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A Comparison of NTU Values in a Cooperative Game with Incomplete Information☆

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Abstract

Several value-like solution concepts are computed and compared in a cooperative game with incomplete information and non-transferable utility.

Keywords: Cooperative games, incomplete information, non-transferable utility.

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1. Introduction

By introducing the concept of "virtual utility", Myerson (1984) proposed a general notion of value for cooperative games with incomplete information. The so-called M-value generalizes the Shapley non-transferable utility (NTU) value.[1] Later, the same virtual utility approach was used in Salamanca (2016) to define an alternative value concept called the S-value, which generalizes the Harsanyi NTU value. Both value concepts reflect not only the signaling costs associated with incentive compatibility, but also the fact that individuals negotiate at the interim stage (i.e., after each player has received his private information). De Clippel (2005) and Salamanca (2016) show that the M-value differs from the S-value in that the former is less sensitive to some informational externalities. In this short note we analyze a simple example of an NTU game in which these two solution concepts differ because of the way payoff strategic possibilities of subcoalitions are handled.[2] We also study our example under the assumption of ex-ante negotiation. In that situation, the players make coalitional agreements before they learn their private information, so that they are symmetrically informed at the time of contracting.

2. The Example

Let r be a parameter with $0 < r < 1/2$. For each value of r, we consider the following cooperative game with incomplete information: The set of players is $N = \{1, 2, 3\}$. Player 1 has private information about one of two possible states, $T = \{H, L\}$, which happen with prior probabilities $p(H) = 1 - p(L) = 4/5$. Feasible decisions for coalitions are $D_{\{i\}} = \{d_i\}$ $(i \in N)$, $D_{\{i,j\}} = \{[d_i, d_i], d_{ij}\}$ $(i \neq j)$, $D_N = \{[d_1, d_2, d_3], [d_{12}, d_3], [d_{13}, d_2], [d_{23}, d_1]\}$. Utility functions, $u_i : T \times D_N \to \mathbb{R}$, are given by:

(u_1, u_2, u_3)	H	L
$[d_1, d_2, d_3]$	$(0, 0, 0)$	$(0, 0, 0)$
$[d_{12}, d_3]$	$(50, 50, 0)$	$(40, 40, 0)$
$[d_{13}, d_2]$	$(100r, 0, 100(1-r))$	$(40r, 0, 40(2-r))$
$[d_{23}, d_1]$	$(0, 100r, 100(1-r))$	$(0, 40r, 40(2-r))$

Feasible decisions are understood as follows: Decision d_i denotes player i's non-cooperative option, which leaves him with his reservation utility normalized to zero. When coalition $\{i, j\}$ forms and its members agree on an outcome $d \in D_{\{i,j\}}$, player k (in the complementary coalition) is left alone with the only possibility to choose d_k. Hence, $[d_1, d_2, d_3]$ denotes the outcome in which no player cooperates, and $[d_{ij}, d_k]$ corresponds to the cooperative outcome in which players i and j form a coalition and share the proceeds of cooperation as specified above. No other outcomes are possible.

In this game, player 3 can be considered as weak in the sense that he can only offer players 1 and 2 a payoff that is strictly lower than what they both can get by acting together in coalition $\{1, 2\}$. Then it does appear that coalitions $\{1, 3\}$ and $\{2, 3\}$ are less likely to form than $\{1, 2\}$. Moreover, the smaller r is, the less utility player 3 can transfer to players 1 and 2, and therefore the less likely it should be that $\{1, 3\}$ or $\{2, 3\}$ form.

A *mechanism* for coalition $S \subseteq N$ is a pair of functions

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[1]The Shapley NTU value is sometimes referred to as the λ-transfer value.

[2]Our example is reminiscent of an NTU game with complete information proposed by Roth (1980).

