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Welfare egalitarianism in surplus-sharing problems and convex games*

Pedro Calleja[†], Francesc Llerena[‡], and Peter Sudhölter[§]

Abstract

We show that the constrained egalitarian surplus-sharing rule, which divides the surplus so that the poorer players' resulting payoffs become equal but not larger than any remaining player's status quo payoff, is characterized by Pareto optimality, path independence, both well-known, and less first (LF), requiring that a player does not gain if her status quo payoff exceeds that of another player by the surplus. This result is used to show that, on the domain of convex games, Dutta-Ray's egalitarian solution is characterized by aggregate monotonicity (AM), bounded pairwise fairness, resembling LF, and the bilateral reduced game property (2-RGP) à la Davis and Maschler. We show that 2-RGP can be replaced by individual rationality and bilateral consistency à la Hart and Mas-Colell. We prove that the egalitarian solution is the unique core selection that satisfies AM and bounded richness, requiring that the poorest players cannot be made richer within the core. Replacing "poorest" by "poorer" allows to eliminate AM.

Keywords: Surplus-sharing, egalitarianism, convex TU game
JEL Classification: C71

1 Introduction

The notion of equity has a significant position in distributional problems, where a quantity of a divisible resource (e.g., money) is divided among a set of agents that believe in egalitarianism as a social value.

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In the context of cooperative transferable utility games (games, for short), Dutta and Ray (1989) introduce their *egalitarian solution* which combines coalitional agreements with the Lorenz criterion to promote equality. In rationing models, the *constrained equal awards* rule and the *constrained equal losses* rule implement the idea of egalitarianism in different directions, namely by equalizing awards or losses, respectively. Describing rationing problems as suitable games, the constrained equal awards rule may be regarded as the egalitarian solution. Here, we show that the egalitarian solution is supported by a suitable new rule in the framework of surplus-sharing problems. In this setting, authors have paid attention to *resource egalitarianism* instead of *welfare egalitarianism*, that is, in distributing equally among players the total resource to be divided, without taking into consideration to equalize the welfare of the agents ex-post, i.e., after the allocation process.[1] To recognize this latter aspect, we introduce the *constrained egalitarian* surplus-sharing rule. Imagine a situation where there is a resource to be divided among a set of agents that are ranked with respect to (w.r.t.) a *reference point*, representing some objective (and measurable) feature. First, agents with the lowest ranking receive everything until they become equal to the second lowest ranked agents, and so forth until the resource is exhausted. The constrained egalitarian rule has some resemblance with the *constrained outcome-egalitarian* rule, introduced by Moreno-Ternero and Roemer (2012) in a more complex model, where agents possess the capability to transform wealth into non-transferable outcomes.

In this paper, we show that the constrained egalitarian rule is characterized by three properties: *Pareto optimality*, *path independence* (Moulin, 1987) requiring that the assigned payoffs remain unchanged when applying the rule consecutively to any partition of the resource, and *less first*, a new property requiring that a player does not gain if her status quo payoff exceeds that of another player by the surplus. We observe that the egalitarian solution distributes any growth in the value of the grand coalition following the path recommended by the constrained egalitarian rule. This fact drives our investigation to search for axiomatizations of the egalitarian solution on the domain of convex games[2] using *aggregate monotonicity* (Megiddo, 1974), a very natural property requiring that no player suffers if only the grand coalition becomes richer. Together with aggregate monotonicity, we use *bounded pairwise fairness*, a new property that is reminiscent of less first in the surplus-sharing setting, and the standard requirement of

[1] See Moreno-Ternero and Roemer (2012) for a concise exposition of these two conceptions of distributive justice.

[2] Outside the class of convex games, the existence of the egalitarian solution is not guaranteed. Recently, Dietzenbacher et al. (2017) introduce the *procedural egalitarian solution*, a (single-valued) solution defined for arbitrary games that coincides with the egalitarian solution on the class of convex games.

consistency (à la Davis and Maschler, 1965, and à la Hart and Mas-Colell, 1989). Finally, we provide two additional characterizations without consistency. To do so, we introduce *bounded richness*, requiring that the poorest players cannot be made richer unless the remaining players receive less than what they can guarantee by cooperation. Strengthening bounded richness, replacing "poorest" by "poorer", allows to eliminate aggregate monotonicity. Up to our knowledge, aggregate monotonicity has not been employed before in any of the existing characterizations of the egalitarian solution and the unique characterization till now that does not employ any consistency property was provided by Arin et al. (2003).[3]

The remainder of the paper is organized as follows. Section 2 contains preliminaries on games and surplus-sharing problems. In Section 3 we study some logical implications among properties and provide a characterization of the constrained egalitarian rule. Section 4 is devoted to the characterization results of the egalitarian solution with consistency (in Subsection 4.1) and without consistency (in Subsection 4.2). The logical independence of each property is extensively discussed in Section 5. Finally, Section 6 concludes with some final remarks.

2 Preliminaries

Let U be a set (the universe of potential players) and $\mathcal{N}$ be the set of coalitions in U (a *coalition* is a nonempty finite subset of U). Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ and $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. We assume that $|U| \geq 3$. Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the set of all real functions on N. An element $x \in \mathbb{R}^N$, $x = (x_i)_{i \in N}$, is a payoff vector for N. For all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$. Given $N \in \mathcal{N}$, for all $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$ for all $i \in N$. For all $\alpha \in \mathbb{R}$, $\alpha_+ = \max\{0, \alpha\}$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that y *weakly Lorenz dominates* x, denoted by $y \succeq_{\mathcal{L}} x$, if $\min\{y(S) \mid S \subseteq N,\ |S| = k\} \geq \min\{x(S) \mid S \subseteq N,\ |S| = k\}$, for all $k = 1, 2, \ldots, n-1$. We say that y *Lorenz dominates* x, denoted by $y \succ_{\mathcal{L}} x$, if at least one of the above inequalities is strict. Given $x \in \mathbb{R}^N$, let $\mathcal{P}(x) = (N_1, N_2, \ldots, N_k)$ denote the ordered partition of N that is determined by $N_1 = \{i \in N \mid x_i \leq x_j \, \forall j \in N\}$ and $N_m = \{i \in N \setminus \bigcup_{j=1}^{m-1} N_j \mid x_i \leq x_j \, \forall j \in N \setminus \bigcup_{j=1}^{m-1} N_j\}$ for all $m = 2, \ldots, k$.

A *transferable utility game (a game)* is a pair (N, v) where $N \in \mathcal{N}$ is the set of players

[3] The first axiomatic characterization was provided by Dutta (1999). Other characterizations can be found in Klijn et al. (2000), Hougaard et al. (2001), and Llerena and Mauri (2017).

and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each $S \subseteq N$ a real number $v(S)$, with $v(\emptyset) = 0$. Given a game (N, v) and $\emptyset \neq N' \subset N$, the *subgame* associated to N' is denoted by (N', v). A game (N, v) is *convex* if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. Throughout the paper, we only consider the set of convex games denoted by Γ. For $t \in \mathbb{R}$ and any game (N, v), denote by (N, v^t) the game that differs from (N, v) at most inasmuch as $v^t(N) = v(N) + t$. Note that (N, v^t) remains convex if (N, v) is convex and $t > 0$. Any $x \in \mathbb{R}^N$ defines the *inessential* game $(N, x) \in \Gamma$ by $x(S) = \sum_{i \in S} x_i$. For $(N, v) \in \Gamma$, define

$$\begin{array}{lcl} X^*(N,v) & = & \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\} - \text{the set of } \textit{feasible} \text{ payoff vectors}, \\ X(N,v) & = & \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\} - \text{the set of } \textit{pre-imputation}, \\ I(N,v) & = & \{x \in X(N,v) \,|\, x_i \geq v(\{i\}) \; \forall\, i \in N\} - \text{the set of } \textit{imputations}, \\ C(N,v) & = & \{x \in X(N,v) \,|\, x(S) \geq v(S) \; \forall\, S \subseteq N\} - \text{the } \textit{core}. \end{array}$$

A *single-valued solution* is a function σ that associates with each $(N, v) \in \Gamma$ a unique element $\sigma(N, v)$ of $X^*(N, v)$. A single-valued solution σ satisfies *Pareto optimality* (PO) if for all $(N, v) \in \Gamma$, $\sum_{i \in N} \sigma_i(N, v) = v(N)$. PO simply says that the worth of the grand coalition should be exhausted.

A *surplus-sharing problem* is a pair (x, t), where $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t \geq 0$.[4]

A *surplus-sharing rule* is a mapping that associates a unique non-negative allocation to each surplus-sharing problem (x, t). Formally, it is a function

$$f : \bigcup_{N \in \mathcal{N}} \mathbb{R}^N \times \mathbb{R}_+ \rightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^N_+$$

that satisfies, for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \geq 0$,

(i) $f(x, t) \in \mathbb{R}^N$,

(ii) $f_i(x, t) \geq 0$ for all $i \in N$ (*non negativity*), and

(iii) $\sum_{i \in N} f_i(x, t) \leq t$ (*feasibility*).[5]

Let $\mathcal{F}$ denote the set of surplus-sharing rules.

