(P) \leqslant v(P), z(S) \geqslant v(S) \text{ for all } S \in \mathcal{F} \setminus 2^P\}$. Defined by weak inequalities, X is a closed polyhedral set. As $z_P \geqslant y_P$ implies $z(R) \geqslant y(R)$ for all $R \subseteq P$,

$$X \subseteq \{z \in \mathbb{R}^N \mid z(N) = v(N), z(R) \geqslant y(R) \text{ for all } R \in \mathcal{F} \cap 2^P, z(T) \geqslant v(T) \text{ for all } T \in \mathcal{F} \setminus 2^P\}.$$

A careful inspection of Remark 3.1 (2) shows that X does neither contain lines nor rays, hence it is compact. As $y \in X$, X is nonempty. Hence, $t = \max\{z(P) \mid z \in X\}$ exists. Choose $z \in X$ such that $z(P) = t$. As $z \succ_P x$, it suffices to prove that $z \in M(v)$. For this purpose, assume, on the contrary, that there exists $z' \in X(v)$ such that $z' \succ_Q z$ for some $Q \in \mathcal{F}$. As $z \succ_P x$, by Lemma 4.1, we have $Q \subseteq P$ and $z' \succ_Q x$. By definition of P, we must have $Q = P$, implying that $z' \in X$, $z'_P \gg z_P$, and $z'_P \leqslant v(P)$ which is impossible by choice of z. □

Lemma 4.3. *Let $x \in C(v)$, $y \in X(v)$, and $\mathcal{F}$ satisfy* (2.2). *Then x dom y if and only if $x \succ y$.*

Proof. If $x \succ_P y$ for some $P \in \mathcal{F}$, then $x \operatorname{dom}_P y$. Conversely, if x dom y, then let S be a minimal (w.r.t. inclusion) coalition such that $x \operatorname{dom}_S y$. Then $S \neq N$ because $y(N) = v(N)$. By minimality of S, $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. As $x \in C(v)$, $x(S) = v(S)$. Hence, S is strictly vital-exact so that $S \in \mathcal{F}$ by (2.2). Thus, $x \succ_S y$. □

Proposition 4.4. *Let (N, v) be a balanced game and let $\mathcal{F}$ satisfy* (2.2) *and* (2.3). *Then $M(v) = C(v)$ if and only if (N, v) has a stable core.*

Proof. As $C(v) \subseteq M(v)$ is true in general, in view of Remark 2.1 (2), the if-part follows from Lemma 4.3. Moreover, Lemma 4.2 and Lemma 4.3 imply the only-if-part. □

5 The Solution $M(\cdot)$ of Maximal Elements

The elements of $M(v)$ being maximal for the outvoting relation, we may consider them as the "best" payoff vectors, so that one could think of $M(v)$ as a solution of the game (N, v). In what follows we investigate some properties of the solution $M(\cdot)$.

Let U be a set with $U \supseteq \{1, 2, 3, 4\}$ and Γ^b be the set of balanced games (N, v) such that $N \subseteq U$. For regarding $M(\cdot)$, assigning $M(v)$ to each game $(N, v) \in \Gamma^b$, as a solution on Γ^b_N, the subset of balanced games with player set N, we assume that $\mathcal{F}$ does not depend on the considered game, say, $\mathcal{F} = 2^N \setminus \{N, \emptyset\}$. Hence $\mathcal{F}$ satisfies (2.2) and (2.3) for each $(N, v) \in \Gamma^b_N$.

Note that, by its definition, $M(\cdot)$ satisfies some traditional properties like anonymity, covariance under strategic equivalence, and Pareto optimality. It also satisfies nonemptiness by Lemma 4.2. Finally, it satisfies the following weak dummy properties: Let $(N, u), (N \cup \{i\}, v) \in \Gamma^b$ with $i \in U \setminus N$ such that $u(S) = v(S)$ and $v(S \cup \{i\}) = u(S) + v(\{i\})$ for all $S \subseteq N$.

(1) Then, for each $y \in \mathbb{R}^{N \cup \{i\}}$ with $y_N \in M(u)$ and $y_i = v(\{i\})$, $y \in M(v)$. Indeed, assume, on the contrary, that there exists $z \in X(v)$ such that $z \succ_S^v y$ for some $\emptyset \neq S \subsetneqq N \cup \{i\}$. If $i \in S$, then $z_i > y_i = v(\{i\})$, and if $i \notin S$, then $z_i \geqslant v(\{i\})$ because $\{i\}$ is supposed to be feasible for v. Hence, $S \neq \{i\}$ and, as $z_i \geqslant y_i$ and $z(N \cup \{i\}) = y(N \cup \{i\})$, $S \neq N$. Therefore, there exists $z' \in X(u)$ with $z' \geqslant z_N$ and $z'_{S \setminus \{i\}} = z_{S \setminus \{i\}}$ so that $z' \succ^u_{S \setminus \{i\}} y_N$ contradicting $y_N \in M(u)$.

(2) Then, for each $y \in \mathbb{R}^{N \cup \{i\}}$ with $y \in M(v)$ and $y_i = v(\{i\})$, $y_N \in M(u)$. Indeed, if $z \succ_S^u y_N$, then $z' \gg_S^v y$, where $z'_i = v(\{i\}), z'_N = z$.

(3) If $y \in M(v)$, then $y_i \geqslant v(\{i\})$. Indeed, each $x \in X(v)$ satisfying $x_i < v(\{i\})$ is outvoted via $\{i\}$ by any core element of v.

(4) $M(v)$ nevertheless does not satisfy the dummy (null player) property as shown by an example at the end of this section.

We now compute $M(v)$ for all balanced symmetric 3-person games (N, v). By anonymity and covariance we may assume that $N = \{1, 2, 3\}$ and v is 0-normalized (i.e., $v(\{i\}) = 0$ for all $i \in N$). Now, if $v(N) = 0$, then $v(S) = 0$ for all $S \in \mathcal{F}$ because $C(v) \neq \emptyset$. Hence (N, v) is convex so that $\mathcal{M}(v) = C(v) = \{(0, 0, 0)\}$. Hence, by covariance we may assume that $v(N) = 1$. As $C(v) \neq \emptyset$, the symmetry of the game implies that $v(S) = \alpha$ for all $S \in \mathcal{F}$ with $|S| = 2$ for some $\alpha \leqslant 2/3$. If $\alpha \leqslant 0$, then $C(v) = M(v) = \Delta(N) := \{x \in \mathbb{R}^N_+ \mid x(N) = 1\}$ by convexity of v. If $0 < \alpha < 1/2$, then, again by convexity of v, $C(v) = M(v)$ is the convex hull of

$$\{(0, \alpha, 1-\alpha), (0, 1-\alpha, \alpha), (\alpha, 0, 1-\alpha), (1-\alpha, 0, \alpha), (\alpha, 1-\alpha, 0), (1-\alpha, \alpha, 0)\}.$$

For $\alpha = 1/2$, the game is still convex so that $C(v) = M(v)$ is the convex hull of

$$\{(0, 1/2, 1/2), (1/2, 0, 1/2), (1/2, 1/2, 0)\}.$$

If v is not convex (but balanced), then it has no stable core so that $C(v) \subsetneqq M(v)$.

To analyze the remaining cases, we write $v = v^\alpha$ to emphasize that we consider a one-parameter set of games.

