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Monotonicity and weighted prenucleoli: A characterization without consistency*

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Abstract

A solution on a set of transferable utility (TU) games satisfies *strong aggregate monotonicity* (SAM) if every player can improve when the grand coalition becomes richer. It satisfies *equal surplus division* (ESD) if the solution allows the players to improve equally. We show that the set of weight systems generating weighted prenucleoli that satisfy SAM is open which implies that for weight systems close enough to any regular system the weighted prenucleolus satisfies SAM. We also provide a necessary condition for SAM for symmetrically weighted nucleoli. Moreover, we show that the per capita nucleolus on balanced games is characterized by single-valuedness (SIVA), translation and scale covariance (COV), and equal *adjusted* surplus division (EASD), a property that is comparable but stronger than ESD. These properties together with ESD characterize the per capita prenucleolus on larger sets of TU games. EASD and ESD can be transformed to *independence of (adjusted) proportional shifting* and these properties may be generalized for arbitrary weight systems p to $\mathrm{I(A)S}_p$. We show that the p-weighted prenucleolus on the set of balanced TU games is characterized by SIVA, COV, and IAS_p; and on larger sets by additionally requiring IS_p.

Keywords: TU games, weighted prenucleolus, equal surplus division

JEL Classification: C71

1 Introduction

The core is one of the most important reference solutions for cooperative games. When restricting the attention to transferable utility (TU) games, there are widely accepted nonempty solutions like the prenucleolus, a single-valued solution that is a core selection, i.e., selects an

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element of the core whenever the core is nonempty. As a solution in its own right the prenucleolus may be justified by simple and intuitive axioms. Indeed, in his seminal work Sobolev (1975) proved that the prenucleolus on the set of all TU games with player sets contained in an infinite universe is characterized by *single-valuedness* (SIVA), *anonymity*, *translation and scale covariance* (COV), and the *reduced game property* with respect to (w.r.t.) the Davis-Maschler (Davis and Maschler, 1965) reduced game. It was shown that the *equal treatment property* replaces anonymity (Orshan, 1993) and that, if the equal treatment property is used, then SIVA and the reduced game property may be replaced by *non-emptiness* and the *reconfirmation property* (Orshan and Sudhölter, 2003). However, the infinity assumption of the potential universe of players is crucial in any of the aforementioned axiomatizations of the prenucleolus. It was also shown that it is impossible to suitably modify Peleg's (1986) axiomatization of the prekernel that works for a finite universe of players by replacing the *converse reduced game property* by some minimality principle (see Corollary 3.8 of Orshan and Sudhölter, 2003). Up to our knowledge, there is only one characterization of the (pre)nucleolus in the literature that does not need varying player sets and consistency. Indeed, Oswald et al. (1998) show that the (pre)nucleolus can be characterized with the help of an independence property and continuity. For a detailed comparison with the present characterization we refer to Remark 6.3.

Our characterizations do not employ continuity and they also allow to characterize weighted prenucleoli rather than just the traditional one. For symmetrically weighted prenucleoli there are axiomatizations that are similar to Sobolev's axiomatization of the prenucleolus–only the Davis-Maschler reduced game has to be replaced by a suitable new reduced game, the definition of which depends on the weight system (Kleppe et al., 2016). Hence, in the current article we present axiomatizations of the weighted prenucleoli that avoid references to any kind of reduced game and, instead, make use of the fact that all aforementioned weighted prenucleoli are core selections.

Moreover, in order to treat TU games that have an empty core (are not balanced), we employ some kind of monotonicity property in addition. Well-known monotonicity properties that are also satisfied by the core are *aggregate monotonicity* (AM), *strong aggregate monotonicity* (SAM), and *equal surplus division* (ESD). A solution satisfies AM if there is an element according to which nobody is worse off compared to any proposal of the solution of the original TU game if only the worth of the grand coalition is increased. It satisfies ESD if the additional worth of the grand coalition may be distributed equally among the players. For the formal definitions of these intuitive properties see Section 2. It is well known (see, e.g., Calleja and Llerena, 2017) that the per capita prenucleolus satisfies ESD (hence SAM and AM). However, when considering the class of balanced games, ESD is a rather weak property. Indeed, when diminishing the worth of the grand coalition, a balanced game may become non-balanced. Therefore, an arbitrary balanced game may never arise from another balanced game by just increasing the worth of the grand coalition. In order to receive a stronger property that is similar to ESD, satisfied by the core as well, and applicable to balanced games that do not allow to diminish exclusively the prosperity of the grand coalition, we introduce equal *adjusted* surplus division (EASD). To this end we say that a coalition is *fully exact* (called tight by Oswald et al., 1998) if each core element assigns to this coalition precisely its worth in the game. Now, a solution satisfies equal *adjusted* surplus division (EASD) if, whenever the worth of any fully exact coalition is diminished proportionally so that the new game remains balanced, then adding equal shares to any element of the solution of the new game yields an element of the solution of the original game (see (4) and (5) for the formal definition).

We show that the per capita nucleolus on the set of balanced games with coinciding player sets N of $n \geqslant 2$ elements is axiomatized by SIVA, COV, and EASD. On all games with player sets N, ESD is needed in addition to characterize the per capita prenucleolus.

It turns out that ESD and EASD can be translated to *independence of proportional shifting* (IPS) and independence of *adjusted* proportional shifting (IAPS). A game arises from another game by proportional shifting if the worth of any proper coalition is increased proportionally to its size. Now, IPS requires that a solution element of the latter game belongs to the solution of the proportionally shifted games as well. The game arises by adjusted proportional shifting if only those coalitions that are not fully exact are shifted, and IAPS refers to the corresponding independence axiom. Hence, the per capita (pre)nucleolus on the set of balanced (all) games is axiomatized by SIVA, COV, IAPS (, and IPS).

It turns out that IPS and IAPS may be generalized to any weight system p. Instead of shifting proportionally, the shifting of a coalition has to be proportional to the inverse weight of this coalition. We prove that each weighted (pre)nucleolus is characterized by SIVA, COV, and suitably defined independence axiom(s) for the corresponding weight system. Hence, e.g., the (pre)nucleolus is axiomatized without any reference to reduced games.

We now briefly review the contents of the paper. Section 2 offers the necessary notation, recalls the relevant definitions of the considered solutions and related concepts, contains a list of properties of solutions, and provides the well-known Kohlberg criterion. In Section 3 we investigate which weighted nucleoli satisfy SAM. We generalize the inequalities characterizing symmetric weighted nucleoli that satisfy SAM in the 3-person case provided by Housman and Clark (1998) to the n-person case. However, the conditions only remain necessary in the general n-person case. Without assuming symmetry, we show that a weighted (pre)nucleolus satisfies regular SAM as defined by Calleja and Llerena (2017) if and only if the weight system is *regular* (i.e., associated with a positive payoff vector). As a consequence, a weighted prenucleolus satisfies ESD if and only if it is the per capita prenucleolus. We prove the continuity of the mapping that assigns, to each weight system and game, the corresponding weighted prenucleolus and use this continuity to show that the set of weight systems generating weighted prenucleoli that satisfy SAM is open, implying, in particular, that SAM is satisfied by any weighted prenucleolus if the weights are close enough to some regular weight system. Section 4 is devoted to the axiomatization of the per capita (pre)nucleolus without making use of any reduced game property. Section 5 presents the new properties IPS and IAPS, shows that these properties are equivalent to ESD and EASD, respectively, for solutions that satisfy translation covariance (TCOV), and that TCOV is crucial. In Section 6, IPS and IAPS are generalized to IS_p and IAS_p, the corresponding properties depending on the weight system p. Thus, the p-weighted nucleolus on the set of balanced games is characterized by SIVA, COV, and IAS_p, whereas IS_p is needed in addition to characterize the p-weighted prenucleolus on the unrestricted set of TU games on N. Hence, Theorem 6.2 applied to a weight system p assigning the same weight to any coalition provides an axiomatization of the traditional (pre)nucleolus for a fixed set of n players. Moreover, it should be highlighted that non-symmetrically weighted (pre)nucleoli have not been characterized before. Finally, Section 7 offers some expansions, remarks, and comments.