Note that $f_i(x, 0) = 0$ for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $i \in N$. A surplus-sharing rule $f \in \mathcal{F}$ satisfies *Pareto optimality* ($\mathbb{PO}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \in \mathbb{R}_+$, $\sum_{i \in N} f_i(x, t) = t$. $\mathbb{PO}$ requires that the resource t should be exhausted. Examples of

[4] Usually, in the definition of a surplus-sharing problem the condition $x \in \mathbb{R}^N_+$ is imposed. Here, we consider a more general class of problems in which no restriction on x is required.

[5] Other models incorporate additional requirements in defining a surplus-sharing rule (see, for instance, Moulin, 1987).

Pareto optimal surplus-sharing methods are the *equal sharing rule* f^{EQ} and the *proportional sharing rule* f^{PR}. Formally, for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \geq 0$, f^{EQ} is defined by setting $f_i^{EQ}(x,t) = \frac{t}{|N|}$ for all $i \in N$, and f^{PR} is defined by setting $f_i^{PR}(x,t) = \frac{x_i}{\sum_{j \in N} x_j} \cdot t$ for all $i \in N$, whenever $\sum_{j \in N} x_j \neq 0$.

3 The constrained egalitarian rule

A surplus-sharing rule just distributes an amount t of a divisible resource (e.g., money) among a set of players N that are differentiated by the *reference point* (or status quo) $x \in \mathbb{R}^N$ which, depending on the situation, can denote the opportunity cost of the players, but also their individual endowment or other objective references. In this setting, several rules[6] have been established and characterized but none of them cares about diminishing inequalities concerning the ex-post allocation process. Obviously, if we ignore the initial status quo, f^{EQ} weakly Lorenz dominates every other Pareto optimal rule f, i.e., $f^{EQ}(x,t) \succeq_{\mathcal{L}} f(x,t)$ for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \geq 0$. However, it is not difficult to find instances of surplus-sharing problems $(x,t) \in \mathbb{R}^N \times \mathbb{R}_+$ and Pareto optimal rules $f \in \mathcal{F}$ where $x + f^{EQ}(x,t)$ is Lorenz dominated by $x + f(x,t)$. In this section, we introduce the *constrained egalitarian* surplus-sharing rule, denoted by f^{CE}, and show that the final outcome $x + f^{CE}(x,t)$ weakly Lorenz dominates any other Pareto optimal distribution $x + f(x,t)$. We also provide an axiomatic justification of f^{CE}.

Definition 1. *The* constrained egalitarian surplus-sharing rule, *f^{CE}, is defined by*

$$f_i^{CE}(x,t) = (\lambda - x_i)_+ \text{ for all } N \in \mathcal{N}, x \in \mathbb{R}^N, t \in \mathbb{R}_+, \text{ and } i \in N, \tag{1}$$

where $\lambda \in \mathbb{R}$ is determined by $\sum_{k \in N} (\lambda - x_k)_+ = t$.

Thus, f^{CE} treats equals (w.r.t. the status quo) equally, and makes unequal agents equal as far as this is possible. That is, it distributes the surplus to the poorer agents so that their payoffs become equal but not larger than the remaining agents' status quo payoffs. The following remark explains how to calculate λ for any $x \in \mathbb{R}^N$ and $t > 0$, and it will be useful in our proofs.

Remark 1. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, $t > 0$, and λ be such that $f_i^{CE}(x,t) = (\lambda - x_i)_+$ for all $i \in N$. Choose $i_1, \ldots, i_n$, where $n = |N|$, such that $\{i_1, \ldots, i_n\} = N$ and $x_{i_1} \leq \cdots \leq x_{i_n}$. For $k \in \{1, \ldots, n\}$ define $\alpha_k(t) = \alpha_k = x(\{i_1, \ldots, i_k\}) - k x_{i_k} + t$ and observe that

[6] See, for instance, Moulin, 1987; Chun, 1989; Pfingsten, 1991; Pfingsten, 1998; Young, 1988.

$\alpha_1 = t > 0$ and, for $k < n$, $\alpha_k - \alpha_{k+1} = k(x_{i_{k+1}} - x_{i_k})$, hence $\alpha_1 \geq \cdots \geq \alpha_n$. Now, with $k_0 = \max\{k \in \{1,\dots,n\} \mid \alpha_k > 0\}$, we get

$$\lambda = \frac{\alpha_{k_0}}{k_0} + x_{i_{k_0}} = \frac{x(\{i_1,\dots,i_{k_0}\}) + t}{k_0}.$$

Hence, $\lambda = x_{i_k} + f^{CE}_{i_k}(x,t) < x_{i_{k'}} = x_{i_{k'}} + f^{CE}_{i_{k'}}(x,t)$, for all $k = 1,\dots,k_0$ and all $k' = k_0+1,\dots,n$.

Making use of Remark 1, we show that f^{CE} weakly Lorenz equalizes the initial differences among players in the status quo.

Lemma 1. *For all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \in \mathbb{R}_+$,*

$$x + f^{CE}(x,t) \succ_L x + z, \tag{2}$$

where $z \in \mathbb{R}^N_+, z(N) = t$, and $z \neq f^{CE}(x,t)$.

Proof. Let $i_1,\dots,i_n,k_0$, and λ be defined as in Remark 1, $y = x + f^{CE}(x,t)$, and $y' = x + z$. Let $\{j_1,\dots,j_{k_0}\} = \{i_1,\dots,i_{k_0}\}$ such that $y'_{j_1} \leq \cdots \leq y'_{j_{k_0}}$. Moreover, let $j_k = i_k$ for $k = \{k_0+1,\dots,n\}$. Then, for each $k \in \{1,\dots,n\}$, $\min\{y'(S) \mid S \subseteq N, |S| = k\} \leq y'(\{j_1,\dots,j_k\})$. Moreover, as $y_{i_j} = x_{i_j}$ for all $j \in \{k_0+1,\dots,n\}$, we have $y'_{i_j} \geq y_{i_j}$ so that, by $y'(N) = y(N)$ and $y_{j_1} = \cdots = y_{j_{k_0}} = \lambda$ we conclude that $y'(\{j_1,\dots,j_k\}) \leqslant y(\{i_1,\dots,i_k\})$ for all $k \in \{1,\dots,n\}$. Finally, as $z \neq f^{CE}(x,t)$, there is $k \in \{1,\dots,n\}$ such that $y_{i_k} \neq y'_{j_k}$ so that $y'(\{j_1,\dots,j_{k_1}\}) < y(\{i_1,\dots,i_{k_1}\})$ where k_1 is minimal in $\{1,\dots,n\}$ such that $y_{i_{k_1}} \neq y'_{j_{k_1}}$. Hence, $y \succ_{\mathcal{L}} y'$. □

Lemma 1 has the following immediate consequence.

Corollary 1. *For all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $f \in \mathcal{F}$ that satisfies $\mathbb{PO}$,*

$$x + f^{CE}(x,t) \succeq_L x + f(x,t). \tag{3}$$

In order to characterize f^{CE}, let us first recall some acceptable and well-known properties. A surplus-sharing rule $f \in \mathcal{F}$ satisfies

- *Path independence* ($\mathbb{PI}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t, t' \in \mathbb{R}_+$, $f(x, t+t') = f(x,t) + f(x + f(x,t), t')$;
- *Resource monotonicity* ($\mathbb{RM}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t, t' \in \mathbb{R}_+$ with $t' > t$, $f(x,t') \geq f(x,t)$;

- *Equal treatment of equals* ($\mathbb{ET}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $i, j \in N$, if $x_i = x_j$ then $f_i(x,t) = f_j(x,t)$.
- *Consistency* ($\mathbb{CO}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $\emptyset \neq S \subset N$, $f_S(x,t) = f\left(x_S, t - \sum_{i \in N \setminus S} f_i(x,t)\right)$.

Moulin (1987) introduces $\mathbb{PI}$, which requires that, regardless of the partition of the total surplus, its distribution may be dynamically obtained by applying the surplus-sharing rule consecutively to the given elements of the partition. $\mathbb{RM}$ is a sort of solidarity condition requiring that nobody is worse off when there is more to be divided, and $\mathbb{ET}$ imposes that equal players (w.r.t. the status quo) should receive the same amount of the resource. $\mathbb{CO}$ forces the solution to coincide in both the original and the reduced problem that results when some players leave. Except for $\mathbb{PI}$, the remaining properties are not employed in our characterization result but they will be useful in our findings.

We now introduce a new property that captures how differently non-identical agents (w.r.t. the status quo) should be treated. A surplus-sharing rule $f \in \mathcal{F}$ satisfies

- *Less first* ($\mathbb{LF}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $i, j \in N$ with $x_i - x_j \geq t$, $f_i(x,t) = 0$.

$\mathbb{LF}$ demands that if the relative welfare difference at the status quo between two agents $i, j \in N$ is large enough to exceed the total amount to be divided, i.e., $x_i - x_j \geq t$, then the agent with higher welfare gets nothing. Similar priority requirements can be found in Moulin (2000) or Timoner and Izquierdo (2016) in the context of rationing problems with asymmetries or ex-ante conditions, respectively.