(1) $1/2 < \alpha < 2/3$ (see Figure 5.1): Then $C(v^\alpha)$ is the convex hull of

$$\{(1-\alpha, 1-\alpha, 2\alpha-1), (1-\alpha, 2\alpha-1, 1-\alpha), (2\alpha-1, 1-\alpha, 1-\alpha)\}.$$

Let $X = \{x \in \mathbb{R}^N \mid x(N) = 1, |\{i \in N \mid x_i \geqslant 1-\alpha\}| \geqslant 2\}$. We claim that $X \cup C(v^\alpha) = M(v^\alpha)$. In order to show that $M(v^\alpha) \setminus C(v) \subseteq X$ let $x \in X(v^\alpha) \setminus C(v^\alpha)$ such that $|\{i \in N \mid x_i > 1-\alpha\}| < 2$. By symmetry we may assume $x_1 \leqslant x_2 < 1-\alpha < x_3$. Then x is outvoted via $\{1, 2\}$ by a suitable convex combination of $(1-\alpha, 2\alpha-1, 1-\alpha)$ and $(2\alpha-1, 1-\alpha, 1-\alpha)$. For the other inclusion, let $y \in X$ and assume $y_1 \leqslant y_2 \leqslant y_3, y_2 \geqslant 1-\alpha$, and $y \notin C(v^\alpha)$. Then y cannot be outvoted by some $z \in X(v^\alpha)$ via a singleton $\{i\}$ because $z(N) = 1$ and $z(S) \geqslant \alpha$ for $S \in \mathcal{F}$ with $|S| = 2$ implies $z \in C(v^\alpha)$, i.e., $z_i > 0 = v^\alpha(\{i\})$. Now, y cannot be outvoted via $\{2, 3\}$ as this would imply

$z_2 + z_3 > 2 - 2\alpha > 2/3 > v^\alpha(\{2,3\})$. Hence, it remains to prove that y can neither be outvoted via $\{1,2\}$ nor via $\{1,3\}$. To this end, assume that $z \succ_{\{1,2\}} y$. Hence, $z_1 > y_1$, $z_2 > y_2$, $z_1 + z_3 \geqslant \alpha$, $z(N) = 1$. Therefore, $z_2 = 1 - z_1 - z_3 \leqslant 1 - \alpha \leqslant y_2$, a contradiction. Similarly we may show that y cannot be outvoted via $\{1,3\}$.

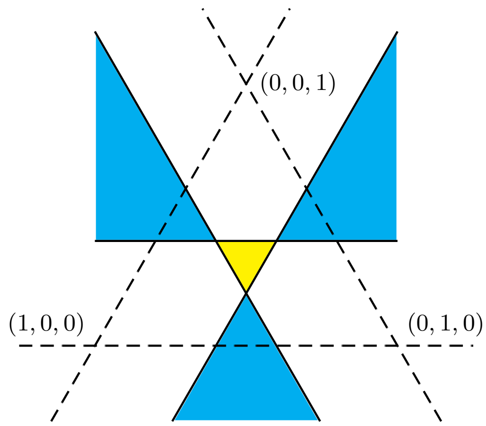

Figure 5.1: $\alpha = 0.6$. The triangle in dashed lines is $I(v)$. Yellow: $C(v^\alpha)$. Blue: X. $M(v^\alpha)$ is the union of the blue and yellow regions.

(2) $\alpha = 2/3$. Then $C(v^\alpha) = \{(1/3, 1/3, 1/3)\}$. As under (1) we may show that $M(v^\alpha) = \{x \in X(v^\alpha) \mid |\{i \in N \mid x_i \geqslant 1/3\}| \geqslant 2\}$.

The foregoing example permits to draw the following conclusions:

- $M(v)$ may not be convex. Indeed, let $1/2 < \alpha \leqslant 2/3$. For $\varepsilon > 0$, $(1-\alpha+\varepsilon, 1-\alpha+\varepsilon, 2\alpha-1-2\varepsilon), (1-\alpha+\varepsilon, 2\alpha-1-2\varepsilon, 1-\alpha+\varepsilon) \in \mathcal{M}(v^\alpha)$, but the midpoint $(1-\alpha+\varepsilon, (\alpha-\varepsilon)/2, (\alpha-\varepsilon)/2)$ is not a member of $M(v^\alpha)$ (see Figure 5.1).

- $M(v)$ may not be bounded[1]. Indeed, let $1/2 < \alpha \leqslant 2/3$. Then $\{(1-\alpha+\varepsilon, 1-\alpha+\varepsilon, 2\alpha-1-2\varepsilon) \mid \varepsilon > 0\}$ is an unbounded subset of $M(v^\alpha)$ (see Figure 5.1).

- The correspondence $M(\cdot) : \Gamma_N^b \rightrightarrows \mathbb{R}^N$ is not u.h.c. Indeed, let $(\alpha^k)_{k\in\mathbb{N}}$ be an arbitrary real sequence with limit $1/2$ and $1/2 < \alpha^k \leqslant 2/3$. Then $x^k = (2-\alpha^k, 2-\alpha^k, 2\alpha^k - 3) \in M(v^{\alpha^k})$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} v^{\alpha^k} = v^{1/2}$, and $\lim x^k = (3/2, 3/2, -2) \notin M(v^{1/2})$.

Now we are able to show that a dummy player may receive more than her individual contribution in M. Let $(N \cup \{4\}, w)$ be the game that arises from $(N, v^{2/3})$ by adding the null player 4. We claim that $x = (2/5, 2/5, 0, 1/5) \in M(w)$. To this end, observe that $(2/5, 2/5, 1/5), (3/5, 2/5, 0), (2/5, 3/5, 0) \in M(v^{2/3})$. Hence, x cannot be outvoted by a proper coalition of N. As x is individually rational, it cannot be outvoted by a 2-person coalition containing player 4. If it were outvoted by $z \in X(w)$ via N, then

[1] However, if all singletons are exact which is a necessary condition for core stability by Remark 2.1, then $\mathcal{M}(v) \subseteq I(v)$, hence $M(v)$ is bounded.

$z(\{1,2,4\}), z(\{1,3,4\}), z(\{2,3,4\}) \geqslant 1$. Summing up these inequalities yields $2 + z_4 \geqslant 3$ so that $z_4 \geqslant 1$ and, hence, $z(N) \leqslant 0$, which is impossible. It remains to show that x is neither outvoted via $S = \{1,3,4\}$ nor via $T = \{2,3,4\}$. Now, let $z \in X(w)$. If $x_S \ll z_S$, then $z(T) = 1 - z_1 < 3/5 < w(T)$, and, similarly, if $x_T \ll z_T$, then $z(S) < w(S)$.

Due to the foregoing "negative" properties we do not consider $M(\cdot)$ as a solution concept further.

6 General Scheme and First Results

For the remainder, whenever a game (N, v) is given, $\mathcal{F} = \mathcal{F}(v)$ is always assumed to satisfy (2.1), (2.2) and (2.3).

Let (N, v) be a balanced game. This section describes the general idea of how to construct the finite test for core stability and serves as a preparation for the main result, namely Theorem 7.1 and its corollary.

Checking core stability of (N, v) amounts to check whether the following condition holds:

$$\forall y \in X(v) \setminus C(v), \exists x \in C(v), x \text{ dom } y.$$

The condition involves two nested quantifiers on uncountable sets, and a test of dominance which amounts to check linear strict inequalities and equality. The introduction of the outvoting relation permits to replace the previous condition by:

$$\forall y \in X(v) \setminus C(v), \exists x \in C(v), x \succ y, \tag{6.4}$$

where the test is stronger and has better properties, but it still involves linear inequalities, some of them being strict, with the two quantifiers. Recall that the test of nonemptiness of the core reads:

$$\exists x \in X(v), x(S) \geqslant v(S) \forall S \subseteq N,$$

which involves one quantifier on an uncountable set and linear inequalities. It is well-known (Bondareva 1963, Shapley 1967) that the above problem is solved by means of the Farkas' lemma and Krein-Milman theorem, and amounts to checking a finite set of linear inequalities (balancedness conditions, see Remark 3.1 (1)). Our approach to solve (6.4) is similar: We replace each test involving one quantifier on an uncountable set by a finite balancedness condition, hence leading to two nested (but finite) balancedness conditions. More precisely:

- Step 1: We solve for a given $y \in X(v) \setminus C(v)$: $\exists x \in C(v), x \succ y$. This is done in Theorem 6.5 below, giving a first finite balancedness condition (6.5) on collections of 2^N, close to the classical one.
- Step 2: We partition $X(v) \setminus C(v)$ into a finite number of blocks $X_{\mathcal{S}}(v)$, where $\mathcal{S} \subseteq 2^N$. Hence, (6.4) becomes: $\forall \mathcal{S}, \forall y \in X_{\mathcal{S}}(v)$, (6.5) holds.
- Step 3: The quantifier on the uncountable set $X_{\mathcal{S}}(v)$ is replaced by a balancedness condition, which also encompasses the first balancedness condition. This is given in Theorem 7.1.