2 Notation, definitions and preliminaries

Let N be a finite nonempty set with $n = |N| \geqslant 2$. A *transferable utility game* (with player set N) is a mapping $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. The set of *coalitions* (nonempty subsets of N) is denoted by $\mathcal{F}$ (i.e., $\mathcal{F} = 2^N \setminus \{\emptyset\}$), we often need the set $\mathring{\mathcal{F}} := \mathcal{F} \setminus \{N\}$ of proper coalitions, and the set of all games is denoted by Γ. Any $x \in \mathbb{R}^N$ defines the *inessential* game $x(\cdot) \in \Gamma$ defined by $x(S) = \sum_{i \in S} x_i$ for all $S \in \mathcal{F}$ (and $x(\emptyset) = 0$). For $v \in \Gamma$, define

$$\begin{aligned} X^*(v) &= \{x \in \mathbb{R}^N \mid x(N) \leqslant v(N)\} \ \text{ – the set of } \textit{feasible allocations} \text{ (payoff vectors)},\\ X(v) &= \{x \in \mathbb{R}^N \mid x(N) = v(N)\} \ \text{ – the set of } \textit{preimputations}\text{, and}\\ C(v) &= \{x \in X^*(v) \mid x(S) \geqslant v(S)\ \forall S \in \mathcal{F}\} \ \text{ – the } \textit{core.} \end{aligned}$$

Recall that $\mathcal{B} \subseteq \mathcal{F}$ is *balanced* if there exists $(\delta_S)_{S \in \mathcal{B}}$ such that $\delta_S > 0$ for all $S \in \mathcal{B}$ and $\sum_{S \in \mathcal{B}} \delta_S \mathbb{1}^S = \mathbb{1}^N$ where $\mathbb{1}^S \in \mathbb{R}^N$ is the indicator vector of S for any $S \subseteq N$. In such a case, $(\delta_S)_{S \in \mathcal{B}}$ is a *system of balancing weights* for $\mathcal{B}$. Then $\mathcal{B}$ is a minimal (w.r.t. set inclusion) balanced collection of coalitions if and only if it has a unique collection of balancing weights. In this case, let $(\delta_S^{\mathcal{B}})_{S \in \mathcal{B}}$ denote the unique system of balancing weights.

Remark 2.1. Let $v \in \Gamma$. According to the Bondareva-Shapley Theorem (Bondareva, 1963; Shapley, 1967), $C(v) \neq \emptyset$ if and only if

$$v(N) \geqslant \max \left\{ \sum_{S \in \mathcal{B}} \delta_S^{\mathcal{B}} v(S) \middle| \mathcal{B} \subseteq \mathring{\mathcal{F}} \text{ is minimal balanced} \right\} =: \beta(v).$$

Hence, games that have nonempty cores are called *balanced*.

Denote by Γ^b the set of balanced games. A coalition $S \in \mathcal{F}$ is *exact* (Shapley, 1971; Schmeidler, 1972) at $v \in \Gamma$ if there exists $x \in C(v)$ that is *effective* for S, i.e., $x(S) = v(S)$. Moreover, let us call S *fully* exact if it is exact and all $x \in C(v)$ are effective for S. Let $\mathcal{E}(v)$ denote the set of all fully exact coalitions at v, i.e.,

$$\mathcal{E}(v) = \{S \in \mathcal{F} \mid S \text{ is exact at } v \text{ and } x(S) = v(S) \text{ for all } x \in C(v)\}.$$

Hence, if $v \in \Gamma^b$, then $N \in \mathcal{E}(v)$. Let $\mathring{\mathcal{E}}(v) = \mathcal{E}(v) \setminus \{N\}$. Then for any $S \in \mathcal{F} \setminus \mathcal{E}(v)$ there exists $x^S \in C(v)$ with $x^S(S) > v(S)$. As $C(v)$ is convex, we conclude that there exists $x \in C(v)$ such that $x(S) > v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{E}(v)$. Note that, by the mentioned Bondareva-Shapley Theorem, the following relations are valid for any $v \in \Gamma$:

$$\beta(v) > v(N) \iff \mathcal{E}(v) = \emptyset \iff C(v) = \emptyset \tag{1}$$

$$\beta(v) = v(N) \iff \mathring{\mathcal{E}}(v) \neq \emptyset \tag{2}$$

$$\beta(v) < v(N) \iff \mathcal{E}(v) = \{N\} \tag{3}$$

A *solution* is a mapping σ that assigns a subset $\sigma(v)$ of $X^*(v)$ to any $v \in \Gamma$. Its restriction to a set $\Gamma' \subseteq \Gamma$ is again denoted by σ. Moreover, a solution on Γ' is the restriction to Γ' of some solution.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *single-valuedness* (SIVA) if $|\sigma(v)| = 1$ for all $v \in \Gamma'$;
- *translation covariance* (TCOV) if for all $v, v' \in \Gamma'$ such that $v' - v = x(\cdot)$ for some $x \in \mathbb{R}^N$ (i.e., $v - v'$ is inessential), it holds that $\sigma(v') - \sigma(v) = \{x\}$, i.e., $\sigma(v') = \sigma(v) + \{x\}$, where $A + B$ denotes the Minkowsky sum whenever $A, B \subseteq \mathbb{R}^N$;
- *scale covariance* (SCOV) if $\sigma(\beta v) = \beta\sigma(v)$, whenever $\beta > 0$ and $v, \beta v \in \Gamma'$.

We often compare the solutions applied to two games v and v' such that $v(S) = v'(S)$ for all $S \subsetneq N$. Hence, it is useful to define, for each $v \in \Gamma$ and each $\alpha \in \mathbb{R}$, the game $v^{(\alpha)}$ that arises from v by exclusively diminishing the worth of the grand coalition by $n\alpha$. The game $v^{(\alpha)}$ is called the *α-diminished* game of v, and it is formally defined by

$$v^{(\alpha)}(S) = \begin{cases} v(S) & , \text{if } S \in \mathring{\mathcal{F}}, \\ v(N) - n\alpha & , \text{if } S = N. \end{cases}$$

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *strong aggregate monotonicity* (SAM) if for all $x \in \sigma(v^{(\alpha)})$ there exists $y \in \sigma(v)$ such that $y \gg x$ (i.e., $y_i > x_i$ for all $i \in N$), whenever $\alpha > 0$ and $v, v^{(\alpha)} \in \Gamma'$;
- *equal surplus division* (ESD) if $\sigma(v^{(\alpha)}) + \{\alpha \mathbb{1}^N\} \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v^{(\alpha)} \in \Gamma'$.

SIVA is clearly a desirable property of a normative solution, whereas TCOV and SCOV are widely accepted standard properties, together traditionally called *covariance under strategic equivalence*. Monotonicity properties like SAM and ESD have been discussed, e.g., by Megiddo (1974) who showed that the Davis-Maschler (Davis and Maschler, 1967) bargaining set does not satisfy *aggregate monotonicity* (AM), defined as SAM except that just $y \geqslant x$ is required.
Note that the solutions $C(\cdot), X(\cdot)$, and $X^*(\cdot)$ satisfy all foregoing properties except SIVA (provided that Γ' is rich enough).
We now recall the definition of a "weighted (pre)nucleolus" (see, e.g., Derks and Haller, 1999) and of some relevant results of Kleppe et al. (2016). A *weight system* is a system $p = (p_S)_{S \in \mathring{\mathcal{F}}}$ such that $p_S > 0$ for all $S \in \mathring{\mathcal{F}}$. Let $v \in \Gamma$. The *p-weighted prenucleolus* of v, denoted $\mathcal{PN}^p(v)$, is the set of preimputations x of v that lexicographically minimize the non-increasingly ordered vector $(p_S e(S, x, v))_{S \in \mathring{\mathcal{F}}}$ where $e(S, x, v) = v(S) - x(S)$ is the *excess* of $S \in \mathcal{F}$ at x w.r.t. v. Formally, with $\theta : \mathbb{R}^{\mathring{\mathcal{F}}} \to \mathbb{R}^{2^n-2}$ defined by $\theta_t(x) = \max_{\mathcal{T} \subseteq \mathring{\mathcal{F}}, |\mathcal{T}|=t} \min_{T \in \mathcal{T}} x_T$ for all $t = 1, \ldots, 2^n - 2$,

$$\mathcal{PN}^p(v) = \{x \in X(v) \mid \theta\big((p_S e(S, x, v))_{S \in \mathring{\mathcal{F}}}\big) \leqslant_{lex} \theta\big((p_S e(S, y, v))_{S \in \mathring{\mathcal{F}}}\big) \, \forall y \in X(v)\},$$

where $\leqslant_{lex}$ denotes the lexicographical order on $\mathbb{R}^{2^n-2}$. Note that the p-weighted *nucleolus* of v is defined similarly. Only $X(v)$ is replaced by the set of *imputations*, i.e., by $\{x \in X(v) \mid x_i \geqslant v(\{i\}) \forall i \in N\}$. It is well known that the p-weighted prenucleolus is a singleton that we denote by $\nu^p(v)$. Moreover, the p-weighted nucleolus is a singleton whenever $v(N) \geqslant \sum_{i \in N} v(\{i\})$. If $v \in \Gamma^b$, then both solutions coincide. The foregoing statements may easily be derived from the results of Justman (1977). Replacing p by αp for some positive α does not change the corresponding weighted prenucleolus, and it can be shown that the opposite is also true: $\mathcal{PN}^p = \mathcal{PN}^{p'}$ if and only if p and p' are proportional. If all weights are identical, the corresponding p-weighted (pre)nucleolus (point) is the traditional *prenucleolus* (point) introduced by Schmeidler (1969). If the p_S are proportional to $\frac{1}{|S|}$, i.e., $p_S|S| = p_T|T|$ for all $S, T \in \mathring{\mathcal{F}}$, then we omit the upper index p and simply write $\mathcal{PN}(v) = \{\nu(v)\}$ to denote the *per capita* prenucleolus (see, e.g., Grotte, 1970). Whether or not a (pre)imputation of a game coincides with its p-weighted (pre)nucleolus point can be checked with a suitable modification of Kohlberg's (1971)

"Property I" or "Property II". In order to formulate Property II$_p$ (i.e., Kohlberg's Property II for the p-weighted prenucleolus), we denote, for any $\alpha \in \mathbb{R}$ and $x \in X(v)$,

$$\mathcal{D}^p(\alpha, x, v) = \{S \in \mathring{\mathcal{F}} \mid p_S e(S, x, v) \geqslant \alpha\}.$$

Then x has *Property II$_p$* if, for any $\alpha \in \mathbb{R}$, $\mathcal{D}^p(\alpha, x, v)$ is balanced or empty. Now, Proposition 2.2 of Kleppe et al. (2016) implies part (i) of the following remark.

