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Generic Finiteness of Equilibrium Distributions for Bimatrix Outcome Game Forms*

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Abstract

We provide sufficient and necessary conditions for the generic finiteness of the number of distributions on outcomes, induced by the completely mixed Nash equilibria associated to a bimatrix outcome game form. These equivalent conditions are stated in terms of the ranks of two matrices constructed from the original game form.

Keywords: Outcome game form, Completely mixed Nash equilibrium, Generic finiteness

JEL Classification: C72

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1 Introduction

It has long been established (Rosenmüller 1971, Wilson 1971, Harsanyi 1973) that for normal form games with an arbitrary number of players, if the payoffs can be independently perturbed, there is generically a finite number of equilibria. For the case of extensive form games Kreps and Wilson (1982) show that the equilibrium distributions on terminal nodes are generically finite.

The equivalent result for outcome games has turned out to be difficult to elucidate. On the one hand, Govindan and McLennan (2001) were the first to provide an example of a game form for which the Nash equilibria of the games associated to an open set of utility profiles induce a continuum of probability distributions on outcomes. Their example made use of three players and six outcomes. Examples of game forms with the same feature have appeared since: in Kukushkin, Litan, and Marhuenda (2008) with two players and four outcomes and in Litan, Marhuenda, and Sudhölter (2015) with three players and three outcomes.

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On the other hand, there are a number of results that point towards the paucity of such examples. Govindan and McLennan (2001) proved that for games with two outcomes and any number of players the number of equilibrium distributions on outcomes is generically finite. Similar results have been obtained for two player, three outcomes games (González-Pimienta 2010), sender-receiver cheap-talk games (Park 1997), zero sum or common interest games (Govindan and McLennan 1998, Litan and Marhuenda 2012) and games with three players and two strategies each (Litan, Marhuenda, and Sudhölter 2015).

Clarifying for what types of game forms the number of probability distributions on outcomes induced by the Nash equilibria of the associated game is generically finite remains an open problem. In the present work we address this question and provide a partial answer. We find sufficient and necessary conditions for the generic finiteness of the number of distributions on outcomes, induced by the completely mixed Nash equilibria associated to a bimatrix outcome game form. These are specified in terms of the ranks of two matrices constructed from the original game form and can be checked automatically.

2 Outcome game forms with two players

We follow very closely the notation and set up adopted in Litan, Marhuenda, and Sudhölter (2015) and Litan and Marhuenda (2012). Let $S^1 = \{1, 2, \dots, m\}$ and $S^2 = \{1, 2, \dots, n\}$ be the two players' sets of pure strategies. Let $S = S^1 \times S^2$ and consider a finite set of outcomes Ω . We denote by $\Delta(\Omega)$ (resp. $\Delta_+(\Omega)$) the set of (resp. strictly positive) probability measures on Ω . An *outcome game form* is a function $\phi : S \rightarrow \Delta(\Omega)$. We write $\phi = (\phi^\omega)_{\omega \in \Omega}$ and for each outcome $\omega \in \Omega$ we regard ϕ^ω as an $m \times n$ matrix, the entries of which are the probabilities that $\phi(i, j)$ ($1 \leq i \leq n$, $1 \leq j \leq m$) assigns to the outcome $\omega \in \Omega$.

Agents have a utility function on outcomes $u \in \mathbb{R}^\Omega$, which extends to a von-Neumann-Morgenstern expected utility function. For each $u \in \mathbb{R}^\Omega$ we assign the matrix

$$u(\phi) = \sum_{\omega \in \Omega} u(\omega)\phi^\omega.$$

Given two profiles of utilities on outcomes $u^1, u^2 \in \mathbb{R}^\Omega$ for the players, the matrices $u^1(\phi)$ and $u^2(\phi)$ define the two-person game $(u^1(\phi), u^2(\phi))$. A pair of strategies $(x, y) \in \Delta(S^1) \times \Delta(S^2)$ is a *Nash equilibrium* (NE) of that game if $xu^1(\phi)y \geq u^1(\phi)_{i \cdot}y$ and $xu^2(\phi)y \geq xu^2(\phi)_{\cdot j}$ for all $i \in S^1$ and $j \in S^2$. Here, $u(\phi)_{i \cdot}$ (resp. $u(\phi)_{\cdot j}$) denote the i -th row (resp. j -th column) of the matrix $u(\phi)$. Throughout the paper, $xu(\phi)$ (resp. $u(\phi)y$) are regarded as elements of \mathbb{R}^n (resp. \mathbb{R}^m) and the scalar product of $z, \bar{z} \in \mathbb{R}^k$ is written as $z \cdot \bar{z}$ or simply as $z\bar{z}$. The pair (x, y) is a *completely mixed NE* (CMNE) if, in addition, $x \in \Delta_+(S^1)$ and $y \in \Delta_+(S^2)$.

The strategies $x \in \Delta(S^1)$ and $y \in \Delta(S^2)$ of the players induce a probability distribution on Ω that assigns the probability $x\phi^\omega y$ to the outcome $\omega \in \Omega$.