Let $S \in \mathcal{F}$ and $\mathcal{B} \subseteq 2^N \setminus \{N, \emptyset\}$ be a minimal balanced collection. We say that $\mathcal{B}$ is *associated* with S, if there exists $i \in S$ such that $\{i\} \in \mathcal{B}$ and $\mathcal{B} \subseteq \{\{j\} \mid j \in S\} \cup \{N \setminus S\} \cup \left(\mathcal{F} \setminus 2^S\right)$. Let $\mathbb{B}_S$ denote the set of minimal balanced collections associated with S. Moreover, let (N, v^S) be the game the definition of which differs from that of (N, v) only inasmuch as $v^S(N \setminus S) = v(N) - v(S)$.

Theorem 6.1. *Let (N, v) be a balanced game, $y \in \mathbb{R}^N$, and $S \in \mathcal{F}$. Then y is outvoted by some preimputation via S (i.e., there exists $x \in X(v)$ such that $x \succ_S y$) if and only if, for every $\mathcal{B} \in \mathbb{B}_S$,*

$$\sum \{\lambda^{\mathcal{B}}_{\{i\}} y_i \mid i \in S, \{i\} \in \mathcal{B}\} + \sum_{T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}} \lambda^{\mathcal{B}}_T v^S(T) < v(N). \tag{6.5}$$

Proof. The only-if-part: Let $S \in \mathcal{F}$ and $x \in X(v)$ such that $x \succ_S y$. As $x(S) \leqslant v(S)$, $x(N \setminus S) = x(N) - x(S) \geqslant v(N) - v(S) = v^S(N \setminus S)$. As $x_S \gg y_S$ and $x(T) \geqslant v(T)$ for all $T \in \mathcal{F} \setminus 2^S$, we conclude that, for each $\mathcal{B} \in \mathbb{B}_S$,

$$\begin{aligned} v(N) = x(N) = \textstyle\sum_{T \in \mathcal{B}} \lambda^{\mathcal{B}}_T x(T) &= \textstyle\sum \{\lambda^{\mathcal{B}}_{\{i\}} x_i \mid i \in S, \{i\} \in \mathcal{B}\} + \sum_{T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}} \lambda^{\mathcal{B}}_T x(T) \\ &> \textstyle\sum \{\lambda^{\mathcal{B}}_{\{i\}} y_i \mid i \in S, \{i\} \in \mathcal{B}\} + \sum_{T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}} \lambda^{\mathcal{B}}_T v^S(T). \end{aligned}$$

The if-part: Let $S \in \mathcal{F}$ that satisfies (6.5) for each $\mathcal{B} \in \mathbb{B}_S$. As $\mathbb{B}_S$ is finite, there exists $\varepsilon > 0$ such that $v(N) \geqslant \sum \{\lambda^{\mathcal{B}}_{\{i\}} (y_i + \varepsilon) \mid i \in S, \{i\} \in \mathcal{B}\} + \sum_{T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}} \lambda^{\mathcal{B}}_T v^S(T)$ for all $\mathcal{B} \in \mathbb{B}_S$. It is sufficient to show that there exists $x \in \mathbb{R}^N$ such that $x_i \geqslant y_i + \varepsilon$ for all $i \in S$, $x(S) \leqslant v(S)$, $x(T) \geqslant v(T)$ for all $T \in \mathcal{F} \setminus 2^S$, and $x(N) = v(N)$. As $x(N) = v(N)$, $x(S) \leqslant v(S)$ may be replaced by $x(N \setminus S) \geqslant v(N) - v(S)$. As $v(N \setminus S) \leqslant v(N) - v(S)$ by balancedness, $x \succ_S y$ if the linear program

$$\begin{aligned} &\min x(N) \\ &\text{subject to} \begin{cases} x_i \geqslant y_i + \varepsilon \text{ for all } i \in S, \\ x(N \setminus S) \geqslant v(N) - v(S), \\ x(T) \geqslant v(T) \text{ for all } T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\}), \text{ and} \\ x(N) \geqslant v(N) \end{cases} \end{aligned} \tag{6.6}$$

has an optimal solution and the value $v(N)$. Hence, by the duality theorem (Franklin 1980, p. 62), it suffices to show that the dual program of (6.6) has an optimal solution and its value is also $v(N)$. Now, the dual program is:

$$\begin{aligned} &\max \textstyle\sum_{i \in S} \lambda_{\{i\}} (y_i + \varepsilon) + \lambda_{N \setminus S} (v(N) - v(S)) + \sum_{T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\})} \lambda_T v(T) + \lambda_N v(N) \\ &\text{subject to} \begin{cases} \sum_{i \in S} \lambda_{\{i\}} \mathbb{1}^{\{i\}} + \lambda_{N \setminus S} \mathbb{1}^{N \setminus S} + \sum_{T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\})} \lambda_T \mathbb{1}^T + \lambda_N \mathbb{1}^N = \mathbb{1}^N \text{ and} \\ \lambda_{\{i\}} \geqslant 0 \text{ for all } i \in S, \lambda_{N \setminus S}, \lambda_N \geqslant 0, \lambda_T \geqslant 0 \text{ for all } T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\}). \end{cases} \end{aligned} \tag{6.7}$$

Now, with $\lambda_N = 1$ and $\lambda_R = 0$ for all other variables, we have a feasible solution (with value $v(N)$). Hence, we may assume that there is a further feasible solution. However, each further feasible solution satisfies $\lambda_N < 1$ so that we just have to make sure that the value of the derived linear program

$$\begin{aligned} &\max \textstyle\sum_{i \in S} \lambda_{\{i\}} (y_i + \varepsilon) + \lambda_{N \setminus S} (v(N) - v(S)) + \sum_{T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\})} \lambda_T v(T) \\ &\text{subject to} \begin{cases} \sum_{i \in S} \lambda_{\{i\}} \mathbb{1}^{\{i\}} + \lambda_{N \setminus S} \mathbb{1}^{N \setminus S} + \sum_{T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\})} \lambda_T \mathbb{1}^T = \mathbb{1}^N \text{ and} \\ \lambda_{\{i\}} \geqslant 0 \text{ for all } i \in S, \lambda_{N \setminus S}, \lambda_T \geqslant 0 \text{ for all } T \in \mathcal{F} \setminus (2^S \cup \{N \setminus S\}) \end{cases} \end{aligned} \tag{6.8}$$

is not larger than $v(N)$. Let $\mathcal{B}_0 = \{\{i\} \mid i \in S\} \cup \{N \setminus S\} \cup (\mathcal{F} \setminus (2^S \cup \{N \setminus S\}))$, let $(\lambda_T)_{T\in\mathcal{B}_0}$ be a feasible solution of (6.8), and let $\widetilde{\mathcal{B}} = \{T \in \mathcal{B}_0 \mid \lambda_T > 0\}$. Then $\widetilde{\mathcal{B}}$ is a balanced collection and $(\lambda_T)_{T\in\widetilde{\mathcal{B}}}$ is a system of balancing weights. Hence, $\widetilde{\mathcal{B}}$ is a union of minimal balanced collections $\mathcal{B} \subseteq \widetilde{\mathcal{B}}$ such that $(\lambda_T)_{T\in\widetilde{\mathcal{B}}}$ is a convex combination of the systems $(\lambda_T^{\mathcal{B}})_{T\in\widetilde{\mathcal{B}}}$, where $\lambda_T^{\mathcal{B}} = 0$ for all $T \in \widetilde{\mathcal{B}}\setminus\mathcal{B}$. If there exists $i \in S$ such that $\{i\} \in \mathcal{B}$, then $\mathcal{B} \in \mathbb{B}_S$ so that $\sum_{i\in S} \lambda_{\{i\}}^{\mathcal{B}}(y_i+\varepsilon) + \lambda_{N\setminus S}^{\mathcal{B}}(v(N)-v(S)) + \sum_{T\in\mathcal{F}\setminus(2^S\cup\{N\setminus S\})} \lambda_T^{\mathcal{B}} v(T) \leqslant v(N)$ by our assumption on ε. Otherwise $\mathcal{B} \subseteq \{N \setminus S\} \cup \mathcal{F} \setminus 2^S$. As S is exact, there exists $z \in C(v)$ with $z(S) = v(S)$ so that $z(N \setminus S) = v(N) - v(S)$. Therefore $v(N) = z(N) = \sum_{T\in\mathcal{B}} \lambda_T^{\mathcal{B}} z(T) \geqslant \sum_{T\in\mathcal{B}} \lambda_T^{\mathcal{B}} v^S(T) = \sum_{T\in\widetilde{\mathcal{B}}} \lambda_T^{\mathcal{B}} v^S(T)$. □