Remark 2.2. Let $v \in \Gamma, p$ be a weight system, and $x \in X(v)$.

(i) Then $x = \nu^p(v)$ if and only if x has Property II$_p$.

(ii) Let $x = \nu^p(v)$. If v is balanced, as x minimizes the largest p-excess, then $x \in C(v)$ and $x(T) > v(T)$ for all $T \in \mathcal{F} \setminus \mathcal{E}(v)$.

It is well-known that the weighted prenucleoli satisfy SIVA, TCOV, and SCOV. The specialty of the per capita prenucleolus is that it also satisfies ESD because for any $v \in \Gamma$, any $\alpha \geqslant 0$, and $x \in \mathbb{R}^N$, and any $S \in \mathring{\mathcal{F}}$,

$$\frac{1}{|S|} e(S, x, v^{(\alpha)}) = \frac{1}{|S|} e(S, x + \alpha \mathbb{1}^N, v) + \alpha,$$

i.e., the per capita excesses of $v^{(\alpha)}$ at x and those of v at $x + \alpha \mathbb{1}^N$ coincide up to a constant.

3 On strong aggregate monotonicity and weighted prenucleoli

In this section, we introduce *regular* weight systems that result in weighted prenucleoli satisfying SAM. We show that the set of weight systems generating weighted prenucleoli that satisfy SAM is open. Hence, a p-weighted prenucleolus satisfies SAM if, e.g., p is close enough to a regular weight system. Moreover, as a consequence of a more general statement, we show that the per capita prenucleolus is the unique weighted prenucleolus that satisfies ESD. We also provide a necessary condition for symmetric weight systems to generate aggregate monotonic weighted prenucleoli. Say that a solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *regular strong aggregate monotonicity* (RSAM) if there exists $z \in \mathbb{R}^N$ with $z \gg 0$ and $z(N) = n$ such that $\sigma(v^{(\alpha)}) + \{\alpha z\} \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v^{(\alpha)} \in \Gamma'$. In this case we say that σ satisfies RSAM w.r.t. z.

Note that, for single-valued solutions, our version of RSAM coincides with the corresponding property introduced by Calleja and Llerena (2017). The following theorem characterizes the weighted prenucleoli that satisfy RSAM.

Theorem 3.1. *Let $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$, p be a weight system, and let $z \in \mathbb{R}^N$ satisfy $z(N) = n$ and $z \gg 0$. The solution $\mathcal{PN}^p$ on Γ' satisfies* RSAM *w.r.t. z if and only if $z(Q)p_Q = z(R)p_R$ for all $Q, R \in \mathring{\mathcal{F}}$.*

Proof. In order to show the if part let $\widehat{p}_Q = \frac{1}{z(Q)}$ for all $Q \in \mathring{\mathcal{F}}$, $\alpha > 0$, $v, v^{(\alpha)} \in \Gamma'$, and $x = \nu^{\widehat{p}}(v^{(\alpha)})$. Then

$$\widehat{p}_S e(S, x + \alpha z, v) = \frac{v^{(\alpha)}(S) - x(S) - \alpha z(S)}{z(S)} = \widehat{p}_S e(S, x, v^{(\alpha)}) - \alpha \text{ for all } S \in \mathring{\mathcal{F}}$$

and $x + \alpha z(N) = v(N)$ so that $\nu^{\widehat{p}}(v) = x + \alpha z$ by Remark 2.2.

In order to show the only-if part, let p be a weight system such that there are $Q, R \in \mathring{\mathcal{F}}$ satisfying $z(Q)p_Q \neq z(R)p_R$. It remains to show that $\mathcal{PN}^p$ does not satisfy RSAM w.r.t. z. Choose $\alpha \geqslant \max\{z(T)p_T \mid T \in \mathring{\mathcal{F}}\}$. We now define a collection $\mathcal{S}$ of coalitions by

$$\mathcal{S} = \begin{cases} \{Q, R\} \cup \{\{i\} \mid i \in N \setminus (Q \cup R)\} & , \text{if } Q \cap R = \emptyset, \\ \{Q, R, (Q \cup R) \setminus (Q \cap R)\} \cup \{\{i\} \mid i \in N \setminus (Q \cup R)\} & , \text{if } Q \cap R \neq \emptyset. \end{cases}$$

Note that $\mathcal{S}$ is minimal balanced (with unique balancing weights $\delta_Q = \delta_R = \delta_{\{i\}} = 1$ for all $i \in N \setminus (Q \cup R)$ in the case $Q \cap R = \emptyset$ and $\delta_Q = \delta_R = \delta_{(Q \cup R)\setminus(Q \cap R)} = \frac{1}{2}$ and $\delta_{\{i\}} = 1$ for all $i \in N \setminus (Q \cup R)$ in the case $Q \cap R \neq \emptyset$).

Define $v \in \Gamma$ by

$$v(S) = \begin{cases} -\frac{\alpha}{p_S} & , \text{if } S \in \mathring{\mathcal{F}} \setminus \mathcal{S}, \\ 0 & , \text{if } S \in \mathcal{S}, \\ n & , \text{if } S = N. \end{cases}$$

Moreover, let $v' = v^{(1)}$, i.e., $v'(S) = v(S)$ for all $S \in \mathring{\mathcal{F}}$ and $v'(N) = 0$. Let $x = 0 \in \mathbb{R}^N$. Then $p_S e(S, x, v') = 0$ for all $S \in \mathcal{S}$ and $p_T e(T, x, v') = -\alpha$ for all $T \in \mathring{\mathcal{F}} \setminus \mathcal{S}$. Hence, v' and v are balanced. As $\mathcal{S}$ is balanced, it is straightforward to deduce from (i) of Remark 2.2 that $x = \nu^p(v')$. Let $y = x + z$. It remains to show that $y \neq \nu^p(v)$. Let $\rho = \max\{p_S e(S, y, v) \mid S \in \mathring{\mathcal{F}}\}$ and $\mathcal{T} = \{T \in \mathring{\mathcal{F}} \mid p_T e(T, y, v) = \rho\}$. Note that, for any $S \in \mathcal{S}$, $p_S e(S, y, v) = -p_S z(S) \geqslant -\alpha$. Also, for any $T \in \mathring{\mathcal{F}} \setminus \mathcal{S}$, $p_T e(T, y, v) = -\alpha - p_T z(T) < -\alpha$ so that $\mathcal{T} \subseteq \mathcal{S}$. Now, $p_R z(R) \neq p_Q z(Q)$ implies $\mathcal{T} \subsetneqq \mathcal{S}$. As $\mathcal{S}$ is minimal balanced, $\mathcal{T} = \mathcal{D}^p(\rho, y, v)$ is not balanced. Hence, Remark 2.2 shows that $y \neq \nu^p(v)$ so that $\mathcal{PN}^p$ does not satisfy RSAM w.r.t. z. □

Applied to $z = \mathbb{1}^N$, Theorem 3.1 yields the following corollary.

Corollary 3.2. *For any $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$, the per capita prenucleolus is the unique weighted prenucleolus that satisfies* ESD.

For $n = 2$, any weighted prenucleolus satisfies SAM. Now we turn to SAM assuming $n > 2$. Let $p = (p_S)_{S \in \mathring{\mathcal{F}}}$ be a weight system. Recall that p is *symmetric* if p_S may only depend on $s = |S|$, i.e., we write $p_S = p(s)$ for all $S \in \mathring{\mathcal{F}}$ in this case. Note that a p-weighted prenucleolus satisfies the equal treatment property if and only if it is symmetric (see Kleppe et al., 2016, Theorem 3.3). Therefore, mainly symmetrically weighted prenucleoli were discussed in the literature. Housman and Clark (1998, Theorem 3) show that for a symmetric weight system p in the case $n = 3$ the p-weighted prenucleolus satisfies SAM if $p(1) > p(2)$ (and AM if $p(1) \geqslant p(2)$). We now show that the opposite is also true.

