On the submodularity of multi-depot traveling salesman games

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On the submodularity of multi-depot traveling salesman games

Trine Tornøe Platz*

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Abstract

The Steiner traveling salesman problem (STSP) is the problem of finding a minimum cost tour for a salesman that must visit a set of locations while traveling along costly streets before returning to his starting point at the depot. A solution to the problem is a minimum cost tour that both starts and ends at the depot and visits all the required locations. If different players are associated with the destinations to be visited, the STSP induces a cooperative traveling salesman (TS) game that poses the question of how to allocate the total cost of the tour between the different players involved. This cost allocation problem can be tackled using tools and solutions from cooperative games.

The purpose of this paper is to generalise the notion of a traveling salesman (TS) game to allow for multiple depots in the underlying STSP and to analyse the submodularity of such multi-depot TS games. A multi-depot STSP can be represented by a connected (di)graph in which a fixed set of nodes are denoted depots, and a non-negative weight function is defined on the edges of the graph. The submodularity of multi-depot TS games is analysed by characterising graphs and digraphs that induce submodular multi-depot TS games for any position of the depots and for at least one position of the depots, respectively.

Keywords: Traveling salesman problem, cooperative game, submodularity

JEL Classification Number: C71.

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1 Introduction

The Steiner traveling salesman problem (STSP) is a routing problem studied in the operations research literature.[1] In its most simple form it models the situation in which a salesman affiliated with a fixed depot (warehouse) has to visit a number of locations before returning to the warehouse. There is a cost associated with traveling along the roads that connect the locations, and a solution to the problem is a minimum cost tour that visits the required locations while starting and ending at the depot. This basic model has been extended in a number of directions, for example by allowing for multiple salesmen, multiple warehouses, and by adding restrictions on visiting times as well as capacity.

The STSP can be represented by a graph in which a fixed vertex is denoted the depot, and a weight function is defined on the edges. If the vertices to be visited are associated with different players, the STSP induces a cooperative cost allocation game denoted a traveling salesman game (Potters et al., 1992) in which the cost of a coalition of players is the cost of a minimum weight tour that visits all players in the coalition. Once the cooperative game is defined, tools and solution concepts from cooperative game theory can be applied to analyze and solve the cost allocation problem.

The purpose of this paper is to extend the notion of a traveling salesman game to allow for multiple depots in the underlying STSP and to consider the submodularity of such multi-depot TS games. The multi-depot setup considered here may be interpreted as a situation in which several depots exist, each with their own salesman/vehicle of unlimited capacity, such that an optimal tour may be a collection of subtours each originating from a different depot and returning to it's point of origin.

The focus on submodularity is motivated by the desirable properties of submodular games. For one thing, submodular games are totally balanced, implying that the core of every subgame is non-empty. The Shapley value is in the core of and is the barycenter of the core, Shapley (1971). Furthermore, several solution concepts coincide (the nucleolus and the kernel, the bargaining set and the core, Maschler et al. (1971), and others can be more easily computed for this class of games than is generally the case.

In the following, a multi-depot TS problem with k depots will be denoted a k-depot TS problem. Furthermore, a graph G is said to be globally (locally) k-TS submodular

[1]The Steiner traveling salesman problem generalizes the traditional traveling salesman problem by allowing multiple visits to the same vertices and by allowing the set of nodes to be visited to be only a subset of the nodes of the graph.

if the game induced by a k-TS problem on G is submodular for every (some) location of the k depots and for every weight function. This paper characterizes the classes of globally and locally k-TS submodular graphs and digraphs, respectively.

For the standard case with just one depot in the underlying STSP, Herer and Penn (1995) provided a characterization of the undirected graphs that induce submodular traveling salesman games. This result was extended to the directed case in Granot et al. (2000). The games analyzed in these papers, can be seen as special cases of the multi-depot TS game for which $k = 1$. In a related string of research, Granot et al. (1999) characterized for the case of Chinese postman games both CP-balanced and CP-submodular graphs. In Granot and Hamers (2004), the authors distinguished between global and local requirements and characterized the class of locally CP-submodular graphs as well as locally TS-submodular graphs. The approach of analyzing both global and local requirements is followed in the present paper. Another related paper is Platz and Hamers (2015) that introduce multi-depot Chinese postman games and characterize the classes of (globally and locally) k-CP balanced and k-CP submodular graphs. The modelling of the multi-depot TS game in the present paper follows a similar approach, but due to the different combinatorial nature of the two problems, the results and proofs differ.

The results of this paper show that an undirected graph G is globally k-TS submodular for $k \in \{2, ..., |V(G)| - 3\}$ if and only if G does not contain a path of 5 vertices or more. Likewise, directed graphs are globally k-TS submodular only if they do not contain specific forbidden structures. For both undirected and directed graphs, the class of globally k-TS submodular graphs are a proper subset of the class of locally k-TS subodular graphs.

In analyzing properties of games by characterizing classes of graphs that induce games with nice properties, I follow an established line of literature on cooperative games arising from underlying optimization problems. A few examples of such classes of games are minimum cost spanning tree games (Granot and Huberman, 1981), sequencing games (Curiel et al., 1994), Chinese postman games (Hamers et al., 1999), and minimum colouring games (Deng et al., 2000). An overview can be found in Curiel (2010).

The paper is structured as follows. Section 2 presents some useful terms and notation. In section 3, multi-depot traveling salesman games are introduced. Results are presented in sections 4 and 5 for undirected and directed graphs respectively.

2 Preliminaries

From cooperative games, we recall the following definitions: In a cooperative (cost) game (N, c), $N = \{1, \ldots, n\}$ denotes the finite playerset, and $c : 2^N \rightarrow \mathbb{R}$ is a function that assigns to every coalition $S \subseteq N$ a *cost* $c(S)$, with $c(\emptyset) = 0$. Let $x \in \mathbb{R}^N$ be an allocation of $c(N)$ between the players. A game (N, c) is monotonic if $c(S) \leq c(T)$ for all $S \subset T \subseteq N$, and it is subadditive if $c(S \cup T) \leq c(S) + c(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. The core of a game (N, c) is given by

$$core(N, c) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

The game (N, c) is submodular if

$$c(T \cup \{i\}) - c(T) \leq c(S \cup \{i\}) - c(S) \tag{2.1}$$

for all $i \in N$ and all $S \subset T \subseteq N \setminus \{i\}$.