Let (N, v) be a balanced game. By (2.3), for each $x \in X(v) \setminus C(v)$ there exists $P \in \mathcal{F}$ such that $x(P) < v(P)$. It follows that $X(v)$ can be partitioned into blocks $X_{\mathcal{S}}(v)$, where $\mathcal{S}$ is the collection of all sets $S \in \mathcal{F}$ where strict inequality occurs. Formally:

$$X_{\mathcal{S}}(v) = \{x \in X(v) \mid x(S) < v(S) \forall S \in \mathcal{S}, x(T) \geqslant v(T) \forall T \in \mathcal{F} \setminus \mathcal{S}\},$$

with $\mathcal{S} \subseteq \mathcal{F}$, and $X(v) = \bigcup_{\mathcal{S}\subseteq\mathcal{F}} X_{\mathcal{S}}(v)$. Note that $X_{\emptyset}(v) = C(v)$ and $X_{\mathcal{S}}(v)$ may be empty for some collections $\mathcal{S} \neq \emptyset$. Let us call $\mathcal{S}$ *feasible collection* (for v) if $\mathcal{F} \supseteq \mathcal{S} \neq \emptyset \neq X_{\mathcal{S}}(v)$ so that

$$X(v) \setminus C(v) = \dot{\bigcup}\{X_{\mathcal{S}}(v) \mid \mathcal{S} \text{ is a feasible collection for } v\}.$$

The previous considerations lead to the following result.

Lemma 6.2. *Let (N, v) be a balanced game and $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$. Then $X_{\mathcal{S}}(v) \cap M(v) = \emptyset$ if and only if for every $y \in X_{\mathcal{S}}(v)$ there exists $S \in \mathcal{S}$ such that, for all $\mathcal{B} \in \mathbb{B}_S$,* (6.5) *holds.*

Proof. Suppose that $X_{\mathcal{S}}(v) \cap M(v) = \emptyset$. Then, by Lemma 4.2, every $y \in X_{\mathcal{S}}(v)$ is outvoted by some preimputation x via some $S \in \mathcal{F}$. Since outvoting implies that $y(S) < x(S) \leqslant v(S)$, we deduce that $S \in \mathcal{S}$. By Theorem 6.1, (6.5) must hold for every $\mathcal{B} \in \mathbb{B}_S$.

Conversely, suppose that for each $y \in X_{\mathcal{S}}(v)$ there exists $S \in \mathcal{S}$ such that, for all $\mathcal{B} \in \mathbb{B}_S$, (6.5) holds. By Theorem 6.1, y is outvoted via S. Hence, $y \notin M(v)$. □

The foregoing lemma implies the following corollary.

Corollary 6.3. *The balanced game (N, v) has a stable core if and only if for each feasible collection $\mathcal{S}$ for v and for each $y \in X_{\mathcal{S}}(v)$ there exists $S \in \mathcal{S}$ such that, for all $\mathcal{B} \in \mathbb{B}_S$,* (6.5) *holds.*

Proof. Suppose the core is stable. Then $M(v) = C(v)$ by Proposition 4.4 so that Lemma 6.2 finishes the only-if-part.

Suppose now that for all feasible $\mathcal{S}$ and all $y \in X_{\mathcal{S}}(v)$ there exists $S \in \mathcal{S}$ such that (6.5) holds for all $\mathcal{B} \in \mathbb{B}_S$. Then, $X_{\mathcal{S}}(v)$ does not intersect $M(v)$ by Lemma 6.2 so that $M(v) = C(v)$, i.e., the core is stable by Proposition 4.4. □

We can slightly refine these results by eliminating some balanced collections $\mathcal{B}$ in each $\mathbb{B}_S$ for which (6.5) is automatically satisfied. Supposing (N, v) is balanced, consider some feasible $\mathcal{S} \subseteq \mathcal{F}$, some $S \in \mathcal{S}$, and $\mathcal{B} \in \mathbb{B}_S$ with the following property:

$$(\mathcal{B} \setminus \{\{i\} \mid i \in S\}) \cap \mathcal{S} = \emptyset \neq (\mathcal{B} \setminus \{\{i\} \mid i \in S\}) \cap \{N \setminus R \mid R \in \mathcal{S}\}. \tag{6.9}$$

We claim that for each $x \in X_{\mathcal{S}}(v)$, $x(T) \geqslant v^S(T)$ for $T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}$ and at least one of the inequalities is strict. To show our claim, note that for each $T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}$ that has the form $T = N \setminus R$ with $R \in \mathcal{S}$ we have, since $x \in X_{\mathcal{S}}(v)$, $x(T) > v(N) - v(N \setminus T) \geqslant v(T)$ by balancedness of (N, v). As there exists such a T, at least one of the mentioned inequalities is strict. For all $T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}$ that are not of the foregoing form, we have $T \in \mathcal{F} \setminus \mathcal{S}$, hence $x(T) \geqslant v(T)$. Therefore,

$$\begin{aligned} v(N) = x(N) = \textstyle\sum_{T \in \mathcal{B}} \lambda_T^{\mathcal{B}} x(T) \quad &= \textstyle\sum\{\lambda_{\{i\}}^{\mathcal{B}} x_i \mid i \in S, \{i\} \in \mathcal{B}\} + \sum_{T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}} \lambda_T^{\mathcal{B}} x(T) \\ &> \textstyle\sum\{\lambda_{\{i\}}^{\mathcal{B}} x_i \mid i \in S, \{i\} \in \mathcal{B}\} + \sum_{T \in \mathcal{B} \setminus \{\{i\} \mid i \in S\}} \lambda_T^{\mathcal{B}} v^S(T)), \end{aligned}$$

which is (6.5).

Let $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$. The foregoing paragraph motivates to call $\mathcal{B} \in \mathbb{B}_S$ *admissible* for $\mathcal{S}$ if (6.9) is not satisfied, i.e., if $\mathcal{B} \setminus \{\{i\} \mid i \in S\}$ contains an element of $\mathcal{S}$ or does not contain a complement of an element of $\mathcal{S}$. Let $\mathbb{B}_S^{\mathcal{S}}$ denote the set of minimal balanced collections associated with S that are admissible for $\mathcal{S}$.

For each $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$ we denote $\mathbb{C}(\mathcal{S}) = \{(\mathcal{B}_S)_{S \in \mathcal{S}} \mid \mathcal{B}_S \in \mathbb{B}_S^{\mathcal{S}} \forall S \in \mathcal{S}\}$. Thus, we have deduced that Lemma 6.2 and Corollary 6.3 are still valid if we require admissibility, so that we have the following corollary.

Corollary 6.4. *Let (N, v) be a balanced game and $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$. Then $M(v) \cap X_{\mathcal{S}}(v) \neq \emptyset$ if and only if there exist a system of balanced collections $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$ and $x \in X_{\mathcal{S}}(v)$ such that*

$$\sum\{\lambda_{\{i\}}^{\mathcal{B}_S} x_i \mid i \in S, \{i\} \in \mathcal{B}_S\} + \sum_{T \in \mathcal{B}_S \setminus \{\{i\} \mid i \in S\}} \lambda_T^{\mathcal{B}_S} v^S(T) \geqslant v(N) \forall S \in \mathcal{S}.$$

7 Main Result

Let (N, v) be a balanced game. Now, we express the necessary and sufficient condition for core stability in Corollary 6.4 by a certain balancedness condition.