Proposition 3.3. *Let p be a symmetric weight system and $s, t \in \{1, \ldots, n-1\}$, $s \neq t$. If $sp(s) \leqslant (t-1)p(t)$, then $\mathcal{PN}^p$ does not satisfy* SAM. *If $sp(s) < (t-1)p(t)$, then $\mathcal{PN}^p$ does not satisfy* AM.

Proof. We may assume $t > 1$. Let $i_0 \in N$, $\mathcal{S} = \{S \subseteq N \setminus \{i_0\} \mid s = |S|\}$, and $\mathcal{T} = \{T \subseteq N \mid i_0 \in T, t = |T|\}$. Note that $\mathcal{S} \cup \mathcal{T}$ is balanced. Indeed, $|\mathcal{T}| = \binom{n-1}{t-1}$ and every $j \in N \setminus \{i_0\}$ is a member of $\binom{n-2}{t-2}$ elements of $\mathcal{T}$ and a member of $\binom{n-2}{s-1}$ elements of $\mathcal{S}$. Therefore it is straightforward to show that, with $\delta_S = \frac{(n-t)(s-1)!(n-s-1)!}{(n-1)!}$ for all $S \in \mathcal{S}$ and $\delta_T = \frac{(t-1)!(n-t)!}{(n-1)!}$ for all $T \in \mathcal{T}$, $\sum_{R \in \mathcal{S} \cup \mathcal{T}} \delta_R \mathbb{1}^R = \mathbb{1}^N$. Moreover, $\{\mathbb{1}^R \mid R \in \mathcal{S} \cup \mathcal{T}\}$ spans $\mathbb{R}^N$ so that, for any $\mathcal{S} \cup \mathcal{T} \subseteq \mathcal{B} \subseteq \mathring{\mathcal{F}}$, we conclude that $\mathcal{B}$ is balanced (see, e.g., Sudhölter, 1997, Remark 2.7(i)). Choose $v \in \Gamma$ by $v(R) = 0$ for all $R \in \mathcal{S} \cup \mathcal{T}$, $v(N) = n$, and $v(S) = \gamma$ for all other $S \in \mathring{\mathcal{F}}$, where γ is some negative constant. Moreover, let $v' = v^{(1)}$, i.e., $v'(S) = v(S)$ for all $S \in \mathring{\mathcal{F}}$ and $v'(N) = 0$. Let $x = 0 \in \mathbb{R}^N$. By Remark 2.2, $x = \nu^p(v')$. With $a = \frac{np(t)}{sp(s)+(n-t)p(t)}$, define $y \in \mathbb{R}^N$ by $y_i = a$ for all $i \in N \setminus \{i_0\}$ and $y_{i_0} = n - (n-1)a$. Then $y \in X(v)$ and $p(|R|)e(R, y, v) = -\frac{snp(s)p(t)}{sp(s)+(n-t)p(t)} = b$ for all $R \in \mathcal{S} \cup \mathcal{T}$ and, if γ is small enough, $p(|Q|)e(Q, y, v) < b$ for all $Q \in \mathring{\mathcal{F}} \setminus (\mathcal{S} \cup \mathcal{T})$. Hence, by Remark 2.2, $y = \nu^p(v)$. Now, is $\mathcal{PN}^p$ satisfies SAM, then $y \gg x$, i.e., $n - (n-1)a > 0$. Inserting the formula for a and eliminating the denominator yields $nsp(s) + n(n-t)p(t) - n(n-1)p(t) > 0$, i.e., $sp(s) > (t-1)p(t)$. Similarly, if $\mathcal{PN}^p$ satisfies AM, we receive $sp(s) \geqslant (t-1)p(t)$. □

Note that Proposition 3.3, applied to $s = 1$ and $t = 3$, reproves that the traditional nucleolus (all weights coincide) does not satisfy aggregate monotonicity, if $n \geqslant 4$ (see also Megiddo, 1974, for $n \geqslant 9$), which was already shown by Hokari (2000) even for the set of convex games.

As mentioned, the opposite statement of Proposition 3.3 is correct for $n = 3$ (Housman and Clark, 1998), but the inequalities $sp(s) > (t-1)p(t)$ for all $s, t \in \{1, \ldots, n-1\}$ do not guarantee that the p-weighted prenucleolus satisfies SAM. For instance, it may be shown similarly to Proposition 3.3 that $2p(3) > p(2)$ is necessary for SAM if $n > 4$.

In general we do not have a characterization of the weight systems p such that the p-weighted prenucleoli satisfy SAM. However, we now show that the set of weight systems p, not necessarily symmetric, generating p-weighted prenucleoli that satisfy SAM is open.

To this end the following two lemmas are needed. Let $\mathcal{W} \subseteq \mathbb{R}^{\mathring{\mathcal{F}}}$ denote the set of all weight systems. A suitable adjustment of the well-known proof of the continuity of the classical prenucleolus allows to show that any weighted prenucleolus is continuous as well. However, we need the following stronger result.

Lemma 3.4. *Let $f : \Gamma \times \mathcal{W} \to \mathbb{R}^N$ defined by $f(v,p) = \nu^p(v)$ for all $v \in \Gamma$ and $p \in \mathcal{W}$. Then f is continuous.*

Proof. Let $v, v^k \in \Gamma$ and $p, p^k \in \mathcal{W}$ for all $k \in \mathbb{N}$ such that $\lim_{k\to\infty} v^k = v$ and $\lim_{k\to\infty} p^k = p$. For each $k \in \mathbb{N}$ let $x^k = \nu^{p^k}(v^k)$.

Claim: The set $\{x^k \mid k \in \mathbb{N}\}$ is bounded. Indeed, let $y \in \mathbb{R}^N$ such that $y(N) \leqslant \inf_{k\in\mathbb{N}} v^k(N)$ which is finite because $\{v^k(N) \mid k \in \mathbb{N}\}$ is bounded. Moreover, $\{p^k_S e(S, y, v^k) \mid S \in \mathring{\mathcal{F}}, k \in \mathbb{N}\}$ is bounded so that $d = \sup_{k\in\mathbb{N}, S\in\mathring{\mathcal{F}}} p^k_S e(S, y, v^k)$ exists. Now, choose $z^k \in \mathbb{R}^N$ such that $z^k \geqslant y$ and $z^k(N) = v^k(N)$. Then $p^k_S e(S, y, v^k) \geqslant p^k_S e(S, z^k, v^k)$ for all $S \in \mathring{\mathcal{F}}$ and $\max_{S\in\mathring{\mathcal{F}}} p^k_S e(S, x^k, v^k) \leqslant \max_{S\in\mathring{\mathcal{F}}} p^k_S e(S, z^k, v^k) \leqslant d$. We conclude that, for all $k \in \mathbb{N}$, we have

$$x^k(N) = v^k(N) \text{ and } p^k_S e(S, x^k, v^k) \leqslant d \text{ for all } S \in \mathring{\mathcal{F}}.$$

In particular, $p^k_{\{i\}}(v^k(\{i\}) - x^k_i) \leqslant d$, i.e., $x^k_i \geqslant v^k(\{i\}) - d/p^k_{\{i\}}$, for all $k \in \mathbb{N}$ and $i \in N$, and our claim follows.

Let $(x^{k_j})_{j\in\mathbb{N}}$ be a convergent subsequence of $(x^k)_{k\in\mathbb{N}}$ and let $x = \lim_{j\to\infty} x^{k_j}$. We have to prove that $x = \nu^p(v)$. Let $\alpha \in \mathbb{R}$ such that $\mathcal{D}^p(\alpha, x, v) \neq \emptyset$. By Remark 2.2 it remains to show that $\mathcal{D}^p(\alpha, x, v)$ is balanced. Choose $\varepsilon > 0$ such that $\mathcal{D}^p(\alpha - 2\varepsilon, x, v) = \mathcal{D}^p(\alpha, x, v)$. As $\lim_{j\to\infty} x^{k_j} = x$, $\lim_{j\to\infty} v^{k_j} = v$, and $\lim_{j\to\infty} p^{k_j} = p$, there exists $J \in \mathbb{N}$ such that

$$p_S e(S, x, v) - \varepsilon < p^{k_j}_S e(S, x^{k_j}, v^{k_j}) < p_S e(S, x, v) + \varepsilon \text{ for all } S \in \mathring{\mathcal{F}} \text{ and } j > J.$$

Therefore, $\mathcal{D}^p(\alpha, x, v) = \mathcal{D}^{p^{k_j}}(\alpha - \varepsilon, x^{k_j}, v^{k_j})$ for all $j > J$. As $x^{k_j} = \nu^{p^{k_j}}(v^{k_j})$ for all $j \in \mathbb{N}$, $\mathcal{D}^p(\alpha, x, v)$ is balanced by Remark 2.2. □