We identify \mathbb{R}^Ω with Euclidean space $\mathbb{R}^{|\Omega|}$. With this identification the mapping $u \mapsto u(\phi)$ is linear from $\mathbb{R}^{|\Omega|}$ into the linear space of $m \times n$ matrices. We say that a subset of \mathbb{R}^Ω is *generic* if it contains an open and dense subset. For $l \in \mathbb{N}$, let d_l denote the vector $(1, \dots, 1) \in \mathbb{R}^l$.

The number of pure NEs of a finite game is finite. In the case of games with a mixed NE, by eliminating those strategies that are played with zero probability, we will focus on the CMNEs of the corresponding subgames. Given two utility profiles $u^1, u^2 \in \mathbb{R}^\Omega$, if a pair $(x, y) \in \Delta_+(S^1) \times \Delta_+(S^2)$ of completely mixed strategies is a NE of the game $(u^1(\phi), u^2(\phi))$, then it is a solution of the following systems of linear

equations.

$$u^1(\phi)y = \alpha_1 d_m, \quad y \cdot d_n = 1 \quad (1)$$

$$xu^2(\phi) = \alpha_2 d_n, \quad x \cdot d_m = 1 \quad (2)$$

for some $\alpha_i \in \mathbb{R}$, the payoff of player $i = 1, 2$.

Definition 2.1. Given an outcome game form ϕ and a pair of utility profiles $u^1, u^2 \in \mathbb{R}^\Omega$, a *quasi-equilibrium* (QE) of the game $(u^1(\phi), u^2(\phi))$ is a pair of solutions $(x, y) = (x(u^2), y(u^1)) \in \mathbb{R}^m \times \mathbb{R}^n$ of the system of equations (1) and (2), for some $\alpha_i = \alpha_i(u^i) \in \mathbb{R}$, $i = 1, 2$.

We say that vector $z \in \mathbb{R}^\Omega$ is a *quasi(probability)-distribution* on Ω for the game $(u^1(\phi), u^2(\phi))$ if there exists a QE (x, y) of that game such that for each $\omega \in \Omega$, $z(\omega) = x\phi^\omega y$.

Thus, the set of CMNE is a subset of the set of all QE of a given game. For the rest of the paper we fix an outcome game form ϕ . Given an $m \times n$ matrix A and $b \in \mathbb{R}^m$, $(A|b)$ denotes the $m \times (n+1)$ matrix that arises from A by adding b as final column and A^t denotes the transpose matrix of A . Let $k = \max\{\text{rank } u(\phi) : u \in \mathbb{R}^\Omega\}$. We rely on the following fact shown by Mas-Colell (2010). There is an open and dense (and hence generic) subset G of \mathbb{R}^Ω such that the following conditions hold.

- (a) For every $u \in G$, we have $\text{rank } u(\phi) = k$. After reordering, if necessary, the strategies of the players we may write

$$u(\phi) = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \quad (3)$$

where $B = B(u)$ is a $k \times k$ matrix with $|B| \neq 0$.¹

- (b) The functions $k_1 = \text{rank}(u(\phi)|d_m)$ and $k_2 = \text{rank}(u(\phi)^t|d_n)$ are constant on G .

Throughout the rest of this exposition, we fix a reordering of the strategies of the players such that (a) holds. Consider the following polynomial on $|\Omega|$ variables,

$$p(u) = |B(u)| d_k B^{-1}(u) d_k, \quad u \in \mathbb{R}^\Omega \quad (4)$$

Whenever $p(u) \neq 0$ we define

$$\alpha(u) = \frac{|B(u)|}{p(u)} \quad (5)$$

We have the following.

Proposition 2.2. *If $k = m = n$, then generically there is at most one QE and, hence, at most one CMNE. Suppose $k < \max\{m, n\}$. Then,*

- (a) *If $k < \max\{k_1, k_2\}$, then for every $u^1, u^2 \in G$ the game $(u^1(\phi), u^2(\phi))$ has no QE and, hence, no CMNE.*

- (b) *If $k_1 = k_2 = k$ and p is the zero polynomial, then for any $u^1, u^2 \in G$ the game $(u^1(\phi), u^2(\phi))$ has no QE and, hence, no CMNE.*

¹When there is no danger of confusion, we will not write explicitly the dependence on the utility u for matrices.

(c) Let $u^1, u^2 \in G$. Suppose $k_1 = k_2 = k$ and for $i = 1, 2$, $p(u^i) \neq 0$. Then, there is a continuum of QEs of the game $(u^1(\phi), u^2(\phi))$. Furthermore, the systems of linear equations (1) and (2) have a solution only if the payoffs are $\alpha_i = \alpha(u^i)$, $i = 1, 2$, with α defined in (5).

Proof. The case $k_1 = k_2 = k$ of the Proposition is a standard result in elementary Linear Algebra. See Lemmas 3.2, 3.3 and 3.4 in Litan and Marhuenda (2012) for the proof of parts (b) and (c).

We prove next part (a). We consider only the case $k_1 = k + 1$. The case $k_2 = k + 1$ is similar. We remark first that, since d_m is not a linear combination of the columns of $u(\phi)$, any $y \in \mathbb{R}^n$ which is a solution of (1) for some $u \in G$, must satisfy $u(\phi)y = 0$.