Let $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$ and $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$. For $S \in \mathcal{S}$ let $z^S \in \mathbb{R}^N$ be given by

$$z_j^S = \begin{cases} \lambda_{\{i\}}^{\mathcal{B}_S} & \text{, if } j = i \text{ for some } i \in S \text{ such that } \{i\} \in \mathcal{B}_S, \\ 0 & \text{, for all other } j \in N. \end{cases}$$

Define

$$Z := Z\left(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}\right) := \{\mathbb{1}^{N \setminus S} \mid S \in \mathcal{S}\} \cup \{\mathbb{1}^T \mid T \in \mathcal{F} \setminus \mathcal{S}\} \cup \{z^S \mid S \in \mathcal{S}\}.$$

Moreover, for each $z \in Z$, define $a_z := a_z\left(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}, v\right) := \max(A \cup B \cup C)$, where

$$\begin{aligned} A \quad &= \{v(N) - v(S) \mid S \in \mathcal{S}, \mathbb{1}^{N \setminus S} = z\}, \\ B \quad &= \{v(T) \mid T \in \mathcal{F} \setminus \mathcal{S}, \mathbb{1}^T = z\}, \text{ and} \\ C \quad &= \left\{ v(N) - \textstyle\sum_{T \in \mathcal{B}_S \setminus \{\{i\} \mid i \in S\}} \lambda_T^{\mathcal{B}_S} v^S(T) \,\middle|\, S \in \mathcal{S}, z = z^S \right\}. \end{aligned}$$

Note that A and B are empty or singletons. Moreover, note that by balancedness of v, $v(S) \leqslant v(N) - v(N \setminus S)$ for all $S \in 2^N$. By applying the exactness of the elements of $\mathcal{F}$, it may also be deduced that $\max\{B, C\} = \max C$ if $B \neq \emptyset \neq C$. Indeed, if $z^S = 1^T$ for some $S \in \mathcal{S}$ and $T \in \mathcal{F} \setminus \mathcal{S}$, then $\mathbb{1}^T + \sum_{R \in \mathcal{B} \setminus \{\{i\} | i \in S\}} \lambda_R^{\mathcal{B}_S} \mathbb{1}^R = \mathbb{1}^N$. Choose $y \in C(v)$ such that $y(S) = v(S)$, hence $y \in C(v^S)$. We conclude

$$v(N) = y(N) = y(T) + \sum_{R \in \mathcal{B} \setminus \{\{i\} | i \in S\}} \lambda_R^{\mathcal{B}_S} y(R) \geqslant v(T) + \sum_{R \in \mathcal{B} \setminus \{\{i\} | i \in S\}} \lambda_R^{\mathcal{B}_S} v^S(R)$$

so that $\max C \geqslant B$. Hence,

$$a_z = \begin{cases} \max C & \text{, if } C \neq \emptyset = A, \\ \max\{A, C\} & \text{, if } C \neq \emptyset \neq A, \\ v(N) - v(S) & \text{, if } z = \mathbb{1}^{N \setminus S} \text{ for some } S \in \mathcal{S}, C = \emptyset, \\ v(T) & \text{, if } z = \mathbb{1}^T \text{ for some } T \in \mathcal{F} \setminus \mathcal{S}, A = \emptyset = C. \end{cases}$$

Denote by $\mathbb{B} := \mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}})$ the set of minimal balanced subsets in Z (see Sect. 3 for the definition of minimal balanced sets) and by $\mathbb{B}_0 = \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}, v)$ the subset of all $Z' \in \mathbb{B}$ for which there exists $S \in \mathcal{S}$ such that $z = \mathbb{1}^{N \setminus S} \in Z'$ and $a_z = v(N) - v(S)$. Finally, for $Z' \in \mathbb{B}$, let $\delta^{Z'} = (\delta_z^{Z'})_{z \in Z'}$ denote the system of balancing weights of Z'.

We now formulate our fundamental result.

Theorem 7.1. *Let (N, v) be a balanced game and $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$. Then the following conditions are equivalent.*

(1) $M(v) \cap X_{\mathcal{S}}(v) = \emptyset$.

(2) For all $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$, with $a_z := a_z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}, v)$ for all $z \in Z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}})$, one of the following conditions holds:

$$\exists Z' \in \mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}) \setminus \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}) \quad : \quad \sum_{z \in Z'} \delta_z^{Z'} a_z > v(N). \tag{7.10}$$

$$\exists Z' \in \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}) \quad : \quad \sum_{z \in Z'} \delta_z^{Z'} a_z \geqslant v(N). \tag{7.11}$$

Proof. Let $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$. Let $\varepsilon \geqslant 0$ and consider the linear program

$$\min x(N)$$
$$\text{subject to} \begin{cases} x(T) \geqslant v(T) \text{ for all } T \in \mathcal{F} \setminus \mathcal{S}, \\ x(N \setminus S) \geqslant v(N) - v(S) + \varepsilon \text{ for all } S \in \mathcal{S}, \text{ and} \\ \sum\{\lambda_{\{i\}}^{\mathcal{B}_S} x_i \mid i \in S, \{i\} \in \mathcal{B}_S\} \geqslant v(N) - \sum_{T \in \mathcal{B}_S \setminus \{\{i\} | i \in S\}} \lambda_T^{\mathcal{B}_S} v^S(T) \text{ for all } S \in \mathcal{S}. \end{cases} \tag{7.12}$$

Note that (7.12) has feasible solutions. Hence, the linear program has either an optimal solution or it is unbounded from below. Therefore, its dual program must also have either an optimal solution with the same optimal value or it has no feasible solution. Now, with $Z = Z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}})$ and

$$a_z^\varepsilon = \begin{cases} a_z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}, v), \text{ if } z \in Z \setminus \{\mathbb{1}^{N \setminus S} \mid S \in \mathcal{S}\}, \\ \max\{a_z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}, v), v(N) - v(S) + \varepsilon\}, \text{ if } z \in \{\mathbb{1}^{N \setminus S} \mid S \in \mathcal{S}\}, \end{cases}$$

the dual program is

$$\begin{aligned}&\max \textstyle\sum_{z\in Z} \delta_z a^\varepsilon_z\\ &\text{subject to} \begin{cases} \sum_{z\in Z} \delta_z z = \mathbb{1}^N \text{ and}\\ \delta_z \geqslant 0 \text{ for all } z \in Z.\end{cases}\end{aligned} \tag{7.13}$$

Sufficiency: Assume that $M(v) \cap X_{\mathcal{S}}(v) \neq \emptyset$. By Corollary 6.4 there exist $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$ and $\varepsilon > 0$ such that there exists a feasible solution x of (7.12) such that $x(N) \leqslant v(N)$. Hence, for each $Z' \in \mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}})$, $\sum_{z\in Z'} \delta^{Z'}_z a^\varepsilon_z \leqslant v(N)$. If $a^\varepsilon_z > a_z$ for some $z \in Z'$, $\sum_{z\in Z'} \delta^{Z'}_z a_z < v(N)$. Otherwise, $\sum_{z\in Z'} \delta^{Z'}_z a^\varepsilon_z = \sum_{z\in Z'} \delta^{Z'}_z a_z \leqslant v(N)$ so that this direction has been shown.