(μ_S, x_S) defined by[3]

$$\mu_S : T \to \Delta(D_S) \qquad x_S : T \to \mathbb{R}^S_-$$
$$t \mapsto \mu_S(\cdot \mid t) \qquad t \mapsto (x^i_S(t))_{i\in S}$$

Both mappings, μ_S and x_S, are measurable w.r.t. the information of the members of S. The component μ_S is a type-contingent lottery on the set of feasible decisions for S, while x_S is a vector of type-contingent utility decrements (*free disposal*). The mechanism μ_S ($S \neq N$) stands as a *threat* to be carried out only if $N \setminus S$ refuses to cooperate with S. We denote by $\mathcal{F}_S$ the set of mechanisms for coalition S.

In this game, efficient allocations can be made incentive compatible, by which incentive constraints are not essential. We shall thus assume that all information is public at the implementation stage, which implies that any mechanism can be enforced once it is agreed upon.[4] As a result, virtual utility specializes to a rescaling of actual utility and one obtains simple expressions for both the M-value and the S-value.[5]

3. Contracting at the Interim Stage

For a given coalition S, we write $u_i(\mu_S, t)$ for the linear extension of the utility functions over $\mu_S(\cdot \mid t)$.[6] We define $u_i((\mu_S, x_S), t) := u_i(\mu_S, t) + x^i_S(t)$ to be player i's expected utility from (μ_S, x_S) conditional on state t. Hence, $u_i(\mu_S, x_S) := \sum_t p(t) u_i((\mu_S, x_S), t)$ denotes i's ex-ante expected utility from (μ_S, x_S). A mechanism $(\bar{\mu}_N, \bar{x}_N)$ is (*interim*) *efficient* for the grand coalition iff there exists a non-negative vector $\lambda = (\lambda^H_1, \lambda^L_1, \lambda_2, \lambda_3)$, such that $(\bar{\mu}_N, \bar{x}_N)$ maximizes the social welfare function

$$\lambda^H_1 u_1((\mu_N, x_N), H) + \lambda^L_1 u_1((\mu_N, x_N), L) + \lambda_2 u_2(\mu_N, x_N) + \lambda_3 u_3(\mu_N, x_N).$$

Thus, λ is normal to the interim Pareto frontier at the utility allocation implemented by $(\bar{\mu}_N, \bar{x}_N)$.

Fix a vector λ of utility weights as above. Given a coalition S and a mechanism (μ_S, x_S), the *virtual utility* of players in state t is defined as

$$v^\lambda_1((\mu_S, x_S), t) := \frac{\lambda^t_1}{p(t)} u_1((\mu_S, x_S), t),$$
$$v^\lambda_j((\mu_S, x_S), t) := \lambda_j u_j((\mu_S, x_S), t), \quad j = 2, 3.$$

Consider the fictitious game in which, conditionally on every state t, virtual utilities are transferable. The worth of coalition $S \subseteq N$ in state $t \in T$, when its members agree on the mechanism (μ_S, x_S), is defined to be

$$W^\lambda_S((\mu_S, x_S), t) := \sum_{i\in S} v^\lambda_i((\mu_S, x_S), t).$$

For a given profile of threats, $\eta = ((\mu_S, x_S))_{S\subseteq N}$, $W^\lambda(\eta, t) := (W^\lambda_S((\mu_S, x_S), t))_{S\subseteq N}$ defines a TU game in state t. Let $\phi_i(W^\lambda(\eta, t))$ denote the Shapley TU value of player i in the game $W^\lambda(\eta, t)$. A mechanism $(\bar{\mu}_N, \bar{x}_N)$ for the grand coalition is (*virtually*) *equitable* if

$$\lambda^t_1 u_1((\bar{\mu}_N, \bar{x}_N), t) = p(t)\phi_1(W^\lambda(\eta, t)), \quad \forall t \in T, \tag{3.1}$$
$$\lambda_j u_j(\bar{\mu}_N, \bar{x}_N) = \sum_{t\in T} p(t)\phi_j(W^\lambda(\eta, t)), \quad \forall j = 2, 3.$$

Definition 1 (**NTU value**)
A mechanism $(\bar{\mu}_N, \bar{x}_N)$ is called a bargaining solution if there exists a strictly positive vector λ such that $(\bar{\mu}_N, \bar{x}_N)$ is efficient and equitable given λ.[7] The interim utility allocation generated by $(\bar{\mu}_N, \bar{x}_N)$ is called an NTU value.