In what follows we highlight some logical implications among the aforementioned properties.

Remark 2. It is immediate to check that $\mathbb{PI}$ implies $\mathbb{RM}$. Moreover, if $f \in \mathcal{F}$ satisfies $\mathbb{RM}$ and $\mathbb{PO}$, then, for all $N \in \mathcal{N}$ and all $x \in \mathbb{R}^N$, $f(x, \cdot) : \mathbb{R}_+ \to \mathbb{R}^N_+$ is a *continuous mapping.*

Remark 2 enables us to prove the following result.

Proposition 1. *The properties* $\mathbb{PO}$*,* $\mathbb{PI}$*, and* $\mathbb{LF}$ *together imply* $\mathbb{ET}$*.*

Proof. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}_+$, and $i, j \in N$ such that $x_i = x_j$. Let $f \in \mathcal{F}$ satisfy $\mathbb{PO}$, $\mathbb{PI}$ and $\mathbb{LF}$.

If $t = 0$, then by $\mathbb{PO}$ (and non negativity of f), $f_i(x,t) = f_j(x,t) = 0$.

If $t > 0$ suppose, w.l.o.g., $f_i(x,t) < f_j(x,t)$. Note that by definition, $f_j(x,t) > 0$. Moreover, since $\mathbb{PI}$ implies $\mathbb{RM}$, for all $0 \le t' \le t$ we have that $f(x,t') \le f(x,t)$. By continuity and $\mathbb{RM}$ of f (see Remark 2), $t^* = \min\{\tau \in \mathbb{R}_+ \mid f_j(x,\tau) = f_j(x,t)\}$ exists and there exists $0 < \hat{t} < t^*$ with $f_j(x,\hat{t}) - f_i(x,\hat{t}) > t^* - \hat{t}$. As $x_i = x_j$, we obtain $t^* - \hat{t} < x_j + f_j(x,\hat{t}) - (x_i + f_i(x,\hat{t}))$. Hence, by $\mathbb{LF}$, $f_j(x + f(x,\hat{t}), t^* - \hat{t}) = 0$. But then, by $\mathbb{PI}$, $f_j(x,t^*) = f_j(x,\hat{t})$ which means that $f_j(x,t) = f_j(x,\hat{t})$, and a contradiction is obtained. □

Remark 3. Let us stress that $\mathbb{PO}$, $\mathbb{RM}$, and $\mathbb{LF}$ together are not enough to guarantee $\mathbb{ET}$. Indeed, select $i \in U$ and define $f \in \mathcal{F}$ as follows. Let $N \in \mathcal{N}, x \in \mathbb{R}^N$, and $t \ge 0$. If $i \notin N$ or $i \in N$ and $x_i > x_j$ for some $j \in N \setminus \{i\}$, define $f(x,t) = f^{CE}(x,t)$.[7] If $i \in N$ and $x_i \le x_j$ for all $j \in N$, define $f_i(x,t) = t$ and $f_j(x,t) = 0$ for all $j \in N \setminus \{i\}$. Then, f satisfies $\mathbb{PO}$, $\mathbb{RM}$, and $\mathbb{LF}$ but not $\mathbb{ET}$.

Making use of both Remark 2 and Proposition 1, we show that $\mathbb{CO}$ is also a consequence of $\mathbb{PO}$, $\mathbb{PI}$, and $\mathbb{LF}$.

Proposition 2. *The properties* $\mathbb{PO}$, $\mathbb{PI}$, *and* $\mathbb{LF}$ *together imply* $\mathbb{CO}$.

Proof. Let f be a surplus-sharing rule satisfying $\mathbb{PO}$, $\mathbb{PI}$, and $\mathbb{LF}$. Let $N \in \mathcal{N}$ with $|N| \ge 3$, $x \in \mathbb{R}^N$, $t > 0$, and $\emptyset \ne S \subset N$. Suppose that $f_S(x,t) \ne f(x_S, t')$, where $t' = \sum_{i \in S} f_i(x,t)$. Then, by $\mathbb{PO}$, there exist $i, j \in S$ such that

$$x_i + f_i(x,t) > x_i + f_i(x_S, t') \tag{4}$$

and

$$x_j + f_j(x,t) < x_j + f_j(x_S, t'). \tag{5}$$

Claim: If $N \in \mathcal{N}, t > 0$, and $i, j \in N$ such that $f_i(x,t) - f_j(x,t) > x_j - x_i$, then $f_i(x,t) = 0$.

To prove the claim assume, on the contrary, that $f_i(x,t) > 0$. By Proposition 1, f satisfies $\mathbb{ET}$. Hence, $x_i \ne x_j$. If $x_i < x_j$, then, by continuity of f (see Remark 2), there exists $0 < t' < t$ such that $f_i(x,t') - f_j(x,t') = x_j - x_i$ because $f_i(x,0) - f_j(x,0) = 0 < x_j - x_i$. With $x' = x + f(x,t')$, $\mathbb{PI}$ yields $x + f(x,t) = x' + f(x', t - t')$. However, as $x'_i = x'_j$, by $\mathbb{ET}$, $x_i + f_i(x,t) = x_j + f_j(x,t)$, a contradiction. Hence, we may assume that $x_i > x_j$. By $\mathbb{RM}$ and continuity, $t_0 = \max\{\tilde{t} \ge 0 \mid f_i(x,\tilde{t}) = 0\}$ is well-defined and $t_0 < t$. If $x_i > x_j + f_j(x,t_0)$, then, by $\mathbb{LF}$, $f_i(x + f(x,t_0), \varepsilon) = 0$ for $\varepsilon > 0$ small enough which contradicts, in view of $\mathbb{PI}$, the definition of t_0. On the other hand, if $x_i \le x_j + f_j(x,t_0)$,

[7] Proposition 3 shows that f^{CE} satisfies $\mathbb{PI}$ and $\mathbb{LF}$.

then by continuity there exists $t_0 \leq t_1 < t$ such that $x_i + f_i(x, t_1) = x_i = x_j + f_j(x, t_1)$. By $\mathbb{PI}$ and $\mathbb{ET}$, we get $x_i + f_i(x, t) = x_j + f_j(x, t)$, a contradiction. This finishes the proof of the claim.

Now, we may distinguish two cases: (a) $x_i + f_i(x_S, t') \geq x_j + f_j(x_S, t')$ and (b) $x_i + f_i(x_S, t') < x_j + f_j(x_S, t')$. In case (a), inequalities (4) and (5) imply $x_i + f_i(x, t) > x_j + f_j(x, t)$, and by the claim we receive $f_i(x, t) = 0$. Thus, from (4) we obtain $0 > f_i(x_S, t')$, contradicting the non-negativity of f. In case (b), again by the claim, we receive $f_j(x_S, t') = 0$. Thus, from (5) we obtain $f_j(x, t) < 0$, contradicting the non-negativity of f. □

Now, we have the necessary intermediate results to characterize f^{CE} by means of $\mathbb{PO}$, $\mathbb{PI}$, and $\mathbb{LF}$. Clearly, f^{CE} satisfies $\mathbb{PO}$. Next, we check that it also meets $\mathbb{PI}$ and $\mathbb{LF}$.

Proposition 3. *The surplus-sharing rule f^{CE} satisfies $\mathbb{PI}$ and $\mathbb{LF}$.*

Proof. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t > 0$. We first show $\mathbb{PI}$. Let $i_1, \dots, i_n$ be defined as in Remark 1, $t = t_1 + t_2$, $t_1, t_2 > 0$,

$$k_0^1 = \max\{k \in \{1, \dots, n\} \mid x(\{i_1, \dots, i_k\}) + t_1 > k x_{i_k}\}$$

and

$$k_0 = \max\{k \in \{1, \dots, n\} \mid x(\{i_1, \dots, i_k\}) + t > k x_{i_k}\}.$$

That is, with

$$\lambda_1 = \frac{x(\{i_1, \dots, i_{k_0^1}\}) + t_1}{k_0^1} \text{ and } \lambda = \frac{x(\{i_1, \dots, i_{k_0}\}) + t}{k_0},$$

we have $f_i^{CE}(x, t_1) = (\lambda_1 - x_i)_+$ and $f_i^{CE}(x, t) = (\lambda - x_i)_+$, for all $i \in N$. Let $y = x + f^{CE}(x, t_1)$. By Remark 1, $y_{i_1} = \dots = y_{i_{k_0^1}} < y_{i_{k_0^1+1}} \leq \dots \leq y_{i_n}$ and $k_0^1 \leq k_0$. As $k_0\lambda - x(\{i_1, \dots, i_{k_0}\}) = t$ and $k_0^1\lambda_1 - x(\{i_1, \dots, i_{k_0^1}\}) = t_1$, we conclude that

$$\begin{aligned} k_0\lambda - y(\{i_1, \dots, i_{k_0}\}) &= k_0(\lambda - \lambda_1) - x(\{i_{k_0^1+1}, \dots, i_{k_0}\}) \\ &= k_0\lambda - x(\{i_1, \dots, i_{k_0}\}) + x(\{i_1, \dots, i_{k_0^1}\}) - k_0^1\lambda_1 \\ &= t - t_1 = t_2 \end{aligned}$$

so that $\mathbb{PI}$ is shown.