Necessity: Assume that, for some $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$, neither (7.10) nor (7.11) is valid. If $\mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) = \emptyset$, then, regardless of the choice of $\varepsilon > 0$, (7.13) has no feasible solution so that there must be a feasible solution x of (7.12) such that $x(N) \leqslant v(N)$. If $\mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) \neq \emptyset$, then, for all $Z' \in \mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}})$, $\sum_{z\in Z'} \delta^{Z'}_z a_z \leqslant v(N)$ and, if $Z' \in \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}})$, then $\sum_{z\in Z'} \delta^{Z'}_z a_z < v(N)$. Hence, we may select $\varepsilon > 0$ such $\sum_{z\in Z'} \delta^{Z'}_z a^\varepsilon_z \leqslant v(N)$. We conclude in any case that there must be a feasible solution x of (7.12) such that $x(N) \leqslant v(N)$. Therefore, there exists $x' \in \mathbb{R}^N$ with $x' \geqslant x$ and $x'(N) = v(N)$. Now, as $x' \geqslant x$, x' is a feasible solution of (7.12) that satisfies $x'(N) = v(N)$. Hence, $x' \in X_{\mathcal{S}}(v)$ so that $\mathcal{S}$ is feasible for v. Thus, $M(v) \cap X_{\mathcal{S}}(v) \neq \emptyset$ by Corollary 6.4. □

Corollary 7.2. *Let (N, v) be a balanced game. Then (N, v) has a stable core if and only if for all $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$ and all $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$, with $a_z := a_z(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}, v)$ for all $z \in Z(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}})$, either* (7.10) *holds or* (7.11) *holds.*

Remark 7.3. It should be noted that the condition "for all $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$" in Corollary 7.2 can be replaced by "for all feasible collections $\mathcal{S}$ for v", which would typically reduce the number of required tests for core stability as often not all nonempty subsets of coalitions in $\mathcal{F}$ are feasible for v. On the other hand, the current form does not require to check in advance which of the nonempty subsets of $\mathcal{F}$ are feasible for v.

We now apply our main theorem to an illustrating example.

Example 7.4. Let $N = \{1, \ldots, 5\}$, $\lambda^1 = (2, 1, 0, 0, 0)$, $\lambda^2 = (0, 0, 1, 1, 1)$, and (N, v) be defined by $v(S) = \min\{\lambda^1(S), \lambda^2(S)\}$ for all $S \subseteq N$. Then (N, v) is exact and $C(v)$ is the convex hull of λ^1 and λ^2 (Raghavan and Sudhölter 2005, Example 4.3). The shape of the core immediately shows that all strongly vital-exact coalitions are also strictly vital-exact:

$$\mathcal{F} := \mathcal{VE}^{\text{strong}}(v) = \mathcal{VE}^{\text{strict}}(v) = \{\{i\} \mid i \in N\} \cup \{\{1, k, \ell\} \mid k, \ell \in \{3, 4, 5\}, k \neq \ell\} \cup \{\{2, i\} \mid i \in \{3, 4, 5\}\}.$$

It is known that the core is not stable, and we now illustrate Theorem 7.1 by checking the condition for nonempty subsets $\mathcal{S} \subseteq \mathcal{F}$. The singletons and the 2-person coalitions in $\mathcal{F}$ are extendable. Remarking that if S is extendable, then every preimputation x such that $x(S) < v(S)$ and $x(T) \geqslant v(T)$ for all $T \subsetneq S$ can be outvoted via S by a core element, it follows that, for each $\mathcal{S}$ that contains at least one singleton or one 2-person coalition in $\mathcal{F}$, each element of $X_{\mathcal{S}}(v)$ is outvoted by a core element. Therefore, we now consider only the cases that $\mathcal{S}$ consists of one, two, or all 3-person coalitions in $\mathcal{F}$.

(1) If $\mathcal{S}$ is the singleton of one 3-person coalition in $\mathcal{F}$, then we may assume, as players 3, 4, and 5 are substitutes, that $\mathcal{S} = \{\{1,3,4\}\}$. If $x \in X_{\mathcal{S}}(v)$, then we claim that $x = (2-a-b, 1-a, a, a, b)$ for some $0 \leqslant a < 1$ and some $b > a$ such that $a + b \leqslant 2$. Indeed, for $j \in \{3,4\}$, $x_1 + x_j + x_5 \geqslant v(\{1,j,5\}) = 2$ and $x_2 + x_{7-j} \geqslant v(\{2, 7-j\}) = 1$. As $3 = v(N) = x_1 + x_j + x_5 + x_2 + x_{7-j}$, the foregoing inequalities are, in fact, equalities. Therefore, $x_3 = x_4 =: a$ and with $b := x_5$, our claim follows because $x \geqslant 0$ and $x_1 + x_3 + x_4 < 2$. Therefore, for any $0 < \varepsilon < \min\{1-a, {}^{(b-a)}/_{2}\}$, $(2-2a-2\varepsilon, 1-a-\varepsilon, a+\varepsilon, a+\varepsilon, a+\varepsilon)$ is a core element that outvotes x via $\{1,3,4\}$. (Note that $\mathcal{S}$ is feasible because, e.g., $(0,1,0,0,2) \in X_{\mathcal{S}}(v)$.)

(2) If $\mathcal{S}$ consists of two 3-person coalitions, then we may assume $\mathcal{S} = \{\{1,3,4\},\{1,3,5\}\}$. Choose $\mathcal{B}_{\{1,3,4\}} = \{\{4\},\{1,3,5\},\{2\}\}$, $\mathcal{B}_{\{1,3,5\}} = \{\{5\},\{1,3,4\},\{2\}\}$, and note that $B_{\{1,3,4\}}$ and $B_{\{1,3,5\}}$ are admissible for $\mathcal{S}$. Also $z^{\{1,3,4\}} = \mathbb{1}^{\{4\}}$ and $a_{z^{\{1,3,4\}}} = v(N) - v(\{1,3,5\}) - v(\{2\}) = 1$. Similarly, $z^{\{1,3,5\}} = \mathbb{1}^{\{5\}}$ and $a_{z^{\{1,3,5\}}} = 1$. The following table shows the resulting pairs (z, a_z) such that a_z is positive:

z	a_z
$(0,0,0,0,1)$	1
$(0,0,0,1,0)$	1
$(1,0,0,1,1)$	2
$(0,1,1,0,0)$	1
$(0,1,0,1,0)$	1
$(0,1,0,0,1)$	1

Let $y = (0, 0.5, 0.5, 1, 1)$ and note that $y \cdot z \geqslant a_z$ for all $z \in Z$. Also, note that $y \cdot \mathbb{1}^{N\setminus\{1,3,4\}} = 1.5 > a_{\mathbb{1}^{N\setminus\{1,3,4\}}} = 1$ and $y \cdot \mathbb{1}^{N\setminus\{1,3,5\}} = 1.5 > a_{\mathbb{1}^{N\setminus\{1,3,5\}}} = 1$. Hence, for all minimal balanced subsets Z' of Z, we have $\sum_{z\in Z'} \delta_z^{Z'} a_z \leqslant \sum_{z\in Z'} \delta_z^{Z'} y \cdot z = y \cdot \sum_{z\in Z'} \delta_z^{Z'} z = y \cdot \mathbb{1}^N = 3 = v(N)$, and if Z' contains $\mathbb{1}^{N\setminus\{1,3,4\}}$ or $\mathbb{1}^{N\setminus\{1,3,5\}}$ the inequality is strict. We conclude that $M(v)$ intersects $X_{\mathcal{S}}(v)$, e.g., $(1,1,0,0.5,0.5) \in M(v) \cap X_{\mathcal{S}}(v)$. However, $(1,1,0,0.1,0.9) \in X_{\mathcal{S}}$ is outvoted via $\{1,2,4\}$ by $(1.2, 0.6, 0.4, 0.4, 0.4)$.

(3) The case $\mathcal{S} = \{\{1,3,4\},\{1,3,5\},\{1,4,5\}\}$ is again treated "indirectly". Let $x \in X_{\mathcal{S}}(v)$. In this case we may assume $x_3 \leqslant x_4 \leqslant x_5$. As $x \geqslant 0$, $x_4 + x_5 < 2$ and, hence, $x_4 < 1$. As $x_4 \leqslant x_5$, $x_1 < 2 - 2x_4$. Let $0 < \varepsilon < \min\{1 - x_4, {}^{(2-2x_4-x_1)}/_{2}\}$. Then $(2 - 2x_4 - 2\varepsilon, 1 - x_4 - \varepsilon, x_4 + \varepsilon, x_4 + \varepsilon, x_4 + \varepsilon) \in C(v)$ outvotes x via $\{1,3,4\}$ so that the condition of our theorem is satisfied for $\mathcal{S}$. (Note that $\mathcal{S}$ is feasible because $(0.5, 1, 0.5, 0.5, 0.5) \in X_{\mathcal{S}}(v)$).