Next, we recall some notions from graph theory. An undirected (directed) *graph* $G = (V(G), E(G))$ consist of a non-empty, finite set of *vertices* $V(G)$ and a set of pairs of vertices $E(G)$ called *edges (arcs)*. An edge $\{a, b\}$ in an undirected graph and an arc (a, b) in a directed graph join the vertices a, b and are said to be *incident* to a and b. The vertices a and b are *adjacent*. The *degree* of a vertex a is equal to the number of edges incident to a. An arc (a, b) is directed from a to b and can only be traversed in this direction. A *subdivision* of a graph G is the graph G' that arising by (repeatedly) replacing an edge (arc) in G with a path of length two. A *(directed) walk*, w, in a graph G is a sequence of vertices and edges (arcs) on the form: $v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m$, where $v_0, \ldots, v_m \in V(G)$, $e_1, \ldots, e_m \in E(G)$, and $m \geq 0$ such that $e_j = \{v_{i-1}, v_j\}$ for all $i \in \{1, \ldots m\}$. If $v_0 = v_m$, the walk is said to be *closed*. A closed walk may be empty, $w = \{v_0\}$. A *(directed) path* is a (directed) walk in which no edge (arc) or vertex is visited more than once, except v_0 in the case of $v_0 = v_m$. If there exists an undirected (directed) path between any to vertices in a (directed) graph G, then G is a *(strongly) connected* graph. A closed (directed) walk in which no edge is visited more than once will be denoted a directed circuit, while a (directed) cycle denotes a closed (directed) path, that is a (directed) walk in which no edge (arc) or vertex is visited more than once.

Let G be a graph, and let v_s and v_t be two vertices in G. Then an s - t vertex cut is a set of vertices such that removing these vertices along with the edge (v_s, v_t), if it exists, results in a disconnected graph in which v_s and v_t do not belong to the same component. A *minimal* s - t vertex cut, K, is an s - t vertex cut such that no proper subset of K is an s - t vertex cut.

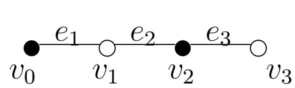
3 Multi-depot traveling salesman games

Let $G = ((V(G), E(G))$ be a (strongly) connected (di)graph, and let $Q \subset V(G)$ be a fixed subset of the vertices of G. An element of Q is called a depot. A multi-depot (Steiner) traveling salesman problem Γ is given by the tuple $\Gamma = \{V^-(G), (G, Q), t\}$ in which $V^-(G) = V(G) \setminus Q$ denotes the set of vertices in G that are not associated with depots, and $t : E(G) \to [0 : \infty)$ is a weight function defined on the edges (arcs) of G.

Let $w = (v_0, e_1, v_1, \ldots, e_m, v_m)$ denote a walk in G. The cost of w equals the sum of the weights of the edges visited, that is $cost(w) = \sum_{i=1}^{m} t(e_i)$. A closed walk that starts and ends at a vertex $v \in Q$ is denoted w_v. A closed walk $w_v = \{v\}$ is said to be empty, and the cost of an empty walk is 0.

Let G be a (strongly) connected (di)graph, and let $\Gamma = \{V^-(G), (G, Q), t\}$ be a multi-depot STSP defined on G. Next, let $S \subseteq V^-(G)$ be a subset of the non-depot vertices of G. Then for a given $Q = \{v_1, \ldots, v_k\}$, an S-tour $d(S)$ is a collection of closed walks $d(S) = \{w_{v_1}, \ldots, w_{v_k}\}$ (some of which may be empty) such that every node in S is visited. The set of all S-tours is denoted $D(S)$. The notion of an S-tour can be illustrated using Figure 1. In Figure 1, depots are located at vertices v_0 and v_2, while v_1 and v_3 are associated with players. Two possible S-tours for $S = \{v_1, v_3\}$ are then $v_2, e_2, v_1, e_2, v_2, e_3, v_3, e_3, v_2$ and the tour consisting of the two subtours v_0, e_1, v_1, e_1, v_0 and v_2, e_3, v_3, e_3, v_2.

Figure 1: S-tours in a multi-depot setting



When depots are located at the vertices of Q, the cost of an S-tour $d(S) = \{w_{v_1}, \ldots, w_{v_k}\}$ is equal to:

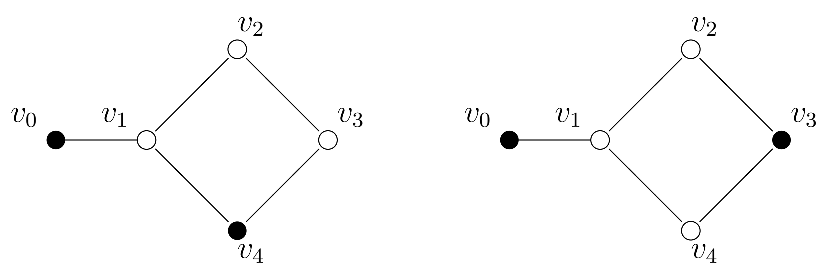
$$C_Q(d(S)) = \sum_{i=1}^{k} cost(w_{v_i})$$

Now, the multi-depot TS game induced by a multi-depot STSP can be defined as follows. Let $G = (V(G), E(G)$ be a (strongly) connected (directed) graph and let $\Gamma = \{V^-(G), (G, Q), t\}$ be a multi-depot STSP defined on G. Then (N, c_Q) is the induced multi-depot TS game in which $N = V^-(G)$ is the set of players, and $c_Q(S)$ is, for any $S \subseteq N$, the cost of a minimum weight S-tour, when the depots are located at the vertices of Q. That is:

$$c_Q(S) = min_{d(S) \in D(S)} C_Q(d(S)).$$

Note that since $N = V^-(G) = V(G) \setminus Q$, different sets of depots imply different sets of players, and in particular, a greater number of depots implies fewer players. An illustration of two 2-depot TS-problems and their induced games are given in the example below.

Example 3.1. The graph in Figure 2 illustrates two different 2-depot TS problems defined on the same graph. For the problem on the left, $Q = \{v_0, v_4\}$ while $Q = \{v_0, v_3\}$ for the problem on the right. For both STSPs assume that $t(\{v_1, v_4\}) = 10$ while $t(e) = 1$ for all other edges in the graph. The two induced 2-depot TS games are shown in the table below. Note that as the location of the depots change, so does the player set.



S	v_1	v_2	v_3	v_1, v_2	v_1, v_3	v_2, v_3	v_1, v_2, v_3
$c_{\{v_0,v_4\}}(S)$	2	4	2	4	4	4	6

S	v_1	v_2	v_4	v_1, v_2	v_1, v_4	v_2, v_4	v_1, v_2, v_4
$c_{\{v_0,v_3\}}(S)$	2	2	2	4	4	4	6

Figure 2: Two 2-TS problems and their induced 2-TS games

We see that with the current choice of weight function, the induced game is not submodular, when the depots are located at v_0 and v_4, since $c_{\{v_0,v_4\}}(v_1, v_2, v_3) - c_{\{v_0,v_4\}}(v_2, v_3) =$

$2 > 0 = c_{\{v_0,v_4\}}(v_1, v_2) - c_{\{v_0,v_4\}}(v_2)$. On the other hand, the induced game is submodular, when $Q = \{v_0, v_3\}$.

While not every k-depot TS game is submodular, it is straightforward to verify that for any multi-depot TS game (N, c_Q) induced by a multi-depot STSP given by $\Gamma(V^-(G), (G, Q), t)$, it holds that $c_Q(S) \leq c_Q(T)$ for all $S \subset T \subseteq N$ (the game is monotonic), and $c_Q(S \cup T) \leq c_Q(S) + c_Q(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$ (the game is subadditive).