Different NTU values can be defined depending on how the vector of threats $((\mu_S, x_S))_{S\subseteq N}$ is determined. According to the M-value, for every coalition S, (μ_S, x_S) is computed solving the following problem:

$$\max_{(\mu_S, x_S)\in\mathcal{F}_S} \sum_{t\in T} p(t) W^\lambda_S((\mu_S, x_S), t). \tag{3.2}$$

Proposition 1 (**M-value**)
For any given $r \in (0, 1/2)$, the unique M-value of this game is the interim utility allocation

$$\left(u^H_1, u^L_1, u_2, u_3\right) = \left(\tfrac{100}{3}, \tfrac{80}{3}, 32, 32\right). \tag{3.3}$$

Proof. The interim Pareto frontier coincides with the hyperplane $\frac{4}{5}U^H_1 + \frac{1}{5}U^L_1 + U_2 + U_3 = 96$ on the individually rational zone. Thus, (3.3) is efficient. Since bargaining solutions are individually rational, an M-value can only be supported by the utility weights $(\lambda^H_1, \lambda^L_1, \lambda_2, \lambda_3) = (4/5, 1/5, 1, 1)$. Hence, virtual and real utilities coincide. After computation of threats according to (3.2), equations in (3.1) yield (3.3). □

The M-value prescribes the same allocation regardless of the value of r. Furthermore, it treats all players symmetrically. This is due to the fact that, by computing threats according to (3.2), coalitions $\{1, 3\}$ and $\{2, 3\}$ can agree on an equitable distribution of the total gains on every state, something that is not possible in the original NTU game. Thus, we may argue that threats in the M-value are not "credible". All that matters for the M-value when measuring the strength of coalitions is the maximum joint gains that can be allocated, and not the restrictions the players face when sharing such gains.

For the example under consideration, the S-value differs from the M-value only on the computation of threats for

[3] This definition is adapted from the mechanisms with sidepayments considered by Myerson (2007). For any finite set A, $\Delta(A)$ denotes the set of probability distributions over A.

[4] Here, the only issue is the revelation of private information at the negotiation stage.

[5] In Myerson's (1984) terminology, the Lagrange multipliers associated with the incentive constraints can be set to zero.

[6] Decisions available to any coalition S do not affect the utilities of the players in $N \setminus S$. Thus, $u_i(\mu_S, t)$ is well defined.

[7] We focus only on non-degenerate values (i.e., those supported by strictly positive utility weights).

two-person coalitions. For any coalition $S = \{1, j\}$ ($j = 2, 3$), the S-value determines (μ_S, x_S) by solving

$$\max_{(\mu_S,x_S)\in\mathcal{F}_S} \sum_{t\in T} p(t) W_S^\lambda((\mu_S, x_S), t)$$
$$\text{s.t.} \quad v_1^\lambda((\mu_S, x_S), t) = v_j^\lambda((\mu_S, x_S), t), \quad \forall t \in T. \tag{3.4}$$

Threats for coalition $S = \{2, 3\}$ are similarly defined, except that the "egalitarian constraints" in (3.4) are replaced by

$$\sum_{t\in T} p(t) v_2^\lambda((\mu_S, x_S), t) = \sum_{t\in T} p(t) v_3^\lambda((\mu_S, x_S), t). \tag{3.5}$$

The egalitarian constraint (3.5) reflects the fact that players 2 and 3 cannot make an agreement contingent on player 1's private information, so that utility comparisons inside $\{2, 3\}$ have to be made in expected terms.