8 Core Super-Stability

Shellshear and Sudhölter (2009) present several sufficient conditions for core stability, the weakest of which is called *vital-exact extendability*. Let (N,v) be a balanced game. A coalition $S \in 2^N \setminus \{N, \emptyset\}$ is called *extendable* if, for each $x \in C(v_{|S})$ there (recall that $(S, v_{|S})$ is the subgame of (N,v) with player set S) there exists $y \in C(v)$ such that $y_S = x$. If all subcoalitions of N are extendable, then the core is stable

(Kikuta and Shapley 1986). However, extendability remains sufficient for core stability if it is weakened to *vital-exact* extendability, requiring just that all strongly vital-exact coalitions must be extendable. By means of examples, Shellshear and Sudhölter (2009) show that core stability is less demanding than vital-exact extendability, which is less demanding than extendability of all vital exact coalitions, which is less demanding than extendability of all coalitions. However, for several remarkable classes of games, vital-exact extendability is necessary (and sufficient) for core stability. Though it is not known if vital-exact extendability can be further relaxed so that is becomes necessary and remains sufficient for core stability in general, Corollary 7.2 allows to construct a finite test for core stability. This section is devoted to study a more demanding property than core stability, called *core super-stability*. It is then shown that this property is equivalent to *vital extendability* requiring that all vital subcoalitions are extendable.

Let (N, v) be a game. A vector $x \in \mathbb{R}^N$ is *N-feasible* if $x(N) \leqslant v(N)$. Now, if $v(N) \geqslant \sum_{i \in N} v(\{i\})$, then, by Remark 2.1, the core is stable if and only if every preimputation that is not in the core may be dominated by a core element. Hence, the core is stable if and only if every N-feasible vector that is not in the core is dominated by a core element. Indeed, the if-part is obvious. For the only-if-part, let $y \in \mathbb{R}^N \setminus C(v)$ satisfy $y(N) \leqslant v(N)$. If there exists $S \subsetneqq N$ with $y(S) < v(S)$, then there exists $y' \in \mathbb{R}^N$ such that $y'_S = y_S$, $y' \geqslant y$, and $y'(N) = v(N)$. As the core is stable, y' is dominated by some core element via some proper coalition so that y is dominated via the same coalition. If $y(S) \geqslant v(S)$ for all $S \in 2^N \setminus \{\emptyset, N\}$, then $y(N) < v(N)$ and $y + \frac{v(N)-y(N)}{|N|}\mathbb{1}^N$ is a core element that dominates y via N.

We now define "super-stability" of the core by relaxing N-feasibility. A vector $x \in \mathbb{R}^N$ is *aspiration-feasible* (Bejan and Gómez 2012) if, for each $i \in N$ there exists $S \subseteq N$ such that $i \in S$ and $x(S) \leqslant v(S)$. Note that Bennett (1983) uses the word "anticipation" for a vector that is aspiration-feasible (and, in her context of a nonnegative game, nonnegative). An *upper vector* is a vector $x \in \mathbb{R}^N$ such that $x(S) \geqslant v(S)$ for all $S \subseteq N$ (van Gellekom, Potters, and Reijnierse 1999). Let $U(v)$ denote the set of upper vectors. An *aspiration* (Bennett 1983) is an aspiration-feasible upper vector, i.e., it is an upper vector x satisfying $x(S) = v(S)$ for all S in a covering of N. Let $Asp(v)$ denote the set of aspirations. Observe that $C(v) \subseteq Asp(v)$.

Remark 8.1. (1) Let $x \in \mathbb{R}^N \setminus U(v)$ and choose an arbitrary minimal coalition $P \subseteq N$ such that $x(P) < v(P)$. Then there exists $y \in U(v)$ such that y dominates x via P. Indeed, for $i \in P$, put $y_i = x_i + \frac{v(P)-x(P)}{|P|}$, and, for $j \in N \setminus P$, put y_j large enough, e.g., $y_j = \max_{S \subseteq N} v(S) - \min_{S \subseteq N} v(S)$. For $S \subsetneqq P$, $y(S) \geqslant v(S)$ because $y_P \gg x_P$ and $x(S) \geqslant v(S)$. For $S \subseteq N$ with $S \setminus P \neq \emptyset$, $y(S) \geqslant (v(S) - v(S \cap P)) + y(S \cap P) \geqslant v(S)$. Finally, $y(P) = v(P)$ so that y has the desired properties.

(2) For each $y \in U(v)$ there exists $z \in Asp(v)$ such that $z \leqslant y$. Indeed, for each $z \in U(v)$ denote $T(z) = \bigcup\{S \subseteq N \mid z(S) = v(S)\}$. Now, let $z \in U(v)$ such that $z \leqslant y$ and $T(z)$ is maximal (w.r.t. inclusion). It remains to show that $T(z) = N$. Assume, on the contrary, that there exists $j \in N \setminus T(z)$. Let $\widehat{z} \in \mathbb{R}^N$ differ from z only inasmuch as $\widehat{z}_j = z_j - \min\{z(S) - v(S) \mid S \subset N, j \in S\}$. Then $\widehat{z} \in U(v)$, $z \leqslant y$, and $T(\widehat{z}) \supseteq T(z) \cup \{j\}$ which is impossible by the maximality of $T(z)$.

The foregoing remark has the following implications. Recall that a game (N, v) has a large core if, for every $y \in U(v)$, there exists $x \in C(v)$ such that $x \leqslant y$ (Sharkey 1982).

Corollary 8.2. *(1) Any $x \in \mathbb{R}^N \setminus U(v)$ is dominated by an aspiration.*

(2) The set of aspirations coincides with the core, i.e., $Asp(v) = C(v)$, if and only if (N, v) has a large core.

Say that the game (N, v) has a *super-stable core* if each aspiration-feasible vector that is not an aspiration is dominated by a core element.

Proposition 8.3. *The game (N, v) has a super-stable core if and only if each $x \in \mathbb{R}^N \setminus U(v)$ is dominated by some core-element.*

Proof. The if-part is true because an aspiration-feasible vector that is not an aspiration is also not an upper vector. For the converse, let $x \in \mathbb{R}^N \setminus U(v)$. Define the game (N, u) by $u(S) = \min\{x(S), v(S)\}$. Hence, $x \in U(N, u)$. By Remark 8.1 (2) there exists $y \in Asp(N, u)$ such that $y \leqslant x$. Hence, $y(S) = x(S)$ whenever $x(S) < v(S)$. As $u \leqslant v$, y is an aspiration-feasible vector of (N, v), but not an aspiration. Let $z \in C(v)$ dominate y via some coalition S. As $y(T) \geqslant v(T)$ whenever $x(T) \geqslant v(T)$, we deduce $x(S) < v(S)$, which implies $y(S) = x(S)$, which in turn implies $x_S = y_S$, so that z dominates also x via S. □

Proposition 8.4. *A TU game has a super-stable core if and only if the game is vital extendable.*

Proof. To show the if part, assume that (N, v) is a vital extendable TU game and let $y \in \mathbb{R}^N \setminus U(v)$. Let S be a minimal (w.r.t. inclusion) coalition such that $y(S) < v(S)$. Define $x_S \in \mathbb{R}^S$ by $x_i = y_i + \frac{v(S)-y(S)}{|S|}$ for all $i \in S$. Then $x_S \in C(v_{|S})$ and $x(T) > v(T)$ for all proper subcoalitions T of S so that S is vital. Hence, x_S can be extended to some $x \in C(v)$. Consequently, x dominates y via S.