We now turn to the analysis of multi-depot traveling salesman games. Both undirected and directed graphs will be considered, and in both cases, classes of graphs that give rise to submodular multi-depot TS games are characterized. Since any subadditive two-player game is submodular, only games of at least three players are considered.Therefore, $|V(G)| \geq k + 3$ for all k-depot STS problems considered throughout the paper.

4 k-TS submodular undirected graphs

4.1 Globally k-TS submodular graphs

Before proceeding to the characterization of graphs, a definition and a few useful observations are stated.

Let P_5 denote a path with 5 vertices. Let P_5^F denote the structure illustrated in Figure 3, consisting of a path P_5 where depots are located only at the endpoints.

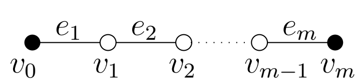


Figure 3: The P_5^F structure on a path of length 4 or more

Definition 4.1. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot TS problem defined on a connected graph G. Then G is *P_5^F-free with respect to Q*, if no path $v_0, e_1, v_1, \ldots, e_m, v_m$ exists in Γ such that $v_0, v_m \in Q$, $v_1, \ldots, v_{m-1} \in V^-(G)$ and $m \geq 4$.

That is, a graph G is P_5^F-free with respect to a set of depots Q, if G does not contain a structure like Figure 3, when depots are located at the vertices in Q. Next, observe that the presence of a P_5^F structure is incompatible with submodularity of the induced multi-depot TS game.

Lemma 4.1. Let $G = (V(G), E(G))$ be a connected graph, let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot STSP defined on G, and let (N, c_Q) be the induced cooperative game. If (N, c_Q) is submodular for all weight functions t, then G is P_5^F-free wrt. Q.

Proof. Assume on the contrary that G is not P_5^F-free with respect to Q. Then there exists a path $P := v_0, e_1, v_1, \ldots, e_m, v_m$ in G, such that $v_0, v_m \in Q$, $v_1, \ldots, v_{m-1} \in V^-(G)$ and $m \geq 4$. Let $t(e_i) = 1$ for $i \in \{1, 2, 3\}$, let $\sum_{i=4}^{m} t(e_i) = 1$, and let $t(e) = 10$ for all other edges in G. Then $c_Q(v_1, v_2, v_3) - c_Q(v_1, v_2) = 6 - 4 = 2$ while $c_Q(v_2, v_3) - c_Q(v_2) = 4 - 4 = 0$, violating (2.1), and c_Q is not submodular. □

Another useful observation is in order.

Lemma 4.2. Let $G = (V(G), E(G))$ be a connected, undirected graph, and let $k \in \{2, \ldots, |V(G)| - 3\}$. If G is a star graph, then G is globally k-TS-submodular.

Proof. Let G be a star graph, in which v_c is the single vertex with degree larger than 1. v_c will be referred to as the center vertex. For any $v \in V(G) \setminus v_c$, let e_v denote the edge connecting v to the center vertex. Consider $\Gamma = (V^-(G), (G, Q), t)$ with $|Q| = k$, and let (N, c_Q) be the induced k-TS game on G. Now, if $S = \emptyset$, then $c(S \cup v) - c(S) = c(v)$, and it follows from the subadditivity of c that $c_Q(T \cup v) - c_Q(T) \leq c_Q(v) = c_Q(S \cup v) - c_Q(S)$ for all $S \subset T \subseteq N \setminus \{v\}$. On the other hand, if $S \neq \emptyset$, then $v_c \in d(S)$, and for all $S \subset T \subseteq N \setminus \{v\}$, we have

$$c_Q(T \cup v) - c_Q(T) = c_Q(S \cup v) - c_Q(S) = \begin{cases} 0 & \text{if } v = v_c \\ 2t(e_v) & \text{otherwise} \end{cases},$$

and (2.1) holds. □

We are now ready to characterize globally k-TS submodular graphs for $1 < k < |V(G)| - 2$. Recall that for $k = 1$, a characterization of globally 1-depot TS-submodular graphs was given in Herer and Penn (1995), and for $k \in \{|V(G)| - 2, |V(G)| - 1\}$, all connected graphs are globally k-TS submodular, since there are at most two players in the induced game.

Theorem 4.1. *Let $G = (V(G), E(G))$ be a connected, undirected graph, and let $k \in \{2, \ldots, |V(G)| - 3\}$. Then G is globally k-TS-submodular if and only if G contains no path of length 4.*

Proof. Let G be a connected, undirected graph, let $\Gamma = (V^-(G), (G, Q), t)$ be a k-depot TS problem on G, and let (N, c_Q) be the induced k-TS game on G.

First, for the 'only if' part assume that G contains a path of length four. Then there exists a pair (v_i, v_{i+4}) of distinct vertices in G such that there is a path $P := v_i, e_{i+1}, v_{i+1}, \dots, e_{i+4}, v_{i+4}$, from v_i to v_{i+4} visiting four edges. However, since $|Q| = k \leq |V(G)| - 3$, we can then choose a Q such that $v_i, v_{i+4} \in Q$, and $v_{i+1}, \dots, v_{i+3} \in V^-(G)$, implying that G is not P_5^F-free with respect to Q. It then follows from Lemma 4.1 that the induced game is not submodular for all weight functions and hence, that G is not globally k-TS submodular.

Now, consider the 'if' part and assume that G contains no path of length four. Since $|V(G)| \geq k + 3$, G has at least 5 vertices. Next, note that the only graphs with five vertices or more that do not contain a path of length four are: star graphs with at least four pendant vertices, graphs obtained by joining two star graphs by adding an edge between the two center vertices (a double star), graphs obtained from a stargraph by adding a single edge between two pendant vertices. The latter two types of graphs are illustrated in Figure 4.

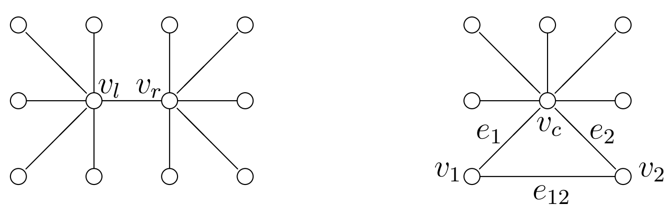


Figure 4: Globally k-TS submodular graphs

From Lemma 4.2, star graphs are globally k-TS submodular. It remains to be shown that for both graph types in Figure 4, the induced cooperative game is submodular for every $Q \subset V(G)$ with $|Q| = k$ and every weight function. The two types of graphs are considered separately.

First, let G be a double star graph. Let the two nodes with a degree higher than 1 be denoted v_l (left-center) and v_r (right-center) respectively, as in Figure 4. For $v_i \in V(G) \setminus \{v_l, v_r\}$ let e_i denote the edge incident to v_i. Then, if $S = \emptyset$, we have $c_Q(S \cup v) - c_Q(S) = c(v)$ for any $v \in V^-(G)$, and it follows from subadditivity that $c_Q(T \cup v) - c_Q(T) \leq c(v) = c_Q(S \cup v) - c_Q(S)$ for all $S \subset T \subseteq V^-(G) \setminus \{v\}$.