To show $\mathbb{LF}$, suppose there are $i, j \in N$ with $x_i - x_j \geq t$ and $f_i^{CE}(x, t) > 0$. Since $x_i \geq x_j$, $f_i^{CE}(x, t) \leq f_j^{CE}(x, t)$ and thus $f_j^{CE}(x, t) > 0$. This means that $x_i + f_i^{CE}(x, t) = x_j + f_j^{CE}(x, t)$ (see Remark 1), which implies $x_i - x_j = f_j^{CE}(x, t) - f_i^{CE}(x, t) \geq t$. But then $f_j^{CE}(x, t) > t$, contradicting $\mathbb{PO}$. Hence, $f_i^{CE}(x, t) = 0$. □

The above three propositions lead to the following characterization.

Theorem 1. *The unique surplus-sharing rule that satisfies* $\mathbb{PO}$, $\mathbb{PI}$, *and* $\mathbb{LF}$ *is* f^{CE}.

Proof. By Proposition 3, f^{CE} satisfies $\mathbb{PI}$ and $\mathbb{LF}$. $\mathbb{PO}$ is obvious. Conversely, let $f \in \mathcal{F}$ denote a rule satisfying these properties. By Proposition 1 and Proposition 2 it also satisfies $\mathbb{ET}$ and $\mathbb{CO}$. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t > 0$.

- If $|N| = 1$, by $\mathbb{PO}$, $f = f^{CE}$.
- If $N = \{i, j\}$ and $x = (x_i, x_j) \in \mathbb{R}^N$ suppose, w.l.o.g., $x_i \geq x_j$. We distinguish two cases:

  (i) $x_i - x_j \geq t$. By $\mathbb{PO}$ and $\mathbb{LF}$, $f_i(x,t) = 0$ and $f_j(x,t) = t$. Thus, $f = f^{CE}$.

  (ii) $0 \leq x_i - x_j < t$. By $\mathbb{PI}$,

  $$f(x,t) = f(x, x_i - x_j) + f(x + f(x, x_i - x_j), t - (x_i - x_j)).$$

  By $\mathbb{LF}$ and $\mathbb{PO}$, $f_i(x, x_i - x_j) = 0$ and $f_j(x, x_i - x_j) = x_i - x_j$.
  Since $x + f(x, x_i - x_j) = (x_i, x_i)$, by $\mathbb{ET}$ and $\mathbb{PO}$ we receive,

  $$f(x + f(x, x_i - x_j), t - (x_i - x_j)) = \left(\frac{t - (x_i - x_j)}{2}, \frac{t - (x_i - x_j)}{2}\right).$$

  Hence, $f(x,t) = \left(\frac{t-(x_i-x_j)}{2}, \frac{t+(x_i-x_j)}{2}\right) = f^{CE}(x,t)$.

- If $|N| \geq 3$ and $f(x,t) \neq f^{CE}(x,t)$, then, by $\mathbb{PO}$, there exist $i, j \in N$ such that $f_i(x,t) > f_i^{CEG}(x,t)$ and $f_j(x,t) < f_j^{CEG}(x,t)$. Let $S = \{i, j\}$. Suppose that $f_i(x,t) + f_j(x,t) \geq f_i^{CE}(x,t) + f_j^{CE}(x,t)$. Taking into account that $f = f^{CE}$ for the two agent case, $f(x_S, f_i(x,t) + f_j(x,t)) = f^{CE}(x_S, f_i(x,t) + f_j(x,t))$. By $\mathbb{CO}$ and $\mathbb{RM}$ we obtain

  $$\begin{aligned} f_j(x,t) &= f_j(x_S, f_i(x,t) + f_j(x,t)) \\ &= f_j^{CE}(x_S, f_i(x,t) + f_j(x,t)) \\ &\geq f_j^{CE}(x_S, f_i^{CE}(x,t) + f_j^{CE}(x,t)) \\ &= f_j^{CE}(x,t), \end{aligned}$$

  which contradicts the assumption $f_j(x,t) < f_j^{CEG}(x,t)$. If, on the contrary, $f_i(x,t) + f_j(x,t) < f_i^{CE}(x,t) + f_j^{CE}(x,t)$, then we get the contradiction $f_i^{CE}(x,t) > f_i(x,t)$. Hence, we conclude that $f = f^{CE}$.

$\square$

4 Game theoretical support: the Dutta-Ray solution

Translating $\mathbb{LF}$ into a new property for games and combining it with aggregate monotonicity, we provide a number of characterizations of the *egalitarian solution* of Dutta and Ray (1989) on the class of convex games. We use the central observation (Lemma 3) that the egalitarian solution obeys f^{CE} when dividing an increase in the value of the grand coalition.

In order to recall the definition of the egalitarian solution, the following lemma is useful.

Lemma 2. *Let $(N, v) \in \Gamma$ and denote $\mu = \max_{\emptyset \neq S \subseteq N} \frac{v(S)}{|S|}$. If $\emptyset \neq S, T \subseteq N$ are such that $v(S) = \mu|S|$ and $v(T) = \mu|T|$, then $v(S \cup T) = \mu|S \cup T|$.*

Proof. Note that, by convexity of (N, v), $v(S \cup T) + v(S \cap T) \geq v(S) + v(T) = \mu(|S| + |T|)$ and, by definition of μ, $v(S \cap T) \leq \mu(|S \cap T|)$. Therefore, $v(S \cup T) + \mu(|S \cap T|) \geq \mu(|S| + |T|) = \mu(|S \cup T| + |S \cap T|)$ and, hence, $v(S \cup T) \geq \mu(|S \cup T|)$ so that $v(S \cup T) = \mu(|S \cup T|)$ by the definition of μ. □

Let $(N, v) \in \Gamma$ and denote

$$\mu(v) = \max_{\emptyset \neq S \subseteq N} \frac{v(S)}{|S|} \text{ and } S(v) = \bigcup \{S \in 2^N \setminus \{\emptyset\} \mid v(S) = \mu(v)|S|\}.$$

By Lemma 2, $\mu(v)|S(v)| = v(S(v))$. Now, we are able to introduce the definition of the egalitarian solution of (N, v), denoted by $L(N, v)$. Namely, let $(S_1, \ldots, S_m)$ be the ordered partition of N that is recursively determined by the requirement that $S_k = S(v_k)$, where $S_0 = \emptyset$ and for all $k = 1, \ldots, m$, $N_k = N \setminus \bigcup_{j=0}^{k-1} S_j$ and (N_k, v_k) is defined by $v_k(T) = v(T \cup (N \setminus N_k)) - v(N \setminus N_k)$ for all $T \subseteq N_k$. Note that $v_1 = v$ and $(N_k, v_k) \in \Gamma$ so that S_k is well defined. The egalitarian solution $L(N, v) = \{x^*(N, v)\}$ is given by

$$x_i^*(N, v) = \mu(v_k) = \frac{v_k(S_k)}{|S_k|} \text{ for all } i \in N_k \text{ and all } k = 1, \ldots, m. \quad (6)$$

Remark 4. Let $(N, v) \in \Gamma$. The unique element x^* of $L(N, v)$ satisfies the following properties:

$$\begin{array}{rcl} \sum_{t=1}^{k} x^*(S_t) & = & v\left(\bigcup_{t=1}^{k} S_t\right) \text{ for all } k = 1, \ldots m; \\ x_i^* & = & x_j^* \text{ for all } i, j \in S_k \text{ and all } k = 1, \ldots, m; \\ x_i^* & > & x_j^* \text{ for all } i \in S_t, j \in S_k \text{ and all } 1 \leq t < k \leq m. \end{array}$$

Moreover, according to Theorem 3 of Dutta and Ray (1989) the egalitarian solution L selects the unique core element that Lorenz dominates every other core element. That is, $x^* \in C(N, v)$ and $x^* \succ_{\mathcal{L}} y$ for all $y \in C(N, v) \setminus \{x^*\}$.

Aggregate monotonicity (Megiddo, 1974) will play a distinguished role in our characterizations. A single-valued solution σ satisfies

- *Aggregate monotonicity* (AM) if for all $(N, v) \in \Gamma$ and all $t > 0$, $\sigma(N, v^t) \geq \sigma(N, v)$.

We now show that the egalitarian solution L distributes any variation in the worth of the grand coalition monotonically among players according to f^{CE}, which also provides a game theoretical support for the constrained egalitarian rule.