In order to prove the second lemma it is useful to introduce some notation. An *ordered partition* of $\mathring{\mathcal{F}}$ is a system $(\mathcal{B}_1, \ldots, \mathcal{B}_t)$ for some $t \in \mathbb{N}$ such that, for all $j, k \in \{1, \ldots, t\}$ with $j \neq k$, $\emptyset \neq \mathcal{B}_k \subseteq \mathring{\mathcal{F}}$, $\mathcal{B}_j \cap \mathcal{B}_k = \emptyset$, and $\bigcup_{\ell=1}^t \mathcal{B}_\ell = \mathring{\mathcal{F}}$. Let $\mathbb{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_t)$ be an ordered partition, $p \in \mathcal{W}$, and $x \in \mathbb{R}^N$. For each $j \in \{1, \ldots, t\}$ let $(\mathcal{B}^j_1, \ldots, \mathcal{B}^j_{t_j})$ be the ordered partition of $\mathcal{B}_j$ such that, for all $i, \ell \in \{1, \ldots, t_j\}$ with $\ell < t_j$, (a) $\mathcal{B}^j_i \neq \emptyset$, (b) $p_S x(S) = p_T x(T)$ for all $S, T \in \mathcal{B}^j_i$, and (c) $p_S x(S) < p_T x(T)$ for all $S \in \mathcal{B}^j_\ell$ and $T \in \mathcal{B}^j_{\ell+1}$. We put

$$\mathbb{B}^p_x = (\mathcal{B}^1_1, \ldots, \mathcal{B}^1_{t_1}, \ldots, \mathcal{B}^t_1, \ldots, \mathcal{B}^t_{t_t})$$

and note that $\mathbb{B}^p_x$ is a refinement of $\mathbb{B}$.

Moreover, we say that $\mathbb{B}$ is a *configuration* if $\bigcup_{\ell=1}^k \mathcal{B}_\ell$ is balanced for all $k \in \{1, \ldots, t\}$.

Let $v \in \Gamma, p \in \mathcal{W}$, $x \in \mathbb{R}^N$, and $\mathbb{B}^p_{v,x} = (\mathcal{B}_1, \ldots, \mathcal{B}_t)$ be the ordered partition defined by the requirements that, for all $j, k \in \{1, \ldots, t\}$ with $j < t$, (a) $\mathcal{B}_k \neq \emptyset$, (b) $p_S e(S, x, v) = p_T e(T, x, v)$ for all $S, T \in \mathcal{B}_k$, and (c) $p_S e(S, x, v) > p_T e(T, x, v)$ for all $S \in \mathcal{B}_j$ and $T \in \mathcal{B}_{j+1}$. If $x \in X(v)$, then, by Remark 2.2, $\mathbb{B}^p_{v,x}$ is a configuration if and only if $x = \nu^p(v)$.
The following lemma is needed.

Lemma 3.5. *Let $p^* \in \mathcal{W}$ such that the p^*-weighted prenucleolus satisfies* SAM*. For any configuration $\mathbb{B}$ there exists $\varepsilon > 0$ such that, for any $p \in \mathcal{W}$ with $\|p - p^*\| < \varepsilon$, where $\|\cdot\|$ denotes the Euclidean norm, there exists $x \in \mathbb{R}^N$ such that $x \gg 0, x(N) = n$, and $\mathbb{B}^p_x$ is a configuration.*

Proof. We may assume without loss of generality that $p^*_S < \frac{1}{n}$ for all $S \in \mathring{\mathcal{F}}$. Let $\mathbb{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_t)$ be a configuration and define, for any $p \in \mathcal{W}$, $v^p, w^p \in \Gamma$ by $v^p(S) = w^p(S) = -\frac{j}{p_S}$ for all $S \in \mathcal{B}_j$ and all $j \in \{1, \ldots, t\}$, $v^p(N) = 0$, and $w^p(N) = n$. By Remark 2.2, $\nu^p(v^p) = 0 \in \mathbb{R}^N$. Moreover, let $x^p = \nu^p(w^p)$. Again by Remark 2.2, $\mathbb{B}^p_{w^p,x^p}$ is a configuration. As ν^{p^*} satisfies SAM, $x^{p^*} \gg 0$. By Lemma 3.4 there exists $\varepsilon > 0$ such that $x^p \gg 0$ for all $p \in \mathcal{W}$ with $\|p - p^*\| < \varepsilon$ and we may assume that ε is small enough such that, in addition, $p_S < \frac{1}{n}$ for all $S \in \mathring{\mathcal{F}}$. We conclude that, for all $S \in \mathcal{B}_j$ and all $j \in \{1, \ldots, t\}$, $-j > p_S e(S, x^p, w^p) > -j - 1$ because $n = x^p(N) > x^p(S) > 0$ and $p_S < \frac{1}{n}$, which guarantees that $\mathbb{B}^p_{w^p,x^p} = \mathbb{B}^p_{x^p}$. □

The foregoing lemma enables us to prove the following result.

Theorem 3.6. *The set of weight systems p such that the p-weighted prenucleolus satisfies* SAM *is open.*

Proof. Let $p^* \in \mathcal{W}$ such that the p^*-weighted prenucleolus satisfies SAM. It remains to show that, for any $p \in \mathcal{W}$ close enough to p^*, the p-weighted prenucleolus satisfies SAM as well. By Lemma 3.5, for any configuration $\mathbb{B}$ there exists $\varepsilon(\mathbb{B}) > 0$ such that, for all $p \in \mathcal{W}$ with $\|p - p^*\| < \varepsilon(\mathbb{B})$, there is $x^p \in \mathbb{R}^N, x^p \gg 0, x^p(N) = n$ such that $\mathbb{B}^p_{x^p}$ is a configuration. As the number of configurations is finite, there exists $\varepsilon > 0$ such that $\varepsilon \leqslant \varepsilon(\mathbb{B})$ for all configurations $\mathbb{B}$. Let $\|p - p^*\| < \varepsilon$. We claim that $\mathcal{PN}^p$ satisfies SAM. Assume, on the contrary, that there exists $v \in \Gamma$ and $\alpha_0 < 0$ such that $\nu^p(v^{(\alpha_0)}) - \nu^p(v) \not\gg 0$. By continuity of the p-weighted prenucleolus we may assume that $\nu^p(v^{(\alpha)}) - \nu^p(v) \not\gg 0$ for all α with $\alpha_0 \leqslant \alpha < 0$. Indeed, if $\nu^p(v^{(\alpha_1)}) \gg \nu^p(v)$ for some $\alpha_0 < \alpha_1 < 0$, then $\alpha_2 = \min\{\alpha \in \mathbb{R} \mid \alpha_0 \leqslant \alpha \leqslant \alpha_1, \nu^p(v^{(\alpha)}) \geqslant \nu^p(v^{(\alpha_1)})\}$ is well-defined by continuity so that we may in this case replace v by $v^{(\alpha_2)}$ and α_0 by $\alpha_0 - \alpha_2$. Let $x = \nu^p(v)$ and $\mathbb{B} = \mathbb{B}^p_{v,x}$. Then there exists $\alpha_0 \leqslant \delta < 0$ such that $\mathbb{B}^p_{v^{(\delta)},x-\delta x^p} = \mathbb{B}^p_{x^p}$ so that, by Remark 2.2 and Lemma 3.5, $x - \delta x^p = \nu^p(v^{(\delta)})$. As $\delta < 0$ and $x^p \gg 0$, $x - \delta x^p \gg x$ and the desired contradiction is obtained. □

We call a weight system p *regular* if there exists $z \in \mathbb{R}^N$ with $z \gg 0$ and $z(N) = n$ such that $z(Q)p_Q = z(R)p_R$ for all $Q, R \in \mathring{\mathcal{F}}$. Theorems 3.1 and 3.6 have the following immediate consequence.

Corollary 3.7. *For any $p \in \mathcal{W}$ that is close enough to some regular weight system the p-weighted prenucleolus satisfies* SAM.

4 Characterizations of the per capita prenucleolus

In order to present our axiomatization of the per capita prenucleolus, first on the set of balanced games, without employing any reduced game property, we recall that, for any $v \in \Gamma^b$, by (3), $v^{(\alpha)}$ is balanced if and only if $n\alpha \leqslant v(N) - \beta(v)$. Therefore, if $\beta(v) = v(N)$, there does not exist $\alpha > 0$ such that $v^{(\alpha)}$ is still balanced. In order to define a property that is similar to ESD, but also applicable when $\beta(v) = v(N)$, we define the following modification of $v^{(\alpha)}$, called *adjusted α-diminished* game of v and denoted by $v^{(\alpha,adj)}$. Namely, for $v \in \Gamma^b$ and $\alpha \geqslant 0$, $v^{(\alpha,adj)} \in \Gamma$ is defined by

$$v^{(\alpha,adj)}(S) = \begin{cases} v(S) - \alpha|S| & \text{, if } S \in \mathcal{E}(v), \\ v(S) & \text{, if } S \in \mathcal{F} \setminus \mathcal{E}(v). \end{cases} \tag{4}$$

Note that, if $\alpha > 0$ is small enough, then $v^{(\alpha,adj)}$ remains balanced. Indeed, there exists $x \in C(v)$ such that $x(S) > v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{E}(v)$. Now, define $x^{(\alpha)} = x - \alpha \mathbb{1}^N$ and observe that $x^{(\alpha)}(T) = v^{(\alpha,adj)}(T)$ for all $T \in \mathcal{E}(v)$ and $|x(S) - x^{(\alpha)}(S)| = |S|\alpha < n\alpha$ for $S \in \mathcal{F} \setminus \mathcal{E}(v)$, hence $x^{(\alpha)}(S) \geqslant v(S)$ for α small enough.
A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *equal adjusted surplus division* (EASD) if

$$\sigma\left(v^{(\alpha,adj)}\right) + \{\alpha \mathbb{1}^N\} \subseteq \sigma(v), \tag{5}$$

  whenever $\alpha > 0$ and $v, v^{(\alpha,adj)} \in \Gamma' \cap \Gamma^b$.