Now, assume instead that $S \neq \emptyset$ and note that when coalition S is non-empty, either v_l or v_r must belong to $d(S)$. Assume that $v_l \in d(S)$, and let e^* denote the cheapest edge connecting v_r to a node in $Q \cup v_l$ (symmetric arguments can be applied if $v_r \in d(S)$). Let $v_i \in V^-(G)$ and consider three separate cases:

Case 1. $v_i \in \{v_l, v_r\}$: If $v_i = v_l$ then $c_Q(T \cup v_i) - c_Q(T) = c_Q(S \cup v_i) - c_Q(S) = 0$ since $v_l \in d(S)$. If $v_i = v_r$ then $c_Q(T \cup v_i) - c_Q(T) = c_Q(S \cup v_i) - c_Q(S) = 0$, if $v_r \in d(S)$, and $c_Q(T \cup v_i) - c_Q(T) \leq c_Q(S \cup v_i) - c_Q(S) = 2t(e^*)$ otherwise.

Case 2. $v_i \in V^-(G) \setminus \{v_l, v_r\}$ and v_i is adjacent to v_l: then $c_Q(T \cup v_i) - c_Q(T) = c_Q(S \cup v_i) - c_Q(S) = 2t(e_i)$.

Case 3. $v_i \in V^-(G) \setminus \{v_l, v_c\}$ and v is adjacent to v_r: then $c_Q(T \cup v_i) - c_Q(T) = c_Q(S \cup v_i) - c_Q(S) = 2t(e_i)$ if $v_r \in \{Q \cup d(S)\}$, and $c_Q(T \cup v_i) - c_Q(T) \leq 2t(e_i) + 2t(e^*) = c_Q(S \cup v_i) - c_Q(S)$ otherwise.

Thus, (2.1) holds in all three cases, and the induced game is submodular for every $Q \subset V(G)$ and all weight functions.

Next, consider the graph in the right panel of Figure 4. Let the single vertex of degree larger than 2 be denoted v_c and referred to as the center vertex. Let the two 2-degree vertices be denoted v_1 and v_2 respectively, and let e_{12} denote the edge incident to both v_1 and v_2. Furthermore, for any $v_i \in V(G) \setminus \{v_c\}$, let e_i denote the edge connecting v_i to v_c, and let e^* denote the minimum weight edge connecting v_c to a node in $Q \cup \{v_1, v_2\}$. It remains to be shown that (2.1) holds for all $v \in V^-(G)$ and all $S \subset T \subseteq V^-(G) \setminus v$.

If $S = \emptyset$, then $c_Q(S \cup v) - c_Q(S) = c_Q(v)$ for all $v \in V^-(G)$, and again, it follows from the subaditivity of c_Q that $c_Q(T \cup v) - c_Q(T) \leq c_Q(S \cup v) - c_Q(S)$ for all $S \subset T \subseteq N \setminus v$. Therefore, assume instead that $S \neq \emptyset$.

First, if $v_c \notin d(S)$, then $\{v_1, v_2\}$ are the only nodes visited by $d(S)$, implying that either $S = \{v_1\}$,$v_2 \in Q$, or $S = \{v_2\}$,$v_1 \in Q$. Either way, it must be that $v_c \in d(T)$ for all $S \subset T$. Therefore, if $v = v_c$, then $c_Q(T \cup \{v\}) - c_Q(T) = 0 < 2t(e^*) = c_Q(S \cup \{v\}) - c_Q(S)$. Likewise, for $v_i \in V^-(G) \setminus \{v_c, v_1, v_2\}$, we have $c_Q(T \cup \{v_i\}) - c_Q(T) = 2t(e_i) < 2t(e_i) + 2t(e^*) = c_Q(S \cup \{v_i\}) - c_Q(S)$.

Secondly, if $v_c \in d(S)$, it is trivial that $c_Q(T \cup \{v_c\}) - c_Q(T) = c_Q(S \cup \{v_c\}) - c_Q(S) = 0$, and for $v_i \in V^-(G) \setminus \{v_c, v_1, v_2\}$, we have $c_Q(T \cup \{v_i\}) - c_Q(T) = c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e_i)$. Finally, consider the case of $v = v_1$ (symmetric arguments can be applied to the case of $V = v_2$). Then either a) $v_1 \notin d(S), v_2 \in d(S)$, in which case $c_Q(T \cup \{v_1\}) - c_Q(T) = c_Q(S \cup \{v_1\}) - c_Q(S) = \min\{2t(e_1), 2t(e_{1,2})\}$, or b) $v_1, v_2 \notin d(S)$, implying that $c_Q(S \cup \{v_1\}) - c_Q(S) = \min\{2t(e_1), 2t(e_2) + 2t(e_{1,2})\} \geq c_Q(T \cup \{v_i\}) - c_Q(T)$, where the inequality follows, since $v_c \in d(S) \Rightarrow v_c \in d(T)$. Thus, (2.1) holds. $\square$

Now, consider the following definition:

Definition 4.2. Let G be a connected, undirected graph. Then G fulfills the *cut condition*, if there does not exist for any vertex-pair $v_t, v_s \in V(G)$ a minimal s-t vertex cut in G with cardinality greater than 2.

It follows from Herer and Penn (1995) that an undirected graph G is globally 1-TS submodular if and only if G satisfies the cut condition for all pairs of vertices.

From the proof of Theorem 4.1, it is easy to verify that globally k-TS submodular graphs satisfy the cut condition for $k > 1$. Therefore, the global condition for submodularity of the induced cooperative game has been strengthened by the introduction of multiple depots in the underlying STSP.

4.2 Locally k-TS submodular graphs

Requiring the induced game to be submodular for all possible locations of depots is very restrictive, and in some situations it may be more relevant to ask simply whether there exists at least one location of depots that induces a submodular game for every possible weight function. Recall that a graph $G = (V(G), E(G))$ is locally k-TS submodular if there exists a $Q \subset V(G)$, with $|Q| = k$, such that the induced k-TS game (N, c_Q) is submodular for any weight function.

In Granot and Hamers (2004), it was shown that the class of locally 1-depot TS submodular games is equivalent to the class of globally 1-depot TS submodular games. A similar result does not hold for k-TS submodular games when $k > 1$. Instead, the class of locally k-TS submodular games is a superset of the class of globally k-TS submodular games. The class of locally k-TS submodular graphs can be characterized as follows.

Theorem 4.2. *Let $G = (V(G), E(G))$ be a connected, undirected graph. Let $\Gamma = (V^-(G), (G, Q), t)$, and let $k \in \{2, \ldots, |V(G)| - 3\}$. Let G_1 denote the subgraph induced by all paths between vertices of Q. Then G is locally k-TS submodular if and only if there exists a $Q \subset V(G)$ such that G is P_5^F-free, with respect to Q, and such that the cut condition is satisfied for all vertex pairs (v_s, v_t) in a connected component of $G \setminus E(G_1)$.*

Proof. Consider first the 'if part' and assume that there exists a $Q \subset V(G)$ such that G is P_5^F-free with respect to Q. Let G_1 be the graph induced by all paths between the k vertices of Q and note that since G is P_5^F-free with respect to Q, any one of these paths can visit at most two player vertices in a row, implying that every player-vertex in G_1 is adjacent to at least one depot and at most one other player vertex within G_1, as illustrated in Figure 5.