Lemma 3. *Let $(N, v) \in \Gamma$, $t > 0$, and $L(N, v) = x^*$. Then, $L(N, v^t) = x^* + f^{CE}(x^*, t)$.*

Proof. Suppose that $L(N, v^t) \neq x^* + f^{CE}(x^*, t)$. Then, since $x^* + f^{CE}(x^*, t) \in C(N, v)$, we have that $L(N, v^t) \succ_L x^* + f^{CE}(x^*, t)$. By AM and PO, $L(N, v^t) = x^* + z$, where $z \in \mathbb{R}^N_+$ with $z(N) = t$. Thus, we receive $x^* + z \succ_L x^* + f^{CE}(x^*, t)$, contradicting Lemma 1. Hence, $L(N, v^t) = x^* + f^{CE}(x^*, t)$. □

The above result shows that, on a sequence of convex games with increasing worth of the grand coalition, the egalitarian solution L evolves dynamically in the sense that it assigns an allocation in each subsequent period that is uniquely determined by the allocation of the previous period.

In the following items, we introduce a number of well-established properties on the domain of convex games, some of which will be used in our characterization results. A single-valued solution σ satisfies

- *Constrained egalitarianism* (CE) if for all $(N, v) \in \Gamma$ with $N = \{i, j\}, i \neq j$, and $v(\{i\}) \leq v(\{j\})$, $\sigma_j(N, v) = \max\left\{\frac{v(N)}{2}, v(\{j\})\right\}$ and $\sigma_i(N, v) = v(N) - \sigma_j(N, v)$;
- *Weak continuity* (WC) if for all $(N, v) \in \Gamma$ and all sequences $(\alpha_k)_{k \in \mathbb{N}}$ with limit $v(N)$ such that (a) the games (N, v^k) that differ from (N, v) at most inasmuch as $v^k(N) = \alpha_k$ are convex and (b) $(\sigma(N, v^k))_{k \in \mathbb{N}}$ converges to some $x \in \mathbb{R}^N$, $x = \sigma(N, v)$;
- *Individual rationality* (IR) if for all $(N, v) \in \Gamma$ and all $i \in N$, $\sigma_i(N, v) \geq v(\{i\})$;
- *Core selection* (CS) if for all $(N, v) \in \Gamma$, $\sigma(N, v) \in C(N, v)$.

Note that CE determines the solution for two-player games. IR requires that no single player can improve the payoff proposed by the solution without cooperation, while CS is a sort of secession-proofness property since CS requires that all coalitions receive at least what they can get by themselves. Moreover, WC is a continuity condition that applies for sequences of convex games that only differ in the worth of the grand coalition. It is not difficult to check that PO and AM together imply WC. The egalitarian solution L satisfies all the above properties.

Now we introduce consistency properties that refer to suitable notions of reduced games. A single-valued solution σ satisfies

- the *Reduced game property* (RGP) if for all $(N, v) \in \Gamma$ and all $\emptyset \neq S \subset N$, $(S, v_{S,x}) \in \Gamma$ and $\sigma(S, v_{S,x}) = x_S$ where $x = \sigma(N, v)$ and $(S, v_{S,x})$ is the game defined by $v_{S,x}(S) = v(N) - x(N \setminus S)$ and $v(T) = \max_{Q \subseteq N \setminus S}\{v(T \cup Q) - x(Q)\}$ for all $\emptyset \neq T \subset S$;[8]

- *Consistency* (CON) if for all $(N, v) \in \Gamma$ and all $\emptyset \neq S \subset N$, $(S, v_{S,\sigma}) \in \Gamma$ and $\sigma(S, v_{S,\sigma}) = x_S$ for all $\emptyset \neq S \subset N$ and all $i \in S$ where $x = \sigma(N, v)$ and $(S, v_{S,\sigma})$ is the game defined by

$$v_{S,\sigma}(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} \sigma_i(T \cup (N \setminus S), v) \text{ for all } \emptyset \neq T \subseteq S.$$[9]

  The *bilateral reduced game property* (2-RGP) and *bilateral consistency* (2-CON) only require RGP and CON for $|S| = 2$, respectively.

Roughly speaking, consistency requires that in the corresponding reduced game the original agreement should be confirmed. The above definitions are due to Sobolev (1975) and Hart and Mas-Colell (1989), respectively. The egalitarian solution L satisfies RGP on Γ. Moreover, as was shown by Hokari (2002), it satisfies 2-CON but violates CON on Γ.[10] Following the proofs of Theorem 5.3 and Theorem 5.4 in Dutta (1990), it can be checked that CE together with 2-RGP and 2-CON, respectively, characterize the egalitarian solution L on Γ.

4.1 Characterizations with consistency

To characterize the egalitarian solution L, together with consistency and AM we will impose a new property, that is reminiscent of $\mathbb{LF}$ in the surplus-sharing setting. A single-valued solution σ satisfies

[8] The game $(S, v_{S,x})$ is called *reduced game* of (N, v) w.r.t. S at x and was introduced by Davis and Maschler (1965).

[9] The game $(S, v_{S,\sigma})$ is called the *σ-reduced game* of (N, v) w.r.t. S at σ and was introduced by Hart and Mas-Colell (1989). Note that the set of convex games Γ is closed under taking subgames.

[10] See Example 1 in Hokari (2002).

- *Bounded pairwise fairness* (BPF) if for all $(N,v) \in \Gamma$, all $t > 0$, and all $i,j \in N$ such that $\sigma_i(N,v) - \sigma_j(N,v) \geq t$, $\sigma_i(N,v^t) \leq \sigma_i(N,v)$.

The property of BPF is a priority requirement imposing that, if the difference in payoffs between two players in the initial game (N,v) exceeds the total additional amount t to be divided, then in the game (N,v^t) the richer player can not gain.

Proposition 4. *The egalitarian solution satisfies* BPF*.*

Proof. Let $(N,v) \in \Gamma$, $t > 0$, and $i,j \in N$ such that $L_i(N,v) - L_j(N,v) \geq t$. By Lemma 3, $L(N,v^t) = L(N,v) + f^{CE}(L(N,v),t)$. Since f^{CE} obeys $\mathbb{LF}$, $f_i^{CEG}(L(N,v),t) = 0$ and thus $L_i(N,v^t) = L_i(N,v)$. □

The next result has the flavor of Proposition 1. While in the framework of surplus-sharing problems $\mathbb{PO}$, $\mathbb{RM}$, and $\mathbb{LF}$ together do not imply $\mathbb{ET}$ (see Remark 3), here PO, AM, and BPF are enough to ensure a kind of equal treatment property that only applies when the worth of the grand coalition increases.

Lemma 4. *Let σ be a single-valued solution that satisfies* PO, AM*, and* BPF*. For all $(N,v) \in \Gamma$, all $i,j \in N$, and all $t \in \mathbb{R}_+$, if $\sigma_i(N,v) = \sigma_j(N,v)$, then $\sigma_i(N,v^t) = \sigma_j(N,v^t)$.*

Proof. Let σ be a single-valued solution satisfying PO, AM and BPF. Suppose, on the contrary, there exist $i,j \in N$ such that $\sigma_i(N,v) = \sigma_j(N,v)$ but $\sigma_j(N,v^t) > \sigma_i(N,v^t)$. By PO and AM, σ meets WC. Therefore, there exists a minimal $t^* \in (0,t]$ such that $\sigma_j(N,v^{t^*}) = \sigma_j(N,v^t)$. Hence,

$$\sigma_j(N,v^{t''}) < \sigma_j(N,v^{t^*}) = \sigma_j(N,v^t) \text{ for all } t'' \in [0,t^*). \tag{7}$$

Note that $\sigma_j(N,v) < \sigma_j(N,v^t)$ since, otherwise, $\sigma_i(N,v) = \sigma_j(N,v) = \sigma_j(N,v^t) > \sigma_i(N,v^t)$, contradicting AM. Let $\hat{t} \in (0,t^*)$ such that $2 \cdot \left(v^{t^*}(N) - v^{\hat{t}}(N)\right) \leq \sigma_j(N,v^t) - \sigma_i(N,v^t)$. By PO and AM, we obtain

$$\begin{array}{rcl} 2 \cdot (v^{t^*}(N) - v^{\hat{t}}(N)) & \leq & \sigma_j(N,v^t) - \sigma_i(N,v^t) \\ & \leq & \sigma_j(N,v^{t^*}) - \sigma_i(N,v^{t^*}) \\ & = & \sigma_j(N,v^{\hat{t}}) - \sigma_i(N,v^{\hat{t}}) + \sigma_j(N,v^{t^*}) - \sigma_j(N,v^{\hat{t}}) \\ & & -(\sigma_i(N,v^{t^*}) - \sigma_i(N,v^{\hat{t}})) \\ & \leq & \sigma_j(N,v^{\hat{t}}) - \sigma_i(N,v^{\hat{t}}) + \sum_{j \in N} \left(\sigma_j(N,v^{t^*}) - \sigma_j(N,v^{\hat{t}})\right) \\ & & -(\sigma_i(N,v^{t^*}) - \sigma_i(N,v^{\hat{t}})) \\ & \leq & \sigma_j(N,v^{\hat{t}}) - \sigma_i(N,v^{\hat{t}}) + v^{t^*}(N) - v^{\hat{t}}(N). \end{array} \tag{8}$$

Hence, $v^{t^*}(N) - v^{\hat{t}}(N) \leq \sigma_j(N, v^{\hat{t}}) - \sigma_i(N, v^{\hat{t}})$. Now, by AM and BPF, $\sigma_j(N, v^{t^*}) = \sigma_j(N, v^{\hat{t}})$, contradicting (7). □

Next, we generalize Lemma 3 to show that not only the egalitarian solution L but all single-valued solutions satisfying PO, AM, and BPF distribute any growth in the value of the grand coalition according to f^{CE}.