Note that $C(\cdot), X(\cdot)$, and $X^*(\cdot)$ satisfy EASD.

Theorem 4.1. *On Γ^b the per capita nucleolus is the unique solution that satisfies* SIVA*,* TCOV*,* SCOV*, and* EASD*.*

Now, referring to Sections 5 and 6 here, Theorem 4.1 is implied by Theorem 6.2 (i) applied to a per capita weight system together with Proposition 5.1.

Remark 4.2. (i) Each of the axioms employed in Theorem 4.1 is logically independent of the remaining axioms. Indeed, $X(\cdot)$, i.e., the solution that assigns to each game its set of preimputations, exclusively violates SIVA, and the equal split solution (assigning to each $v \in \Gamma$ the singleton $\{x\}$ defined by $x_i = \frac{v(N)}{n}$) exclusively violates TCOV. Let $\pi : N \to \{1, \ldots, n\}$ be an order of N, i.e., a bijection. For any $v \in \Gamma$, define $a^\pi = a^\pi(v) \in \mathbb{R}^N$ by $a^\pi_i = v(P^\pi_i \cup \{i\}) - v(P^\pi)$ for all $i \in N$, where $P^\pi_i = \{j \in N \mid \pi(j) < \pi(i)\}$. Note that the Shapley value (Shapley, 1953) of v is the average of all $a^\pi(v)$ taken over all orderings of N. Now, by its definition, the solution $\{a^\pi(\cdot)\}$ satisfies SIVA, SCOV, and TCOV so that it violates exclusively EASD. Finally, let $z \in \mathbb{R}^N \setminus \{0\}$ such that $z(N) \leqslant 0$ and define $\sigma(v) = \mathcal{PN}(v) + \{z\}$ for all $v \in \Gamma^b$. Then σ exclusively violates SCOV.

(ii) The foregoing properties of the solutions hold for any Γ' with $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$.

(iii) However, the solution that assigns the per capita prenucleolus to each game in Γ^b and $\{a^\pi(v)\}$ to each game in $v \in \Gamma' \setminus \Gamma^b$ satisfies the four axioms of the theorem because $\{a^\pi(\cdot)\}$ satisfies the former three axioms, Γ^b is closed under strategic equivalence, and EASD only refers to Γ^b.

Corollary 4.3. *Let* $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$. *The per capita prenucleolus on* Γ' *is the unique solution that satisfies* SIVA*,* TCOV*,* SCOV*,* EASD*, and* ESD*.*

Proof. We have seen that $\mathcal{PN}$ satisfies ESD and, by Theorem 4.1, it also satisfies the remaining axioms. For the uniqueness part let σ be a solution that satisfies the axioms. Moreover, let $v \in \Gamma'$. If v is balanced, then $\sigma(v) = \mathcal{PN}(v)$ by Theorem 4.1. Otherwise, let $\alpha = \frac{v(N)-\beta(v)}{n}$. By SIVA, there exists $x \in \mathbb{R}^N$ such that $\sigma(v) = \{x\}$ and, by ESD and SIVA, $\sigma(v^{(\alpha)}) = \{y\}$ where $y = x - \alpha \mathbb{1}^N$. As $v^{(\alpha)}$ is balanced, $\mathcal{PN}(v^{(\alpha)}) = \sigma(v^{(\alpha)})$ so that $x = \nu(v)$ by ESD of $\mathcal{PN}$. □

In view of Remark 4.2 each of the axioms employed in Corollary 4.3 is logically independent of the remaining axioms.
In view of (1), (2), and (3), for solutions on Γ^b, EASD implies ESD. As both the Shapley value and the per capita prenucleolus satisfy SIVA, TCOV, SCOV, and ESD on Γ^b, it follows that ESD does not replace EASD in Theorem 4.1 provided that $n \geqslant 3$.
Finally, we note that, though ESD is weaker than EASD on Γ^b, it is not satisfied by any weighted nucleolus except the per capita nucleolus by Corollary 3.2.

5 Independence of (adjusted) proportional shifting

Let $v \in \Gamma$ and $\alpha \in \mathbb{R}$. We recall the notation of the "α-shift" game w of v (see Definition 4.3 of Sudhölter, 1997):

$$w(S) = \begin{cases} v(S) + \alpha & \text{, if } S \in \mathring{\mathcal{F}}, \\ v(S) & \text{, if } S = N. \end{cases}$$

We define the *proportional α-shift game* $v_{(\alpha)}$ of v by

$$v_{(\alpha)}(S) = \begin{cases} v(S) + \alpha|S| & \text{, if } S \in \mathring{\mathcal{F}}, \\ v(N) & \text{, if } S = N. \end{cases} \tag{6}$$

Note that

$$v^{(\alpha)} = v_{(\alpha)} - \alpha \mathbb{1}^N(\cdot) \tag{7}$$

Moreover, if $v \in \Gamma^b$, we define the *adjusted proportional* α-shift game of v by

$$v_{(\alpha,adj)}(S) = \begin{cases} v(S) + \alpha|S| & \text{, if } S \in \mathcal{F} \setminus \mathcal{E}(v), \\ v(S) & \text{, if } S \in \mathcal{E}(v), \end{cases} \tag{8}$$

and observe that

$$v^{(\alpha,adj)} = v_{(\alpha,adj)} - \alpha \mathbb{1}^N(\cdot). \tag{9}$$

Now, we are ready to define our independence axioms. A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *independence of proportional shifting* (IPS) if $\sigma\left(v_{(\alpha)}\right) \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v_{(\alpha)} \in \Gamma'$;
- *independence of adjusted proportional shifting* (IAPS) if $\sigma\left(v_{(\alpha,adj)}\right) \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v_{(\alpha,adj)} \in \Gamma' \cap \Gamma^b$.

Hence, by (7) and (9), we have the following result.

Proposition 5.1. *Let $\Gamma' \subseteq \Gamma$ and let σ be a solution on Γ' that satisfies* TCOV*.*

- *If Γ' is closed under translations, i.e., $v \in \Gamma', x \in \mathbb{R}^N$ implies $v + x(\cdot) \in \Gamma'$, then σ satisfies* ESD *if and only if σ satisfies* IPS*.*
- *If $\Gamma' \cap \Gamma^b$ is closed under translations, then σ satisfies* EASD *if and only if σ satisfies* IAPS*.*

The following corollary is a special case (that of a per capita weight system) of Theorem 6.2 of Section 6. The logical independence of each of the employed axioms is also proved in the mentioned section.

Corollary 5.2. *(i) The per capita nucleolus is the unique solution on Γ^b that satisfies* SIVA*,* TCOV*,* SCOV*, and* IAPS*.*

(ii) On an arbitrary Γ', $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$, the per capita prenucleolus is the unique solution that satisfies SIVA*,* TCOV*,* SCOV*,* IAPS*, and* IPS*.*

The remainder of this section is devoted to the discussion of the impact of TCOV in Proposition 5.1.

The solution that assigns to each game v the per capita prenucleolus if $v(N) < 0$ and the equal split solution if $v(N) \geqslant 0$ satisfies the two independence axioms, but violates each of the equal division axioms, provided that $\Gamma' \cap \Gamma^b$ is rich enough. Hence, in general neither IPS implies ESD nor does IAPS imply EASD.

As mentioned, by the Bondareva-Shapley Theorem there exists a smallest real number $\beta(v)$ such that, if $v(N)$ is replaced by $\beta(v)$, the game is balanced. Now, we may consider the solution that assigns the equal split solution to any game v satisfying $\beta(v) < 0$ and the per capita prenucleolus to any other game. This solution satisfies ESD, but violates IPS, provided that Γ' is rich enough. Hence, in general ESD does not imply IPS.