For the remaining implication, assume that (N, v) has a super-stable core. Let S be a vital coalition and $x_S \in C(v_{|S})$. We first assume that $x(T) > v(T)$ for all proper subcoalitions T of S. Let $y_S^k \in \mathbb{R}^S$ be defined by $y_i^k = x_i - \frac{1}{k}$ for all $k \in \mathbb{N}$ and $i \in S$. Moreover, extend y_S^k to some $y^k \in \mathbb{R}^N$ such that $y^k(P) \geqslant v(P)$ for all $P \in 2^N \setminus 2^S$. As $\lim_{k\to\infty} y_S^k = x_S$, there exists $K \in \mathbb{N}$ such that $y^k(P) \geqslant v(P)$ for all $P \in 2^N \setminus \{S\}$ and all $k \geqslant K$. By the super-stability of the core, for each $k \in \mathbb{N}$, as $y^k(S) < x(S) = v(S)$, there exists $z^k \in C(v)$ such that z^k dominates y^k. Hence, for $k \geqslant K$, z^k dominates y^k via S. Moreover, as $\lim_{k\to\infty} y_S^k = x_S$, $z_S > y_S^k$, and $z^k(S) = v(S) = x(S)$ for $k \geqslant K$, $\lim_{k\to\infty} z_S^k = x_S$. By compactness of the core, $(z^k)_{k\in\mathbb{N}}$ has a convergent subsequence. Let z be the limit of this subsequence. Hence, $z_S = x_S$ so that the extendability of x_S has been verified.

Now we can finish the proof. Let $\widetilde{x}_S \in C(v_{|S})$ and $x_S \in C(v_{|S})$ such that $x(T) > v(T)$ for all proper subcoalitions T of S. Then, for all $k \in \mathbb{N}$, $x_S^k = \frac{1}{k}x_S + \frac{k-1}{k}\widetilde{x}_S \in C(v_{|S})$ and $x_S^k(T) > v(T)$ for all proper subcoalitions T of S so that x^k can be extended to some $x^k \in C(v)$ as we have shown before. By compactness of the core the sequence $(x^k)_{k\in\mathbb{N}}$ has a convergent subsequence, and its limit extends $\widetilde{x}_S$ to a core element. □

9 Discussion and concluding remarks

Corollary 7.2 allows to check if a balanced game has a stable core with the help of a finite number of tests. There are finitely many minimal balanced collections of coalitions (and we know how to inductively construct them (Peleg 1965)) so that there are just finitely many triples $(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}, Z')$ that have to be checked. This solves a problem which has remained open for a long time.

This being said, still some further investigation seems to be necessary, mainly for two reasons: The test appears to be very combinatorial, and it involves a strict inequality: $\sum_{z\in Z'} \delta_z^{Z'} a_z > v(N)$. We discuss these points and related ones in what follows.

The first problem which arises is to find all feasible collections $\mathcal{S}$, in order to determine the partition of $X(v)\setminus C(v)$. Indeed, recall that we may restrict the attention to feasible collections for v as explained in Remark 7.3. The following lemma permits to avoid a naive enumeration of all the subcollections of $\mathcal{F}$.

Lemma 9.1. *Let (N, v) be balanced and $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$. Then the following statements are true.*

(1) If $\mathcal{S}$ is feasible, then it does not contain a balanced collection of N.

(2) For $S, S' \in \mathcal{S}$, if $S \cup S' = N$, there is no $x \in X_{\mathcal{S}}(v)$ which can be dominated via S or S'.

Proof. (1) Consider a feasible $\mathcal{S}$ and suppose that a balanced collection $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$ is contained in $\mathcal{S}$. Then for every $x \in X_{\mathcal{S}}(v)$:

$$x(S) < v(S), \quad S \in \mathcal{B}.$$

Multiplying the inequalities by λ_S and summing them yields

$$v(N) = x(N) < \sum_{S\in\mathcal{B}} \lambda_S v(S),$$

which contradicts balancedness of (N, v).

(2) Suppose $S \cup S' = N$. If $X_{\mathcal{S}}(v) = \emptyset$, then nothing has to be shown. Otherwise take $y \in X_{\mathcal{S}}(v)$ and suppose that there is a core element x dominating y via S. Then the side-payment $z = y - x$ satisfies $z_S \ll 0_S$, $z(S) < 0$, and $z(S') < 0$. As $S, S' \neq N$, we have $S \not\subseteq S'$ and $S' \not\subseteq S$. As z is a side-payment, we have $z(S) = -z(N \setminus S) = -z(S' \setminus S)$. It follows that

$$z(S') = z(S\cap S') + z(S'\setminus S) = \underbrace{z(S\cap S')}_{\geqslant z(S)} - z(S) \geqslant 0,$$

a contradiction. □

Note that a consequence of Lemma 9.1 (1) is that $S \in \mathcal{S}$ implies $N \setminus S \notin \mathcal{S}$. Moreover, observe that (2) implies the following simple necessary condition for core stability: No feasible $\mathcal{S}$ can be of the form $\{S, S'\}$ with $S \cup S' = N$. But it may be even superfluous to check some feasible collections. For example, as we already mentioned in Example 7.4, every element of $X_{\mathcal{S}}$ is outvoted when $\mathcal{S}$ contains a minimal coalition S which is extendable.

Unlike the Bondareva-Shapley result on nonemptiness of the core (see (1) of Remark 3.1), we do not know if Corollary 7.2 (even with the condition "for all feasible collection $\mathcal{S}$ for v" rather than "for all $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$") is sharp or not, although we use only minimal admissible balanced collections. In particular, we do not know whether the condition $\sum_{z\in Z'} \delta_z^{Z'} a_z > v(N)$ is necessary or if only the $\ldots \geqslant$ suffices. So far, attempts to find examples where the strict inequality is used have failed. What we could establish, however, is that the set $\mathbb{B}_0$ is never empty when the core is stable. Incidentally, this gives another necessary condition for core stability.

Lemma 9.2. *Let (N, v) be a balanced game with a stable core and $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$. Then $\mathbb{B}_0 = \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) \neq \emptyset$ for all $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$.*

Proof. 1. We claim that for each $S \in \mathcal{S}$, there exists a minimal balanced collection Z' containing $\mathbb{1}^{N\setminus S}$. Indeed, all singletons being strictly vital-exact, they belong either to $\mathcal{S}$ or to $\mathcal{F} \setminus \mathcal{S}$. If $\{i\}$ is in $\mathcal{S}$, observe that $z^{\{i\}} = \mathbb{1}^{\{i\}}$, because $\mathcal{B}_{\{i\}} \ni \{i\}$ and is a minimal balanced collection. Therefore, $Z' = \{\mathbb{1}^{N\setminus S}\} \cup \{\mathbb{1}^{\{i\}} \mid i \in S\}$ is a minimal balanced collection in Z, that we call *S-canonical*.

2. By core stability, Lemma 9.1(2) implies that there exists $S \in \mathcal{S}$ such that $S \cup T \subsetneq N$ for all $T \in \mathcal{S}$. Let Z' be the S-canonical minimal balanced collection. For all $T \in \mathcal{S}$, $z_j^T = 0$ for all $j \in N \setminus T$. As $S \cup T \subsetneq N$ for all $T \in \mathcal{S}$, $z^T \neq \mathbb{1}^{N\setminus S}$. We conclude that the balancing weight $a_{\mathbb{1}^{N\setminus S}}(\mathcal{S}, (\mathcal{B}_T)_{T\in\mathcal{S}})$ must be $v(N) - v(S)$, i.e., $Z' \in \mathbb{B}_0$. □

The presence of a strict inequality in the test prevents to easily answer questions like: Is the set of games with a stable core closed? Is core stability a strong prosperity property? We elaborate on the latter question. According to van Gellekom, Potters, and Reijnierse (1999), a property P on a set of games is a *(strong) prosperity property* if for every game v, there exists a constant $\alpha(v^0)$, where v^0 is the restriction of v to $2^N \setminus \{N\}$, such that v has the property P if and only if $v(N)$ is made greater or equal to $\alpha(v^0)$. P is a weak prosperity property if there exists a constant $\beta(v^0)$ such that v has property P if $v(N) \geqslant \beta(v^0)$. So far, it is known that core stability is a weak prosperity property, but it is not known whether it is a strong one. However, as vital extendability is a strong prosperity property, it follows that core super-stability is also a strong prosperity property.

As concluding remarks, obtaining a sharp form of Corollary 7.2 is a challenging task for future research, as well as establishing efficient algorithmic procedures to make the test easy to use in practice.

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