Lemma 5. *Let σ be a single-valued solution satisfying* PO, AM, *and* BPF. *For all $(N, v) \in \Gamma$ and all $t \in \mathbb{R}_+$, $\sigma(N, v^t) = \sigma(N, v) + f^{CE}(\sigma(N, v), t)$.*

Proof. Let us denote $\sigma(N, v) = x$ and $\sigma(N, v^t) = x^t$. Let $\mathcal{P}(x) = (N_1, N_2, \ldots, N_k)$ be the ordered partition of N as defined in Section 2. We proceed by induction on $|\mathcal{P}(x)|$.

If $k = 1$, by PO, $x_i = \frac{v(N)}{n}$ for all $i \in N$, where $|N| = n$. Hence, by Lemma 4, $x_i^t = x_j^t$ for all $i, j \in N$, and by PO, for all $i \in N$,

$$x_i^t = \frac{v^t(N)}{n} = \frac{v(N)}{n} + \frac{t}{n} = x_i + f_i^{CE}(x, t),$$

where the last equality comes from $\mathbb{ET}$ of f^{CE}.

Induction hypothesis: $x^t = x + f^{CE}(x, t)$ whenever $k < \ell$ for some $\ell \in \mathbb{N}$, $\ell > 1$.

We now assume $k = \ell$. Take $i_1 \in N_1$, with $n_1 = |N_1|$ and $i_2 \in N_2$. We distinguish two cases:

(i) $x_{i_2} - x_{i_1} \geq \frac{t}{n_1}$. By Lemma 4, for all $i, j \in N_1$, $x_i^t = x_j^t$, and AM together with BPF lead to $x_i^t = x_i$ for all $i \in N \setminus N_1$. Now, taking into account that f^{CE} satisfies $\mathbb{LF}$ and $\mathbb{ET}$, we have that $x^t = x + f^{CE}(x, t)$.

(ii) $x_{i_2} - x_{i_1} < \frac{t}{n_1}$. Let $t' = n_1(x_{i_2} - x_{i_1})$ and $\sigma(N, v^{t'}) = x^{t'}$. Note that $t - t' > 0$. By BPF, $x_i^{t'} = x_i$ for all $i \in N \setminus N_1$. By Lemma 4 and PO, $x_i^{t'} = x_i + (x_{i_2} - x_{i_1}) = x_{i_2}$ for all $i \in N_1$. Since $|\mathcal{P}(x^{t'})| = \ell - 1$, by induction hypothesis $x^t = x^{t'} + f^{CE}(x^{t'}, t - t')$. Moreover, from $\mathbb{LF}$ and $\mathbb{ET}$ of f^{CE} we receive $x^{t'} = x + f^{CE}(x, t')$. Finally, from $\mathbb{PI}$ of f^{CE} we obtain

$$\begin{aligned} x^t &= x^{t'} + f^{CE}(x^{t'}, t - t') \\ &= x + f^{CE}(x, t') + f^{CE}(x + f^{CE}(x, t'), t - t') \\ &= x + f^{CE}(x, t). \end{aligned}$$

□

If we additionally impose IR we get CE, which allows us to use, together with consistency, Dutta's (1990) results. With this aim, we first characterize the family of single-valued solutions that satisfies PO, IR, AM, and BPF. To do so, let us first introduce the notion of *convex root game*.

Given $(N, v) \in \Gamma$, the *convex root game* of (N, v), denoted by (N, v_r), is the convex game with the smallest worth of the grand coalition such that $v_r(S) = v(S)$ for all $S \subset N$. That is, $(N, v_r) = (N, v^\tau)$ where τ is such that $(N, v^t) \notin \Gamma$ for all $t < \tau$. Note that (N, v_r) is well defined since $v_r(N) = \max\{v(S) + v(T) - v(S \cap T) \mid S, T \subseteq N, S \cup T = N\}$. Moreover, both games $(N, v), (N, v^t) \in \Gamma$, $t \in \mathbb{R}$, have the same convex root game (N, v_r). By Γ_{root} we denote the set of convex root games.

Definition 2. *An* imputation-selection for convex root games *is a function* $\gamma : \Gamma_{root} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ *such that* $\gamma(N, v) \in I(N, v)$ *for all* $(N, v) \in \Gamma_{root}$.

An imputation-selection for convex root games (cp. Calleja at al., 2012) chooses, for any convex root game, a unique element of its imputation set.

Theorem 2. *A single-valued solution satisfies* PO, IR, AM*, and* BPF *if and only if there exists an imputation-selection for convex root games* γ *such that*

$$\sigma(N, v) = \gamma(N, v_r) + f^{CE}(\gamma(N, v_r), v(N) - v_r(N)), \tag{9}$$

for all $(N, v) \in \Gamma$.

Proof. Let σ be a single-valued solution satisfying PO, IR, AM, and BPF. For all $(N, v) \in \Gamma$, define the imputation-selection for convex root games γ as $\gamma(N, v_r) = \sigma(N, v_r)$. Now, by Lemma 5, we receive $\sigma(N, v) = \gamma(N, v_r) + f^{CE}(\gamma(N, v_r), v(N) - v_r(N))$.

To prove the reverse implication, let σ be a single-valued solution and let γ be an imputation-selection for convex root games such that

$$\sigma(N, v) = \gamma(N, v_r) + f^{CE}(\gamma(N, v_r), v(N) - v_r(N)),$$

for all $(N, v) \in \Gamma$. Clearly, σ satisfies PO, IR, and AM. To check BPF, for all $t > 0$, by $\mathbb{PI}$ of f^{CE} we have

$$\begin{array}{rcl} \sigma(N, v^t) & = & \gamma(N, (v^t)_r) + f^{CE}(\gamma(N, (v^t)_r), v^t(N) - (v^t)_r(N)) \\ & = & \gamma(N, v_r) + f^{CE}(\gamma(N, v_r), v(N) + t - v_r(N)) \\ & = & \gamma(N, v_r) + f^{CE}(\gamma(N, v_r), v(N) - v_r(N)) \\ & & + f^{CE}(\gamma(N, v_r) + f^{CE}(\gamma(N, v_r), v(N) - v_r(N), t)) \\ & = & \sigma(N, v) + f^{CE}(\sigma(N, v), t). \end{array} \tag{10}$$

Let $i, j \in N$ with $\sigma_i(N, v) - \sigma_j(N, v) \geq t$. Then, by $\mathbb{LF}$ of f^{CE}, $f_i^{CE}(\sigma(N, v), t) = 0$. Now, from (10) we receive $\sigma_i(N, v^t) = \sigma_i(N, v)$, which proves BPF. $\square$

As a direct consequence of Theorem 2 we receive CE.

Corollary 2. *Let σ be a single-valued solution satisfying* PO, IR, AM*, and* BPF*. Then, σ also satisfies* CE*.*

Proof. If $(N, v) \in \Gamma$ is a 2-person game, then the convex root game of (N, v) is inessential so that its unique imputation is the unique element of the core. Hence, $\sigma(N, v) = L(N, v)$ by Theorem 2. □

As mentioned, the egalitarian solution L satisfies PO, IR, BPF, and AM. Corollary 2 and Dutta's (1990) results imply that these properties, together with either 2-RGP or 2-CON, characterize L. Remarkably, it turns out that when imposing consistency some of the aforementioned properties become redundant.

Occasionally we consider single-valued solutions on a domain of games Γ' that is a subset of Γ, the domain of all convex games. In this case, in the definitions of the various properties, the requirement that the games belong to Γ must be replaced by the requirement that the games belong to Γ'.

Proposition 5. *On the domain of convex games with at least two players, if the single-valued solution σ satisfies* 2-RGP*, then σ satisfies* CS *as well.*

Proof. Let (N, v) be a convex game. We consider two cases:

(i) $|N| = 2$. By the assumption $|U| \geq 3$ there exists $k \in U \setminus N$. Let $M = N \cup \{k\}$ and (M, w) be the game that arises from (N, v) by adding the null player k, i.e., w is given by $w(S) = v(S \cap N)$ for al $S \subseteq M$. Note that (M, w) is still convex.