Finally, by induction on

$$t(v) = \left| \left\{ \frac{e(S,x,v)}{|S|} \,\middle|\, S \in \mathcal{F} \setminus \mathcal{E}(v) \right\} \right| \text{ where } x = \nu(v), \tag{10}$$

we define a solution σ on Γ^b that satisfies EASD but violates IAPS: If $t(v) = 0$ (i.e., v is inessential), then $\sigma(v) = \mathcal{PN}(v)$ if $v(N) \leqslant 0$, and $\sigma(v) = X(v)$ if $v(N) > 0$. Now, if $t(v) \geqslant 1$, then, with $x = \nu(v)$, let

$$\alpha(v) = -\max \left\{ \frac{e(S,x,v)}{|S|} \,\middle|\, S \in \mathcal{F} \setminus \mathcal{E}(v) \right\} \tag{11}$$

and observe that $\{\alpha \geqslant 0 \mid v^{(\alpha,adj)} \in \Gamma^b\} = [0, \alpha(v)]$ and $\alpha(v) > 0$. Let $v' = v^{(\alpha(v),adj)}$. By EASD of the per capita prenucleolus, $t(v') = t(v) - 1$, hence $\sigma(v')$ is already defined by the inductive hypothesis. We now define $\sigma(v) = \sigma(v') + \{\alpha(v)\mathbb{1}^N\}$. Again by induction on $t(v)$ it may be shown that σ satisfies EASD. However, it does not satisfy IAPS.

Remark 5.3. We now verify that EASD together with SIVA imply IAPS for any solution σ on Γ^b. Indeed, let σ satisfy SIVA and EASD. Let $v \in \Gamma^b$, $\alpha > 0$, and $w = v + \alpha\mathbb{1}^N(\cdot)$. By (9) it suffices to prove that $\sigma(w) = \sigma(v) + \{\alpha\mathbb{1}^N\}$. We proceed by induction on $t(v)$ defined by (10). If $t(v) = 0$, i.e., v is inessential ($\mathcal{E}(v) = \mathcal{F}(v)$), then w is also inessential (i.e., $\mathcal{E}(w) = \mathcal{E}(v)$) so that $v = w^{(\alpha,adj)}$ and the proof is finished by SIVA and EASD. If $t(v) > 0$, then TCOV of the per capita nucleolus guarantees that $\mathcal{E}(v) = \mathcal{E}(w)$, $\alpha(v) = \alpha(w)$, where $\alpha(v)$ and $\alpha(w)$ are defined as in (11), and with $v' = v^{(\alpha(v),adj)}$ and $w' = w^{(\alpha(w),adj)}$, $t(v') = t(w') = t(v) - 1$ and $w' = v' + \alpha\mathbb{1}^N(\cdot)$. Hence, by the inductive hypothesis, $\sigma(w') = \sigma(v') + \{\alpha\mathbb{1}^N\}$ so that, again by SIVA and EASD, $\sigma(v) = \sigma(w)$ and the proof is complete.

6 Generalizing the independence axioms to arbitrary weight systems

Let $p = (p_S)_{S \in \mathring{\mathcal{F}}}$ be a weight system, let $v \in \Gamma$, and $\alpha \in \mathbb{R}$. Define the (α, p)-*shift* game $v_{(\alpha,p)} \in \Gamma$ by

$$v_{(\alpha,p)}(S) = \begin{cases} v(S) + \frac{\alpha}{p_S} & \text{, if } S \in \mathring{\mathcal{F}}, \\ v(N) & \text{, if } S = N, \end{cases} \tag{12}$$

and, if $v \in \Gamma^b$, define the *adjusted* (α, p)-*shift* game $v_{(\alpha,p,adj)} \in \Gamma$ by

$$v_{(\alpha,p,adj)}(S) = \begin{cases} v(S) + \frac{\alpha}{p_S} & \text{, if } S \in \mathcal{F} \setminus \mathcal{E}(v), \\ v(S) & \text{, if } S \in \mathcal{E}(v). \end{cases} \tag{13}$$

Now, we can suitably modify our independence axioms. A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *independence of p-shifting* (IS_p) if $\sigma(v_{(\alpha,p)}) \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v_{(\alpha,p)} \in \Gamma'$;
- *independence of adjusted p-shifting* (IAS_p) if $\sigma(v_{(\alpha,p,adj)}) \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v_{(\alpha,p,adj)} \in \Gamma' \cap \Gamma^b$.

Lemma 6.1. *Let p be a weight system. On any* $\Gamma' \subseteq \Gamma$, $\mathcal{PN}_p$ *satisfies* IS_p *and* IAS_p.

Proof. Let $v \in \Gamma, x = \nu^p(v), \alpha > 0, v' = v_{(\alpha,p)}$, and $S \in \mathring{\mathcal{F}}$. Then $p_S e(S, x, v') = p_S e(S, x, v) + \alpha$ so that IS_p follows from Remark 2.2 (i). Now, assume that v and $w = v_{(\alpha,p,adj)}$ are balanced. It suffices to show that $\nu^p(w) = x$. Hence, we may assume that $\mathcal{E}(v) \subsetneqq \mathcal{F}$. Let $y \in C(w)$. As $w(N) = v(N)$ and $w(S) \geqslant v(S)$ for all $S \in \mathcal{F}$, $y \in C(v)$. We conclude that $y(S) \geqslant v(S) + \frac{\alpha}{p_S}$, i.e., $p_S e(S, y, v) \leqslant -\alpha$ for all $S \in \mathcal{F} \setminus \mathcal{E}(v)$. As x lexicographically minimizes the non-increasingly ordered vector of p-weighted excesses, $p_S e(S, x, v) \leqslant -\alpha$ for all $S \in \mathcal{F} \setminus \mathcal{E}(v)$ as well. Hence, $x \in C(w)$ so that $\gamma_1 = \max\{p_S e(S, x, v) \mid S \in \mathcal{F} \setminus \mathcal{E}(v)\} \leqslant -\alpha$. If $\gamma_1 < -\alpha$, then

$$D^p(\gamma, x, w) = \begin{cases} D^p(\gamma - \alpha, x, v) & \text{, if } \gamma < 0, \\ D^p(0, x, v) & \text{, if } \gamma = 0, \\ \emptyset & \text{, if } \gamma > 0. \end{cases}$$

If $\gamma_1 = -\alpha$, then

$$D^p(\gamma, x, w) = \begin{cases} D^p(\gamma - \alpha, x, v) & \text{, if } \gamma \leqslant 0, \\ \emptyset & \text{, if } \gamma > 0. \end{cases}$$

In any case, $x = \nu^p(w)$ by Remark 2.2 (i). □

Theorem 6.2. *(i) The p-weighted nucleolus is the unique solution on* Γ^b *that satisfies* SIVA, TCOV, SCOV, *and* IAS_p.

(ii) On any Γ', $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$, *the p-weighted prenucleolus is the unique solution that satisfies* SIVA, TCOV, SCOV, IAS_p, *and* IS_p.

Proof. (i) It is well known that $\mathcal{PN}^p$ satisfies SIVA, TCOV, and SCOV, and by Lemma 6.1 it satisfies IAS$_p$. In order to show uniqueness, let σ be a solution that satisfies SIVA, TCOV, SCOV, and IAS$_p$. Let $v \in \Gamma^b$. By SIVA, $\sigma(v) = \{x\}$ for some $x \in \mathbb{R}^N$. Let $y = \nu^p(v)$. It remains to show that $x = y$. Let $t(v) = |\{p_S e(S, y, v) \mid S \in \mathcal{F} \setminus \mathcal{E}(v)\}|$. We proceed by induction on $t(v)$. If $t(v) = 0$, then v is inessential. By TCOV, we may assume that $v(S) = 0$ for all $S \subseteq N$. By SIVA and SCOV, $\{\gamma x\} = \sigma(\gamma v) = \sigma(v)$ for all $\gamma > 0$. Hence, $x = y = 0 \in \mathbb{R}^N$. If $t(v) > 0$, then let $\alpha = -\max\{p_S e(S, y, v) \mid S \in \mathcal{F} \setminus \mathcal{E}(v)\}$. By IAS$_p$ of $\mathcal{PN}^p$, $y = \nu^p(v_{(\alpha,p,adj)})$ so that $t(v_{(\alpha,p,adj)}) = t(v) - 1$. By our induction hypothesis $\sigma(v_{(\alpha,p,adj)}) = \{y\}$. Hence, by IAS$_p$ of σ, $x = y$.

(ii) $\mathcal{PN}^p$ satisfies SIVA, TCOV, SCOV, and IAPS$_p$ and IS$_p$ by Lemma 6.1. To show the uniqueness part, let σ be a solution that satisfies the five foregoing axioms. Let $v \in \Gamma'$ and $y = \nu^p(v)$. By SIVA, $\sigma(v) = \{x\}$ for some $x \in \mathbb{R}^N$ and it remains to show that $x = y$. If $v \in \Gamma^b$, then $x = y$ by part (i). Hence, we may assume that v is not balanced. Choose $\alpha \geqslant \max\{p_S e(S, y, v) \mid S \in \mathring{\mathcal{F}}\}$ and observe that $v' = v_{(-\alpha,p)}$ is balanced because $y \in C(v')$. As $v = v'_{(\alpha,p)}$, by IS$_p$ of $\mathcal{PN}^p$, $y = \nu^p(v')$. We have already proved that $\mathcal{PN}$ coincides with σ on balanced games so that $\sigma(v') = \mathcal{PN}(v') = \{y\}$ and the proof is finished by IS$_p$ of σ.