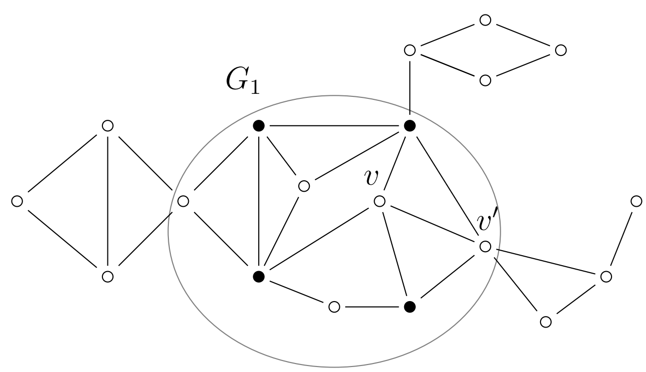


Figure 5: G locally k-TS submodular

It remains to be shown that (2.1) holds for all players $v \in V^{-}(G)$, all $S \subset T \subseteq N \setminus \{v\}$ and all weight functions t. Consider $v \in V^{-}(G)$. We distinguish between two cases.

Case 1. $v \in V(G_1)$. Let $S \subset V^{-}(G)$, and let $P_S^*(v)$ denote a minimum cost path from v to a vertex in $Q \cup d(S)$. Note that $P_S^*(v)$ may: be empty, in case $v \in d(S)$; contain a single edge connecting v to an adjacent vertex in $Q \cup d(S)$; or contain exactly two edges, connecting v to a depot via an adjacent player vertex in G_1. Therefore, $v \in d(S)$ implies $v \in d(T)$ for all $v \in V^{-}(G) \cap V(G_1)$. Indeed, if $v \in S$, then $v \in T$, and the conclusion follows trivially. On the other hand, if $v \in d(S)$ while $v \notin S$, then there exists a player vertex $v' \in (d(S) \cap V(G_1))$ such that $v \in P_S^*(v')$. Now, if $v \in T$, the conclusion follows. If $v \notin T$, then since $v' \in T$, and v is the only player vertex in $V(G_1)$ adjacent to v', it will also hold that $v \in P_T^*(v')$, e.g. see Figure 5.[2] Therefore, $\sum_{e \in P_T^*} 2t(e) \leq \sum_{e \in P_S^*} 2t(e)$, and we conclude that $c(T \cup v) - c(T) = \sum_{e \in P_T^*} 2t(e) \leq \sum_{e \in P_S^*} 2t(e) = c(S \cup v) - c(S)$, for all $S \subset T \subseteq N \setminus \{v\}$. Thus, condition (2.1) holds.

Case 2. $v \in V^{-}(G) \setminus V(G_1)$. Then there exists a vertex $v_0 \in V(G_1)$ that must be visited on any tour visiting v. For $S \subseteq V^{-}(G) \setminus \{v\}$, let $P_S^*(v_0)$ denote the cheapest (and possibly empty) path connecting v_0 to a vertex in $Q \cup d(S)$. Note that if $v_0 \in d(T)$ then either $v_0 \in d(S)$, in which case $\sum_{e \in P_T^*(v_0)} 2t(e) = \sum_{e \in P_S^*(v_0)} 2t(e) = 0$, or $v_0 \notin d(S)$

[2]Note that v' may be adjacent to player vertices in $V(G) \setminus V(G_1)$ as well, as illustrated in Figure 5. This will, however, have no influence on $P_T^*(v')$, since any minimum cost tour that visits such players must pass through v'.

implying that $\sum_{e\in P_T^*(v_0)} 2t(e) = 0 \leq \sum_{e\in P_S^*(v_0)} 2t(e)$. Now, assume $v_0 \notin d(T)$. Consider the graph $G \setminus E(G_1)$, and let G_0 denote the connected component containing v and v_0. Furthermore, let (N', c'_{v_0}) be the 1-depot TS-game defined on G_0 in which v_0 is the single vertex associated with a depot, such that $N' = V(G_0) \setminus \{v_0\}$, and c' is the restriction of c to coalitions in N'. Then, $c_Q(S \cup \{v\}) - c_Q(S) = c'_{v_0}(\{S \cap V(G_0)\} \cup \{v\}) - c'_{v_0}(\{S \cap V(G_0)\} + \sum_{e\in P_S^*(v_0)} 2t(e)$ for all $S \subset N \setminus \{v\}$. Since the cut condition holds for all $v_s, v_t \in V(G_0)$, it follows from Herer and Penn (1995) and Granot and Hamers (2004) that (N', c'_{v_0}) is submodular, which in turn implies that

$$\begin{aligned} c_Q(T \cup \{v\}) - c_Q(T) &= c'_{v_0}(\{T \cap V(G_0)\} \cup \{v\}) - c'_{v_0}(\{T \cap V(G_0)\}) + \sum_{e\in P_T^*(v_0)} 2t(e) \\ &\leq c'_{v_0}(\{S \cap V(G_0)\} \cup \{v\}) - c'_{v_0}(\{S \cap V(G_0)\}) + \sum_{e\in P_S^*(v_0)} 2t(e) \\ &= c_Q(S \cup \{v\}) - c_Q(S), \end{aligned} \tag{4.1}$$

for all $v \in V^-(G) \setminus V(G_1)$ and all $S \subset T \subseteq N \setminus \{v\}$, where the inequality follows from the submodularity of (N', c'_{v_0}) and the fact that $\sum_{e\in P_T^*(v_0)} 2t(e) \leq \sum_{e\in P_S^*(v_0)} 2t(e)$ holds for all $S \subset T \subseteq N \setminus \{v\}$.

For the 'only if' part: First, if G is not P_5^F-free with respect to Q, it follows from Lemma 4.1 that the induced game is not submodular for every weight function. Next, let G_0 be a connected component in $G \setminus E(G_1)$, and assume that there exists an $s-t$ vertex cut of cardinality three or more for some $v_s, v_t \in G_0$. In Granot and Hamers (2004), it was shown that a graph G is locally 1-TS submodular if and only if the cut condition is satisfied for all $v_s, v_t \in V(G)$. It therefore follows that there exists no location of a single depot v_0 in G_0, such that the induced 1-depot TS game (N', c'_{v_0}) on G_0 is submodular for all weight functions. That is, $c'_{v_0}(\{T \cap V(G_0)\} \cup \{v\}) - c'_{v_0}(\{T \cap V(G_0)\}) \leq c'_{v_0}(\{S \cap V(G_0)\} \cup \{v\}) - c'_{v_0}(\{S \cap V(G_0)\})$ does not hold for all t. If we furthermore choose $S \subset T \subseteq N \setminus \{v\}$ such that $\sum_{e\in P_T^*(v_0)} 2t(e) = \sum_{e\in P_S^*(v_0)} 2t(e)$, then (4.1) does not hold, and (N, c_Q) is not submodular for all weight functions. Therefore, there does not exist a Q with $|Q| = k$ such that every induced game on G is submodular, and G is not locally k-TS submodular.