Proposition 2 (**S-value**)
For a given $r \in (0, 1/2)$, the unique S-value of this game is the interim utility allocation

$$\left(u_1^H, u_1^L, u_2, u_3\right) \tag{3.6}$$
$$= \left(50 - \tfrac{100}{3} r \left(\tfrac{88-88r}{96-88r}\right), 40 - \tfrac{80}{3} r \left(\tfrac{88-44r}{96-88r}\right), 48 - \tfrac{88}{3} r, \tfrac{176}{3} r\right).$$

Proof. The same reasoning as in the proof of Proposition 1. □

The S-value gives less to player 3 compared to the M-value. This is due to the fact that two-person coalitions with player 3 cannot fully distribute the total gains from cooperation in an equitable way. This lack of transferability increases as long as r decreases to 0, which explains why the S-value converges to the allocation $(50, 40, 48, 0)$ as r vanishes. It seems that the S-value reflects the power structure of this game better than the M-value, in particular for a small r.

4. Contracting at the Ex-ante Stage

When contracting takes place at the ex-ante stage, players face a cooperative game under incomplete information but with symmetric uncertainty. Then we may apply both the Shapley NTU value and the Harsanyi NTU value to the characteristic function of this game.[8]

The set of feasible payoff allocations for each coalition $S \subseteq N$ is given by $U(S) = \{(u_i(\mu_S, x_S))_{i\in S} \mid (\mu_S, x_S) \in \mathcal{F}_S\}$. Then the ex-ante characteristic function of this game is:

$$U(\{i\}) = \{u_i \mid u_i \le 0\}, \quad \forall i \in N,$$
$$U(\{1,2\}) = \{(u_1, u_2) \mid u_1 \le 48,\ u_2 \le 48\},$$
$$U(\{i,3\}) = \{(u_i, u_3) \mid u_i \le 88r,\ u_3 \le 96 - 88r\},\ (i = 1, 2),$$
$$U(N) = comp(\{u_{12}, u_{13}, u_{23}\}),$$

where $u_{13} := (88r, 0, 96 - 88r)$, $u_{23} := (0, 88r, 96 - 88r)$, $u_{12} := (48, 48, 0)$ and, for any finite set A, $comp(A)$ denotes the comprehensive hull of A.

[8]The reader is referred to McLean (2002) for definitions of these two solutions.

Proposition 3 (**Ex-ante Shapley NTU value**)
For any $r \in (0, 1/2)$, the unique Shapley NTU value of the game (U, N) is the utility allocation[9]

$$(u_1, u_2, u_3) = (32, 32, 32). \tag{4.1}$$

Like the M-value, the Shapley NTU value is independent of r. Moreover, it treats all players symmetrically and ignores the fact that coalitions $\{1, 3\}$ and $\{2, 3\}$ cannot agree on an equitable distribution of the gains.

Proposition 4 (**Ex-ante Harsanyi NTU value**)
For a given $r \in (0, 1/2)$, the unique Harsanyi NTU value of the game (U, N) is the utility allocation[10]

$$(u_1, u_2, u_3) = \left(1 - \frac{22r}{36 - 33r}\right) u_{12} + \frac{11r}{36 - 33r} u_{13} + \frac{11r}{36 - 33r} u_{23}. \tag{4.2}$$

For every r, the weight of the outcome u_{12} of coalition $\{1, 2\}$ is the largest. Furthermore, it increases to 1 as r decreases to 0; thus the probability of player 3 getting into a coalition converges to 0. Therefore, the Harsanyi NTU value prescribes an outcome that better captures the lack of transferable utility in this game.

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[9]The value allocation is supported by the utility weights $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 1)$. As in the case of interim negotiation, here we exclusively deal with non-degenerated values.

[10]The same utility weights as in the Shapley NTU value support this value allocation.