Claim: If (N, v) is inessential, then $\sigma(N, v)$ is the unique element of $C(N, v)$.
In order to show the claim, note that (M, w) is inessential. Let $y \in \mathbb{R}^M$ be defined by $y_i = w(\{i\})$ for all $i \in M$, hence $y(S) = w(S)$ for all $S \subseteq M$. Moreover, let $x = \sigma(M, w)$. For any $i \in M$, by the definition of the Davis-Maschler reduced game, $w_{M\setminus\{i\},x}(\{j\}) \geq w(\{i, j\}) - x_i = y(\{i, j\}) - x_i$ for both $j \in M \setminus \{i\}$ and $w_{M\setminus\{i\},x}(M \setminus \{i\}) = w(M) - x_i = y(M) - x_i$. By 2-RGP, $(M \setminus \{i\}, w_{M\setminus\{i\},x})$ is convex so that $\sum_{j\in M\setminus\{i\}} w_{M\setminus\{i\},x}(\{j\}) \leq w_{M\setminus\{i\},x}(M \setminus \{i\})$. We conclude that

$$\sum_{j\in M\setminus\{i\}} y(\{i, j\}) - x_i = y(M) + y_i - 2x_i \leq y(M) - x_i,$$

hence $x_i \geq y_i$ for all $i \in M$. Now, as $x(M) \leqslant w(M) = y(M)$, we have $x = y$. Finally, since $(N, w_{N,y}) = (N, v)$, the claim follows from 2-RGP.

Now let $x = \sigma(M, w)$, $i \in N$, and $N = \{i, j\}$. By 2-RGP, $(M \setminus \{i\}, w_{M\setminus\{i\},x})$ is convex and $x_{M\setminus\{i\}} = \sigma(M \setminus \{i\}, w_{M\setminus\{i\},x})$. By definition of the Davis-Maschler reduced game,

$$w_{M\setminus\{i\},x}(\{j\}) = \max\{w(\{i,j\}) - x_i, w(\{j\})\} = \max\{v(N) - x_i, v(\{j\})\},$$

$$w_{M\setminus\{i\},x}(\{k\}) = \max\{w(\{i,k\}) - x_i, w(\{k\})\} = \max\{v(\{i\}) - x_i, 0\},$$

and

$$w_{M\setminus\{i\},x}(M \setminus \{i\}) = w(M) - x_i = v(N) - x_i$$

so that 2-RGP implies $x_i \geq v(\{i\})$ and $v(N) - x_i \geq v(\{j\})$. We conclude that $(M \setminus \{i\}, w_{N\setminus\{i\},x})$ is inessential and thus, by 2-RGP and our claim, $x_j = v(N) - x_i$ and $x_k = 0$. Therefore, $x_N \in C(N, v)$ and the proof is finished by 2-RGP.

(ii) $|N| \geq 3$. Let $x = \sigma(N, v)$ and assume that $x \notin C(N, v)$. If $x(N) < v(N)$ select any $S \subseteq N$ with $|S| = 2$. By 2-RGP, $(S, v_{S,x}) \in \Gamma$ and $x_S = \sigma(S, v_{S,x})$. Now $v_{S,x}(S) = v(N) - x(N \setminus S) > x(S)$ so that $x_S \notin C(S, v_{S,x})$ which contradicts case (i). Therefore, we may assume that $x(N) = v(N)$ and $x(T) < v(T)$ for some $\emptyset \neq T \subsetneq N$ so that there exist $i \in T$ and $j \in N \setminus T$. Let $S = \{i, j\}$ and observe that $v_{S,x}(\{i\}) \geq v(T) - x(T \setminus \{i\}) > x_i$ by definition of the Davis-Maschler reduced game. Therefore x_S is not individually rational for $(S, v_{S,x})$ and the desired contradiction is obtained by 2-RGP and case (i).

□

Since CS implies both PO and IR, combining Corollary 2, Proposition 5, and Dutta's (1990) Theorem 5.3, we obtain the following characterization.

Theorem 3. *On the domain of convex games with at least two players, the egalitarian solution L is the unique single-valued solution that satisfies* AM, BPF*, and* 2-RGP*.*

Remark 5. If we consider the set of all games, including all 1-person games, Theorem 3 does not hold. Indeed, let $(N, v) \in \Gamma$ and $\varepsilon > 0$. Define the single-valued solution ρ as follows: $\rho(N, v) = L(N, v)$ if $|N| \geq 2$, and $\rho(N, v) = v(N) - \varepsilon$ otherwise. Then, ρ satisfies AM, BPF, and 2-RGP, but $\rho(N, v) \neq L(N, v)$.

Unfortunately, we do not know if 2-CON implies CS. However, if we additionally impose IR, then we can show that at least PO is also satisfied.

Proposition 6. *On the domain of convex games with at least two players, if the single-valued solution σ satisfies* IR *and* 2-CON*, then σ satisfies* PO *as well.*

Proof. Let $(N, v) \in \Gamma$. If $|N| = 1$, the proof is finished by IR (and feasibility).

If $|N| = 2$, by the assumption $|U| \geq 3$ there exists $k \in U \setminus N$. Let $M = N \cup \{k\}$ and (M, w) be the game that arises from (N, v) by adding the null player k, i.e., w is given by $w(S) = v(S \cap N)$ for al $S \subseteq M$. Note that (M, w) is still convex. Recall that, if (N, v) is inessential, then $\sigma(N, v)$ is the unique element of $C(N, v)$ by IR. Let $x = \sigma(M, w)$, $i \in N$, and $N = \{i, j\}$. Then $w_{M\setminus\{i\},\sigma}(\{j\}) = v(N) - \sigma_i(N, v)$ and $w_{M\setminus\{i\},\sigma}(\{k\}) = v(\{i\}) - \sigma_i(\{i, k\}, w) = 0$, where the last equation follows because $(\{i, k\}, w)$ is inessential. By IR and 2-CON, $x_j \geq v(N) - \sigma_i(N, v)$ and $x_k \geq 0$. Let $y = \sigma(N, v)$. As $y(N) \leq v(N)$ and $x_j \geq v(N) - y_i$ and, analogously, $x_i \geq v(N) - y_j$, we have $v(N) \geq x(M) \geq 2v(N) - y(N) + x_k \geq v(N) + x_k \geq v(N)$ so that all inequalities must be equations, i.e., $x_i + x_j = v(N)$ and $x_k = 0$. Hence, x is Pareto optimal.

If $|N| \geq 3$, assume that $x = \sigma(N, v)$ satisfies $x(N) < v(N)$, then, for any $S \subseteq N$ with $|S| = 2$, $x(S) < v_{S,\sigma}(S) = v(N) - x(N \setminus S)$, a contradiction. □

Now, combining Corollary 2, Proposition 6, and Dutta's (1990) Theorem 5.4, we get the following characterization.

Theorem 4. *The egalitarian solution* L *is the unique single-valued solution that satisfies* IR, AM, BPF, *and* 2-CON.

4.2 Characterizations without consistency

Finally, we provide two additional characterizations of the egalitarian solution L without employing any reduced game property. In the first one, BPF and consistency are replaced by the following property. A single valued solution σ satisfies

- *Bounded richness* (BR) if for all $(N, v) \in \Gamma$, $\sum_{i\in N\setminus S} \sigma_i(N, v) \leq v(N \setminus S)$, where $S = \{i \in N | \sigma_i(N, v) \leq \sigma_j(N, v) \, \forall \, j \in N\}$.

Thus, BR requires that the poorest players cannot be made richer by payoff transfers from the rest of the players unless the coalition of these richer players keeps less than what it can guarantee by cooperation.

Theorem 5. *The egalitarian solution* L *is the unique single-valued solution that satisfies* CS, AM*, and* BR*.*

Proof. It is well known that the egalitarian solution L satisfies CS and AM. Note that CS implies PO. By Remark 4 it also satisfies BR. To show uniqueness, let σ be a single-valued solution satisfying these properties. Let $(N, v) \in \Gamma$. Denote $x = \sigma(N, v)$. By

CS, $x \in C(N,v)$. Let $x^* = L(N,v)$, m, S_0, S_k, N_k and (N, v_k) for $k = 1, \dots, m$ be defined in (6) and the preceding paragraph. It remains to show that $x = x^*$. Let $\alpha = \min\{x_i \mid i \in N\}$ and $S = \{i \in N \mid x_i = \alpha\}$. We proceed by induction on m.

If $m = 1$ then, by PO of x and x^*, $\alpha \leq \frac{v(N)}{|N|} = x_j^*$ for all $j \in N$. Hence, by BR and CS, $v(N \setminus S) = x(N \setminus S) = v(N) - x(S) \geq v(N) - x^*(S) = x^*(N \setminus S)$. We conclude that $x(S) = x^*(S), S = N$ and $x = x^*$.

Induction hypothesis: $\sigma(N,v) = L(N,v)$ whenever $m < l$ for some $l \in \mathbb{N}$, $l > 1$.