□

5 k-TS submodular directed graphs

5.1 Globally k-TS submodular digraphs

We now turn to consider directed graphs. First, note that a directed cycle C with arc set $E(C)$ is globally k-TS submodular for all $k \in \{2, \ldots, |V(G)| - 3\}$, since $c(S) = \sum_{e \in E(C)} t(e)$ for all $S \subseteq N$. For digraphs in general, any induced game is submodular if $k \geq |V(G)| - 2$, since the game is then either a one or a two-player subadditive game. Granot et al. (2000) consider the case of $k = 1$ and provide the equivalence theorem below. The graphs referred to as F_1 and F_2 are illustrated in Figure 6.

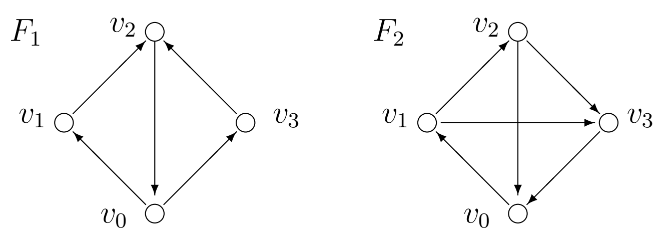


Figure 6: Forbidden subgraphs F_1 and F_2

Theorem 5.1. *[Granot et al. (2000)] Let G be a strongly connected digraph. Then the following three statements are equivalent.*

- *G is globally TS-submodular*
- *G does not contain a subdivision of F1 or F2*
- *G is a 1-sum of harmonic digraphs each of which is an outerplanar graph with a directed cycle on its outer boundary*[3]

If a directed graph G contains a subdivision of F_1 or F_2, then for every 1-depot STSP on G it is possible to choose a location of the single depot such that the induced game is not submodular for all weight functions t. Therefore, G is not globally 1-depot TS submodular. A similar result holds for the case of $k > 1$ as shown below.

[3] In Granot et al. (2000), a digraph G is said to be *harmonic* if every pair of cycles in G visit their common vertices in the same order, i.e., all pairs of cycles are *in harmony*. An *outerplanar graph* is a graph that can be embedded in the plane such that no edges cross, and such that all vertices of the graph lie on the boundary of the outer face of the embedding.

Lemma 5.1. Let $G = (V(G), E(G))$ be a strongly connected, directed graph, and let $k \in \{2, \dots, |V(G)| - 3\}$. If G is globally k-TS submodular, then G does not contain (a subdivision of) F_1 or F_2.

Proof. To arrive at a contradiction, assume first that G contains (a subdivision of) F_1, and let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot STSP on G. Referring to the graph in the left panel of Figure 6, consider the vertices $\{v_0, v_1, v_2, v_3\} \in V(F_1)$. For any $\{i, j\}, i \neq j$, let P_{ij} denote the directed path from vertex v_i to vertex v_j. Let $E(P_{ij})$ denote the set of arcs in P_{ij}. Next, construct the following weight function: let t be such that $\sum_{e \in E(P_{ij})} t(e) = 1$ for all $\{i, j\} \in \{\{0, 1\}, \{1, 2\}, \{0, 3\}, \{3, 2\}, \{2, 0\}\}$, and let $t(e) = 100$ for all other arcs in G. Since we consider $k \leq |V(G)| - 3$, we can choose $Q \subset V(G)$ such that $v_0 \in Q$ and $v_1, v_2, v_3 \in V^-(G)$. Now, let $S = \{v_2\}$ and $T = \{v_1, v_2\}$. Then we see that $c(T \cup v_3) - c(T) = 6 - 3 = 3 > 0 = 3 - 3 = c(S \cup v_3) - c(S)$, and the induced game is not submodular. Next, assume instead that G contains (a subdivision of) F_2, refer to the right panel of Figure 6, and let $\sum_{e \in E(P_{ij})} t(e) = 1$ for all $\{i, j\} \in \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 0\}, \{1, 3\}\}$, and let $t(e) = 100$ for all other arcs in G. Then, if $S = \{v_1\}$, and $T = \{v_1, v_2\}$, we see that $c(T \cup v_3) - c(T) = 4 - 3 = 1 > 0 = 3 - 3 = c(S \cup v_3) - c(S)$. Again, the induced game is not submodular, therefore, G is not globally k-TS submodular for any $k \in \{2, \dots |V(G)| - 3\}$. $\square$

Before moving on, some notation is required. Let a *1-sum* of two graphs G_1 and G_2 be the graph that is obtained by joining G_1 and G_2 by coalescing one vertex from G_1 with one vertex of G_2. The vertex joining the two former graphs will be referred to as the *link vertex*, and each of the original graphs will be referred to as a *component* in the 1-sum. Figure 7 illustrates.

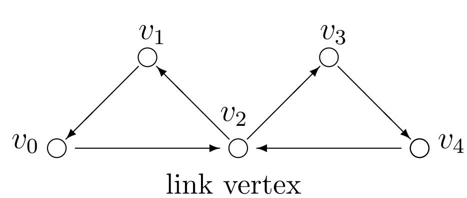


Figure 7: A 1-sum of two directed cycles

As noted above, any directed cycle C is globally k-TS submodular for all $k \in \{2, |V(C)| - 3\}$, and it follows from Theorem 5.1 that 1-sums of directed cycles are globally 1-TS submodular. However, 1-sums of directed cycles or more generally, 1-sums of directed circuits are not globally k-TS submodular for $k > 1$.[4] In fact, if G contains a 1-sum of directed cycles (or circuits) of at least three vertices each, then G is not globally k-TS submodular for any $k \in \{2, \ldots, |V(G)| - 3\}$.

Lemma 5.2. Let $G = (V(G), E(G))$ be a strongly connected, directed graph. Let $k \in \{2, \ldots, |V(G)| - 3\}$. If G contains a 1-sum of two directed circuits C^1 and C^2 such that each circuit is a closed walk of length three or more, then G is not globally k-TS submodular.