□

By means of examples we now show that each of the properties employed in Theorem 6.2 is logically independent of the remaining properties. Indeed, $X(\cdot)$ exclusively violates SIVA, the equal split solution exclusively violates TCOV, and the solution that assigns $\mathcal{PN}^p + \{z\}$, where $z \in \mathbb{R}^N \setminus \{0\}$ satisfies $z(N) \leqslant 0$, to any game v exclusively violates SCOV. To show that IAS$_p$ is logically independent, let, for any $v \in \Gamma$, $\alpha'(v) = \max\{\alpha \in \mathbb{R} \mid v_{(\alpha,p)} \in \Gamma^b\}$ and define $\sigma(v) = \{a^\pi(v_{(\alpha'(v),p)})\}$, where π is some ordering of N (see Remark 4.2 for the definition of $a^\pi(\cdot)$). Notice that $v_{(\alpha'(v_{(\alpha,p)}),p)} = v_{(\alpha'(v),p)}$ for all $v \in \Gamma$ and $\alpha \in \mathbb{R}$. Hence, σ is well-defined. Thus, σ satisfies IS$_p$ and SIVA. It is straightforward to check that σ also satisfies TCOV and SCOV. Finally, IS$_p$ is exclusively violated by the solution that coincides with the p-weighted prenucleolus on Γ^b and with $\{a^\pi(\cdot)\}$ on $\Gamma' \setminus \Gamma^b$, provided that $\Gamma' \setminus \Gamma^b \neq \emptyset$, where π is some ordering of N. To show that there is a solution of this kind that does not coincide with $\mathcal{PN}^p$, choose $v \in \Gamma' \setminus \Gamma^b$ and $S \in \mathring{\mathcal{F}}$ with $v(S) > x(S)$, where $x = \nu^p(v)$. Choose $i \in S$ and an ordering π such that $P_i^\pi = S \setminus \{i\}$. Hence, with $y = a^\pi(v)$, we receive $y(S) = v(S)$, hence $a^\pi(v) \neq \nu^p(v)$.

Remark 6.3. Let p be a weight system of the traditional prenucleolus, i.e., $p_S = p_T$ for all $S, T \in \mathring{\mathcal{F}}$. It should be noted that in this case IAS$_p$ is related to the property "relative independence of slack coalitions" of Oswald et al. (1998) that only apparently aims into the opposite direction.

In our notation, and generalizing the property suitably to set-valued solutions, a solution σ on Γ^b satisfies relative independence of slack coalitions if $\sigma(w) \subseteq \sigma(w_{(-\alpha,p,adj)})$ for all $w \in \Gamma^b$ and all $\alpha > 0$. Our IAS$_p$ implies relative independence of slack coalitions because with $v = w_{(-\alpha,p,adj)}$, $w = v_{(\alpha,p,adj)}$. However, the mentioned authors need continuity in addition to characterize the traditional nucleolus on Γ^b. The reason is simple. Namely, for the foregoing games w and v compared by relative independence of slack coalitions, $\mathcal{E}(v) = \mathcal{E}(w)$. For IAS$_p$, the situation may differ: If $\mathcal{E}(v) \subsetneqq \mathcal{F}$, then with $\alpha(v)$ defined by (11), we receive $\mathcal{E}(v) \subsetneqq \mathcal{E}(v_{(\alpha(v),p,adj)})$. Therefore, our property is stronger. Indeed, we will now generalize the set-valued version of relative independence of slack coalitions to arbitrary weight systems and provide an example (Example 6.4) that shows that this property does not replace IAS$_p$ in (i) of Theorem 6.2. Hence, as in the characterization of the traditional (pre)nucleolus by Oswald et al. (1998), an additional property like continuity would be needed when replacing IAS$_p$ by the aforementioned weaker property.

We say that a solution σ on Γ^b satisfies *p-relative independence of slack coalitions* (p-RISC) if $\sigma(v) \subseteq \sigma(v_{(-\alpha,p,adj)})$ for all $v \in \Gamma^b$ and all $\alpha > 0$. The following example shows that p-RISC does not replace IAS$_p$ in (i) of Theorem 6.2 provided $n \geqslant 3$.

Example 6.4. Let $n \geqslant 3$ and $k, \ell \in N, k \neq \ell$. Choose $x \in \mathbb{R}^N$ with $x(N) \leqslant 0$. For any $t \in \mathbb{R}$, define $w_t \in \Gamma^b$ by

$$w_t(S) = \begin{cases} -\frac{t}{p_{\{k\}}} & \text{, if } S = \{k\}, \\ -1 - \frac{t}{p_{\{\ell\}}} & \text{, if } S = \{\ell\}, \\ 0 & \text{, if } S \in \mathcal{F} \setminus \{\{k\}, \{\ell\}\}. \end{cases}$$

Note that the core of w_t is the singleton $\{0\}$ for any $t \geqslant 0$, and it is empty, whenever $t < 0$. Moreover, w_t is strategically equivalent to $w_{t'}$ (i.e., there exists $\beta > 0$ and $z \in \mathbb{R}^N$ such that $w_{t'} = \beta w_t + z(\cdot)$) if and only if $t = t'$. Hence, we may define our solution σ by $\sigma(\beta w_t + z(\cdot)) = \{\beta x + z\}$ for all $\beta, t > 0$ and $z \in \mathbb{R}^N$ and $\sigma(v) = \{\nu^p(v)\}$ for all $v \in \Gamma^b$ that are not strategically equivalent to any w_t, $t > 0$. By construction, σ satisfies SIVA, SCOV, and TCOV. Moreover, if $v \in \Gamma^b$ is not strategically equivalent to some w_t and $\alpha > 0$, then $v_{(-\alpha,p,adj)}$ is also not strategically equivalent to any w_t. Finally, with $v = \beta w_t + z$ for some $t, \beta > 0$ and $z \in \mathbb{R}^N$, we receive $v_{(-\alpha,p,adj)} = \beta w_{t'} + z$, where $t' = t + \alpha/\beta$. Hence, σ satisfies p-RISC.

7 Final remarks

Theorem 5.2 of Kleppe et al. (2016) axiomatizes a weighted prenucleolus exclusively for symmetric weight systems. Hence, Theorem 6.2 may be regarded as an advantage over the mentioned axiomatization as it characterizes also p-weighted (pre)nucleoli when p is not symmetric.

For the special class of nonsymmetric weight systems p that result in p-weighted prenucleoli that satisfy RSAM (see Section 3) we may proceed as in Section 4 and define, for any $z \in \mathbb{R}^N$ with $z \gg 0$ and $z(N) = n$, for any $v \in \Gamma^b$, and any $\alpha \in \mathbb{R}$,

$$v^{(\alpha,z,adj)}(S) = \begin{cases} v(S) & \text{, if } S \in \mathcal{F} \setminus \mathcal{E}(v), \\ v(S) - \alpha z(S) & \text{, if } S \in \mathcal{E}(v). \end{cases} \tag{14}$$

Then we may say that a solution σ on a set Γ' of games satisfies

- *regular adjusted strong aggregate monotonicity* (RASAM) w.r.t. z if $\sigma(v^{(\alpha,z,adj)}) + \{\alpha z\} \subseteq \sigma(v)$, whenever $\alpha > 0$ and $v, v^{(\alpha,z,adj)} \in \Gamma' \cap \Gamma^b$.

Defining the weight system $p(z) = p$ by $p_S = \frac{1}{z(S)}$ for all $S \in \mathring{\mathcal{F}}$, we receive, similarly as in Section 5, $v^{(\alpha)} = v_{(\alpha,p(z))} - \alpha z(\cdot)$ and $v^{(\alpha,z,adj)} = v_{(\alpha,p(z),adj)} - \alpha z(\cdot)$, respectively. As RSAM and RASAM w.r.t. $z = \mathbb{1}^N$ coincide with ESD and EASD, respectively, the following result that is an immediate consequence of Theorem 6.2 generalizes Theorem 4.1 and Corollary 4.3.

Corollary 7.1. *Let $z \in \mathbb{R}^N$ such that $z \gg 0$ and $z(N) = n$. Let p be the weight system defined by $p_S = \frac{1}{z(S)}$ for all $S \in \mathring{\mathcal{F}}$.*

(i) The p-weighted nucleolus is the unique solution on Γ^b that satisfies SIVA*,* TCOV*,* SCOV*, and* RASAM *w.r.t. z.*

(ii) On an arbitrary Γ', $\Gamma^b \subseteq \Gamma' \subseteq \Gamma$, the p-weighted prenucleolus is the unique solution that satisfies SIVA*,* TCOV*,* SCOV*, and* RASAM *and* RSAM*, both w.r.t. z.*

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