We now assume that $m = l$. Put

$$t = |S_m| \left(\frac{v_{m-1}(S_{m-1})}{|S_{m-1}|} - \frac{v_m(S_m)}{|S_m|} \right) > 0$$

and observe that $y^* \in \mathbb{R}^N$ defined by $y_i^* = \max\{x_i^*, v_{m-1}(S_{m-1})/|S_{m-1}|\}$ for all $i \in N$ is the egalitarian solution of (N, v^t). Hence, by induction hypothesis, $\sigma(N, v^t) = y^*$. By AM, $x \leq y^*$. By CS, Remark 4 implies $x_i = x_i^*$ for all $i \in N \setminus S_m$. By PO, $\alpha \leq x_i^*$ for all $i \in N$. Hence, by BR and CS, $v(N \setminus S) = x(N \setminus S) = v(N) - x(S) \geq v(N) - x^*(S) = x^*(N \setminus S) = v(N \setminus S)$. We conclude that $\alpha = \min\{x_i^* \mid i \in N\}$ and, hence, $S = S_m$ and $x = x^*$. □

If we employ a property that is slightly stronger than BR, then even AM becomes redundant. Indeed, a single-valued solution σ satisfies

- *Strong bounded richness* (SBR) if for all $(N,v) \in \Gamma$, $\sum_{i \in N \setminus S} \sigma_i(N,v) \leq v(N \setminus S)$ for all $\alpha \in \mathbb{R}$, where $S = \{i \in N \mid \sigma_i(N,v) < \alpha\}$.

Similarly to BR, SBR requires that players who are poorer than any wealth level α each cannot be made richer by payoff transfers from the rest of the players unless the coalition of these richer players keeps less than what it can guarantee by cooperation.

Theorem 6. *The egalitarian solution L is the unique single-valued solution that satisfies* CS *and* SBR*.*

Proof. Indeed, L satisfies CS and SBR by Remark 4. To show uniqueness, let σ be a single-valued solution satisfying CS and SBR. Let $(N,v) \in \Gamma$. Denote $x = \sigma(N,v)$ and let $x^* = L(N,v)$, m, S_0, S_k, N_k, (N, v_k) for $k = 1, \dots, m$ be defined in (6) and the preceding paragraph. It remains to show that $x = x^*$. Assume, on the contrary, $x \neq x^*$. Let m be minimal such that there exists $i \in S_m$ with $x_i < x_i^* =: \alpha$. Let $S = \{j \in N \mid x_j < \alpha\}$ and $T = N \setminus S$. Hence $T \supseteq \bigcup_{k=1}^{m-1} S_k$ and $x_j \geq x_j^*$ for all $j \in T$. By SBR and CS, $x(T) = v(T) = x^*(T)$, hence $x_j = x_j^* \geq \alpha$ for all $j \in T$. As $i \in S_m \setminus T$,

$T \subset \bigcup_{k=1}^m S_k$, hence $x\left(\bigcup_{k=1}^m S_k\right) < x^*\left(\bigcup_{k=1}^m S_k\right) = v\left(\bigcup_{k=1}^m S_k\right)$, and a contradiction to CS is obtained. □

5 Logical independence of the properties

In this section we show that, except AM in Theorem 3 and Theorem 4, all other properties employed in the characterization results of Section 3, Subsection 4.1, and Subsection 4.2 are logically independent of the remaining properties.

Remark 6. The following examples show that each of the properties in Theorem 1 is logically independent of the remaining properties:

(i) The surplus-sharing rule f^{EQ} satisfies $\mathbb{PO}$, $\mathbb{PI}$ but not $\mathbb{LF}$.

(ii) Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t \geq 0$. Denote $N_1 = \{i \in N \mid x_i \leq x_j \, \forall j \in N\}$. Define $f_k^{\leq}(x,t) = \frac{t}{|N_1|}$ if $k \in N_1$ and $f_k^{\leq}(x,t) = 0$ if $k \in N \setminus N_1$. Then, $f^{\leq}$ satisfies $\mathbb{PO}$ and $\mathbb{LF}$ but not $\mathbb{PI}$.

(iii) For all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \geq 0$, the surplus-sharing rule f^0 defined by $f^0(x,t) = (0,0,\ldots,0)$ satisfies $\mathbb{PI}$ and $\mathbb{LF}$ but not $\mathbb{PO}$.

Remark 7. Each of the properties in Theorem 2 is logically independent of the remaining properties, even for two-person games.

(i) For all $(N,v) \in \Gamma$, and all $i \in N$, the *equal split solution*, ED, is defined by $ED_i(N,v) = v(N)/|N|$. Then, ED satisfies PO, AM, and BPF but not IR.

(ii) Let $\prec$ be a strict total order on U and $\preceq$ its reflexive cover. For all $(N,v) \in \Gamma$ and all $i \in N$ define the *marginal contribution solution relative to* $\prec$ as follows:

$$mc_i^{\prec}(N,v) = v(\{j \in N \mid j \preceq i\}) - v(\{j \in N \mid j \prec i\}).$$

Then, $mc^{\prec}$ satisfies PO, IR, and AM but not BPF.

(iii) Let $i \in U$, $j \in U \setminus \{i\}$, and $(\{i,j\},u)$ be the flat game, i.e., $u(S) = 0$ for all $S \subseteq \{i,j\}$. Define $\sigma_i^1(\{i,j\},u^t) = 6$ and $\sigma_j^1(\{i,j\},u^t) = t - 6$ for all $10 \leq t < 12$, and $\sigma^1(N,v) = L(N,v)$ for all other convex games. Then σ^1 satisfies CS (hence PO and IR) and BPF but not AM.

(iv) For all $(N,v) \in \Gamma$, define the single-valued solution σ^2 by $\sigma_i^2(N,v) = v(\{i\})$ for all $i \in N$. Then, σ^2 satisfies IR, AM, and BPF but not PO.

Remark 8. We do not know whether AM is logically independent of the remaining properties in Theorem 3. Each of the two other properties is logically independent of the remaining properties as is shown by the following examples:

(i) Let $(N, v) \in \Gamma$. Schmeidler's (1969) *nucleolus*[11] $\nu(N, v)$ is a core-selection that is covariant under translations, hence, does not coincide with the egalitarian solution L even on convex root games. Now define the solution $\sigma^3(N, v) = \nu(N, v_r) + f^{CE}(\nu(N, v_r), v(N) - v_r(N))$. Clearly, σ^3 satisfies AM. By Theorem 2, it also satisfies BPF. Since $\sigma^3 \neq L$, it does not satisfy 2-RGP.

(ii) The marginal contribution solution defined in Remark 7 (ii) satisfies AM and 2-RGP but not BPF.

Remark 9. We do not know whether AM is logically independent of the remaining properties in Theorem 4. Each of the three other properties is logically independent of the remaining properties as is shown by the following examples:

(i) The single-valued solution σ^3 defined in Remark 8 (i) satisfies IR, AM, and BPF but not 2-CON.

(ii) The single-valued solution ρ defined in Remark 5 (Subsection 4.1) satisfies AM, BPF, and 2-RGP but not IR.

(iii) The marginal contribution solution defined in Remark 7 (ii) satisfies IR, AM, and 2-CON but not BPF.

Remark 10. The following examples show that each of the properties employed in Theorem 5 are logically independent:

(i) The solution ED defined in Remark 7 (i) satisfies AM and BR but not CS.

(ii) Let i, j, k be pairwise distinct elements of U and define the game (N', u) by $N' = \{i, j, k\}$, $u(\{i\}) = u(\{j\}) = u(\{k\}) = u(\{i, k\}) = u(\{j, k\}) = 0$, $u(\{i, j\}) = u(\{i, j, k\}) = 1$. Now define the single-valued solution σ^4 as follows: $\sigma^4(N, v) = L(N, v)$ for all $(N, v) \in \Gamma$ with $(N, v) \neq (N', u)$, and $\sigma_i^4(N', u) = 2/3, \sigma_j^4(N', u) = 1/3, \sigma_k^4(N', u) = 0$. Then, σ^4 satisfies CS and BR, but not AM.

(iii) The marginal contribution solution defined in Remark 7 (ii) satisfies CS and AM but not BR.

[11] That is, the unique imputation that lexicographically minimizes the non-increasingly ordered vector of excesses $(v(S) - x(S))_{S \subseteq N}$ over the set of imputations.

Remark 11. The following examples show that each of the properties employed in Theorem 6 are logically independent:

(i) The solution ED defined in Remark 7 (i) satisfies SBR but not CS.

(ii) The marginal contribution solution defined in Remark 7 (ii) satisfies CS but not SBR.

6 Final remarks

The properties employed in Theorem 1 may be formulated for surplus-sharing problems of the form $(x, t), x \in \mathbb{R}^N, t \geq 0$, for a fixed society N of agents. Theorem 1 remains valid on such a restricted domain of surplus-sharing problems. Similarly, the other characterizations that do not employ consistency properties, i.e., Theorem 2, Theorem 5, and Theorem 6, may be formulated and remain valid for a fixed player set N. Only in Theorem 3 and Theorem 4 that employ 2-RGP or 2-CON, respectively, we need to vary the set N of players.

We have introduced f^{CE}, the surplus-sharing rule that equalizes the agents' welfare should this be possible when dividing the surplus. Under welfare egalitarianism, our analysis shows that $f^{CE}(x, t)$ weakly Lorenz dominates $f(x, t)$ for any other Pareto optimal proposal $f \in \mathcal{F}$. In future research, it could be interesting to connect resource and welfare egalitarianism within this context, in the line of Moreno-Ternero and Roemer (2012). That is, to define a set of appealing properties that uniquely determine both f^{EQ} and f^{CE}.

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