Proof. To arrive at a contradiction, assume that such a 1-sum exists and note that $|V(C^1)|, |V(C^2)| \geq 3$. (Figure 7 illustrates the case where G contains a 1-sum of a two cycles with three vertices each, but the proof holds for cycles/circuits in general.) Let v_2 denote the link vertex in this 1-sum and let v_0, v_1 be vertices in C^1 such that paths P_{01}, P_{12}, P_{20} exist. Likewise, let v_3, v_4 be vertices in C^2 such that paths P_{23}, P_{34}, P_{42} exist. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot STSP on G. Now, since the number of depots $2 \leq k \leq |V(G)| - 3$, we can choose $Q \subset V(G)$ such that $v_1, v_2, v_3 \in V^-(G)$ while $v_0, v_4 \in Q$, implying that a depot exists in each component of the 1-sum. Next, define the weight function t on G as follows: for any path $P_{ij} \in \{P_{01}, P_{12}, P_{20}, P_{23}, P_{34}, P_{42}\}$ let $\sum_{e \in E(P_{ij})} t(e) = 1$, and let $t(e) = 100$ for all other arcs in G. Then, if $S = \{v_2\}$ and $T = \{v_1, v_2\}$, we see that $c(S \cup v_3) - c(S) = 3 - 3 = 0$, while $c(T \cup v_3) - c(T) \geq 1$. Thus, the induced game is not submodular, and therefore, G is not globally k-TS submodular. $\square$

Let C_2 denote a directed cycle with only 2 vertices. Then from Theorem 5.1 and Lemmas 5.1 and 5.2, we can infer the following:

Theorem 5.2. *Let $G = (V(G), E(G))$ be a strongly connected, directed graph, and let $k \in \{2, \ldots, |V(G)| - 3\}$. If G is globally k-TS submodular, then G is a 1-sum of a harmonic, outerplanar graph with a directed cycle on its outer boundary and (copies of) C_2, such that G does not contain a 1-sum of directed circuits each of length three or more.*

Even for graphs fulfilling the requirements above, whether the graph is globally k-TS submodular may depend on the specific structure of the graph and the number of depots as shown below.

[4] Recall that a directed circuit is a closed walk in which no edge is visited more than once, while a directed cycle denotes a closed path, implying that neither vertices nor edges are visited more than once.

In the remainder of this section, we focus on the case where G is an oriented graph, implying that G contains no bi-directed edges, and hence no C_2. We already know that for G to be k-TS submodular, it must be a harmonic and outerplanar graph with a directed cycle on its outer boundary. While it follows from Granot et al. (2000) that this is sufficient for G to be globally 1-TS submodular, the same does not hold for the case of $k \geq 2$.

Theorem 5.3. *Let $G = (E(G), V(G))$ be a harmonic and outerplanar graph with a directed cycle on its outer boundary. Let G contain a directed cycle C_i such that $G \setminus V(C_i)$ is a collection of weakly connected components for which at least two distinct components contain 2 vertices or more. Let C_0 denote the shortest such cycle in G. Then G is globally k-TS submodular if and only if $k > |V(G)| - (|V(C_0)| + 2)$.*

Proof. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi depot STSP on G. For the if part, it needs to be shown that (2.1) holds for all $v \in V^-(G)$, $S \subset T \subseteq V^-(G) \setminus v$. If $S = \emptyset$, the result follows from the subadditivity of c. Therefore, assume instead that $S \neq \emptyset$. Let $v_i \in V^-(G)$ and consider three cases depending on whether v_i belongs to $d(S)$ and/or $d(T)$:

Case 1. $v_i \in d(S)$: We start by showing that if v_i is visited on a min cost tour of S, then v_i is also visited on a minimum cost tour of T, i.e., $v \in d(S) \Rightarrow v \in d(T)$. To see this, recall that G is an outerplanar graph with a directed cycle on its outer boundary and denote this cycle by C. Let $v_0 \in Q$ be a depot in $d(T)$ and number the vertices of C in the order they are visited in a tour starting from v_0 and ending at $v_{|V(G)|-1} = v_0$.

To arrive at a contradiction, assume that there exists a $v_i \in V^-(G)$ and an $S \subset T \subseteq V^-(G) \setminus v_i$ such that $v_i \in d(S)$ but $v_i \notin d(T)$. Furthermore, let $h = \max\{l | l < i, v_l \in d(T)\}$. Now, since $v_i \notin d(T)$, there exists a vertex pair $v_h, v_j \in d(T)$ such that the arc (e_h, e_j) - denoted e_{hj} - exists, and $e_{h,j} \in d(T)$. Furthermore, since all the vertices of G lie on C, there exist a path from v_h to v_j that visits v_i. Let P_{hij} denote the minimum cost path from v_h to v_j that visits v_i. Then since v_i is not in $d(T)$, we must have $t(e_{hj}) \leq \sum_{e \in P_{hij}} t(e)$. Furthermore, since only the endpoints of P_{hij} are in $d(T)$, no vertices of C that lie between v_h and v_j belong to T, implying that no vertices between v_h and v_j belong to S.

Now, let C_{hj} denote the smallest cycle that visits v_h, v_j and all vertices in S, and note that for $d(S)$ to be a minimum cost tour, it must be that $C_{hj} \cap Q = \emptyset$, while $P_{hij} \cap Q \neq \emptyset$.[5] However, since $C_{hj} \cap Q = \emptyset$, there are only two possible cases for $d(T)$:

[5]It follows from the definition that e_{hj} belongs to c_{hj}.

Case 1.1. There exist a path $v_l, e_{l,l+1}, v_{l+1}, e_{l+1,l+2}, v_{l+2}$ such that $v_l, v_{l+2} \in C_{hj}$, $v_{l+1} \notin C_{hj}$, but $v_{l+1} \in Q \cap d(T)$. Then, since $d(T)$ is a minimum cost tour, it must be that $t(e_{l,l+1}) + t(e_{l+1,l+2}) < \sum_{e \in P_{hij}} t(e)$. This, however, contradicts that $d(S)$ is a minimum cost tour of S.

Case 1.2. There exist a vertex pair $v_l, v_m \in C_{hj}$, such that v_l, v_m are the endpoints of a path P_{lm} of length 3 or more, and such that $P_{lm} \in d(T)$, $P_{lm} \cap Q \neq \emptyset$ and v_l, v_m are the only vertices of P_{lm} that lie on C_{hj}. Then, either $\sum_{e \in P_{lm}} t(e) < \sum_{e \in P_{hij}} t(e)$, which contradicts that $d(S)$ is a minimum cost tour of S, or $P_{lm} \cap T \neq \emptyset$. However, if the latter is true, then $G \setminus V(C_{hj})$ is a collection of weakly connected components for which at least two distinct components contain 2 vertices or more, namely the paths P_{hij} and P_{lm}. We then know that $k > |V(G)| - (|V(C_{hj})| + 2)$, which contradicts that $C_{hj} \cap Q = \emptyset$.

We conclude that there cannot exist a $v_i \in V^-(G)$ and a $S \subset T \subseteq V^-(G) \setminus v_i$ such that $v_i \in d(S)$ and $v_i \notin d(T)$. Now, because $v_i \in d(S) \Rightarrow v_i \in d(T)$, it follows that (2.1) holds trivially, since $c(T \cup v_i) - c(T) = c(S \cup v_i) - c(S) = 0$.

Case 2. $v_i \notin d(S)$, $v_i \in d(T)$. Then $c(S \cup v_i) - c(S) \geq 0 = c(T \cup v_i) - c(T)$, where the (weak) inequality follows from monotonicity, and (2.1) holds.

Case 3. $v_i \notin d(S), v_i \notin d(T)$. Let $g = \max\{l | l < i, v_l \in d(S)\}$, then there exists a vertex pair v_g, v_k, such that the arc e_{gk} exists and belongs to $d(S)$. Furthermore, for the minimum cost path P^S_{gik} from v_g to v_k that visits v_i, it holds that $t(e_{gk}) < \sum_{e \in P^S_{gik}} t(e)$. Likewise, since $v_i \notin d(T)$ there exists a vertex pair v_h, v_j such that $h = \max\{l | l < i, v_l \in d(T)\}$, the arc e_{hj} exists, $e_{hj} \in d(T)$, and $t(e_{hj}) < \sum_{e \in P^T_{hij}} t(e)$, where P^T_{hij} is the minimum cost path from v_h to v_j that visits v_i. From Theorem 5.3 it follows that G does not contain the forbidden structure, F_2, and we therefore have either $g \leq h < i < k \leq j$, or $h \leq g < i < k \leq j$. We consider the two cases separately:

Case 3.1. $h \leq g < i < k \leq j$. Let P^T_{hj} denote the minimum cost path from v_h to v_j (that may or may not be identical to P^T_{hij}). Then since only the endpoints of this path is in $d(T)$, no players in T and hence no players in S lie on C between v_h and v_j. Furthermore, if there exists a depot in the part of C that is common to $d(S)$ and $d(T)$, i.e., if $(d(S) \cap d(T)) \cap Q \neq \emptyset$, this contradicts that $d(S)$ is a minimum cost tour, since a minimum cost tour could visit e_{hj} instead of e_{gk}. Thus, it must be that $(d(S) \cap d(T)) \cap Q = \emptyset$, and that there is a depot on the path from v_h that goes via e_{gk} to v_j. However, we can now use arguments similar to those above to show that this implies a contradiction. First, for $d(T)$ to be a minimum cost tour, there must then exist a vertex pair v_l, v_m in $d(S) \cap d(T)$, such that v_l, v_m are the endpoints of a path for which no other vertices belong to $d(S) \cap d(T)$, and such that at least one of these other vertices is a depot. Now, in case no players of T are on this path (apart from at the endpoints) this contradicts that both $d(S)$ and $d(T)$ are minimum cost tours. On the other hand, if some player of T reside on this path, this implies that $G \setminus (d(S) \cap d(T))$ is a collection of weakly connected components for which at least two such components contain 2 vertices or more, which in turn contradicts that $(d(S) \cap d(T)) \cap Q = \emptyset$.

Case 3.2. $g \leq h < i < k \leq j$. Now, since $g \leq h$ and $j \leq k$, it must be that $\sum_{e \in P_{hij}} t(e) \leq \sum_{e \in P_{gik}} t(e)$. Next, since v_h is the last vertex in $d(T)$ before v_i, and P_{hij} is the minimum weight path from v_h to v_j, we get $c(T \cup v_i) - c(T) = \sum_{e \in P_{hij}} t(e) - t(e_{hj}) \leq \sum_{e \in P_{gik}} t(e) - t(e_{gk}) = c(S \cup v_i) - c(S)$, where the inequality follows since $t(e_{gk}) \leq t(e_{hj}) + \sum_{e \in P_{gik}} t(e) - \sum_{e \in P_{hij}} t(e)$ must hold for $d(S)$ to be a minimum cost tour. Thus, (2.1) holds.

Next, consider the only if part and assume that $k \leq |V(G)| - (|V(C_0)| + 2)$. Recall that C_0 is the shortest directed cycle in G such that $G \setminus V(C_0)$ contains two weakly connected components with at least two vertices each, as illustrated in Figure 8. Let the two components be denoted W_1 and W_2 respectively. For each component W, let P_W denote the shortest path between two vertices of C_0 that visits at least two vertices of W. Next, let v_0 be a vertex in C_0, let v_1, v_2 be vertices on P_{W_1}, and let v_3, v_4 be vertices on $V(P_{W_2})$.

Consider $\Gamma = (V^-(G), (G, Q), t)$, and note that for any $k \leq |V(G)| - (|V(C_0)| + 2)$, we can choose $Q \subset V(G)$ such that $v_1, v_4 \in Q$ while both $v_0, v_2, v_3 \in V^-(G)$, and $V(C_0) \subset V^-(G)$. That is, there are no depots located at any vertices of C_0, and at least one depot is located on both P_{W_1} and P_{W_2}. Next, define a weight function t on G as follows: for any $e \in E(C_0) \cup E(P_{W_1}) \cup E(P_{W_2}))$, let $t(e) = 1$, and let $t(e)$ be arbitrarily high (e.g. $|E(G)|$) for all other arcs in G. Without loss of generality, assume that $|E(P_{W_2})| \geq |E(P_{W_1})|$.

Next, let $S = \{v_0\}$. Then since there are no depots in C_0, a minimum cost tour of S visits $V(C_0) \cup V(W_1)$, and the cost of this tour is $c(S) = |E(C_0)| + |E(P_{W_1})| - 1$. It follows that $c(S \cup v_1) - c(S) = 0$. Furthermore, let $T = \{v_0, v_3\}$. Then we have $c(T) = |E(C_0)| + |E(P_{W_2})| - 1$. However, since any tour of $T \cup v_1$ must visit both $V(P_{W_1}), V(P_{W_2})$, and $V(C_0)$, it follows that $c(T \cup v_1) - c(T) = |E(P_{W_1})| - 1 > 0$. Thus, the induced game is not submodular, and G is not globally k-TS submodular.

$\square$

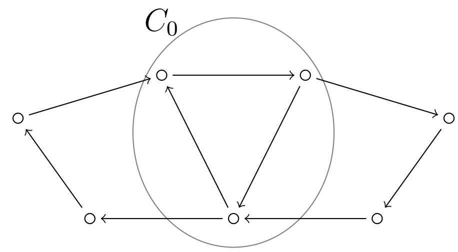


Figure 8: G not globally 2-TS submodular but globally 3-TS submodular

5.2 Locally k-TS submodular digraphs

Recall that for a graph G to be locally k-TS submodular, we only require that one location of k depots in G induces a submodular game for all weight functions. While all globally k-TS submodular graphs are obviously also locally k-TS submodular, it is easy to show that the opposite is not true, and hence that the class of globally k-TS submodular graphs is a proper subset of the class of locally k-TS submodular graphs.

Proposition 5.1. The set of globally k-TS submodular graphs is a proper subset of the set of locally k-TS submodular graphs.

Proof. We need only show that there exists graphs that are locally k-TS submodular but not globally k-TS submodular. To see this, consider the graph G in Figure 9. Since G is a subdivision of F_1, it is not globally k-TS submodular, for $k \in \{1, 2, 3\}$. However, G is locally k-TS submodular for $k = \{2, 3\}$. For $k = 2$, choose $Q = \{v_0, v_1\}$. It can readily be verified that for this location of depots, the induced game is submodular for any weight function. Thus G is locally 2-TS submodular.

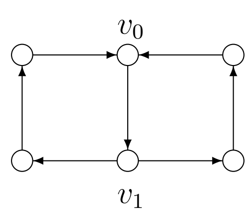


Figure 9: A locally (but not globally) 2-TS submodular graph

□

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