Assume now that there are $u^1, u^2 \in G$ such that the game $(u^1(\phi), u^2(\phi))$ has a QE (x, y) with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Then, $u^1(\phi)y = 0$ by the previous remark. Since, G is open, there exists $\varepsilon > 0$ such that $\bar{u} = u^1 + \varepsilon d_{|\Omega|} \in G$. Moreover, (x, y) is also a QE of the game $(\bar{u}(\phi), u^2(\phi))$ and $\bar{u}(\phi)y = \varepsilon d_m$. But, this contradicts the remark in the previous paragraph. Hence, part (a) of the Proposition follows. \square

Since, $k_1, k_2 \in \{k, k + 1\}$ and we are only interested in the existence of a continuum of CMNEs, from now we consider only games for which the following holds.

Assumption 2.3.

- $k_1 = k_2 = k < \max\{m, n\}$.
- The polynomial p in (4) is not the zero polynomial.

Note that, under the above assumption, the set $U = \{u \in G : p(u) \neq 0\}$ is open and dense in $\mathbb{R}^{|\Omega|}$. Hence, it is generic. Let $\alpha : U \rightarrow \mathbb{R}$ as in (5). Define the functions $y^p : U \rightarrow \mathbb{R}^n$, $y^h : U \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^n$, $x^p : U \rightarrow \mathbb{R}^m$ and $x^h : U \times \mathbb{R}^{m-k} \rightarrow \mathbb{R}^m$ by

$$\begin{aligned}
 y^p(u) &= (\alpha(u)B^{-1}(u)d_k, 0) \\
 y^h(u, v) &= (-B^{-1}(u)C(u)v, v) \\
 x^p(u) &= (\alpha(u)d_k B^{-1}(u), 0) \\
 x^h(u, w) &= (-wD(u)B^{-1}(u), w)
 \end{aligned} \tag{6}$$

The following is proved in Lemmas 3.3 and 3.4 of Litan and Marhuenda (2012).

Lemma 2.4. *Let Assumption 2.3 hold and let $u \in U$. Then,*

- (a) $u(\phi)y^p(u) = \alpha(u)d_m$ and $d_n \cdot y^p(u) = 1$;
- (b) $x^p(u)u(\phi) = \alpha(u)d_n$ and $d_m \cdot x^p(u) = 1$;
- (c) For every $v \in \mathbb{R}^{n-k}$ we have $u(\phi)y^h(u, v) = 0$ and $d_n \cdot y^h(u, v) = 0$;
- (d) For every $w \in \mathbb{R}^{m-k}$ we have $x^h(u, w)u(\phi) = 0$ and $d_m \cdot x^h(u, w) = 0$;

For $V \subset \mathbb{R}^l$ a linear subspace and $a \in \mathbb{R}^l$ we let $aV = Va = \{a \cdot v : v \in V\}$. From now on, we follow the notation of Lemma 2.4. Given $u \in U$ we define $K_1(u) = \{z \in \mathbb{R}^n : u(\phi)z = 0\}$ and $K_2(u) = \{t \in \mathbb{R}^m : tu(\phi) = 0\}$. Since, we are assuming that $k_1 = k_2 = k$, we have that $K_1(u) = \ker u(\phi) = \ker(u(\phi)|d_m)$

and $K_2(u) = \ker u(\phi)^t = \ker (u(\phi)^t | d_n)$. We see from Lemma 2.4 that $K_1(u) = \{y^h(u, v) : v \in \mathbb{R}^{n-k}\}$ and $K_2(u) = \{x^h(u, w) : w \in \mathbb{R}^{m-k}\}$. It follows that $\dim K_1(u) = n - k$, $\dim K_2(u) = m - k$. and $K_1(u) d_n = d_m K_2(u) = 0$. Let $u^1, u^2 \in U$. Any QE (x, y) of the game $(u^1(\phi), u^2(\phi))$ may be written as

$$(x^p(u^2) + x^h(u^2, w), y^p(u^1) + y^h(u^1, v))$$

for some $v \in \mathbb{R}^{n-k}$ and $w \in \mathbb{R}^{m-k}$. When there is no danger of confusion we will not write explicitly the dependence on u^1, u^2, v or w . For each outcome $\omega \in \Omega$, define the following polynomial in the variables $(v, w) \in \mathbb{R}^{n-k} \times \mathbb{R}^{m-k}$

$$q_\omega(v, w) = x^p(u^2) \phi^\omega y^p(u^1) + x^p(u^2) \phi^\omega y^h(u^1, v) + x^h(u^2, w) \phi^\omega y^p(u^1) + x^h(u^2, w) \phi^\omega y^h(u^1, v) \quad (7)$$

The probability that outcome $\omega \in \Omega$ occurs is $q_\omega(v, w)$. Given $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ we define the following subspaces.

$$\begin{aligned} x \phi^\omega K_1(u^1) &= \{x \phi^\omega z : z \in K_1(u^1)\} \\ K_2(u^2) \phi^\omega y &= \{t \phi^\omega y : t \in K_2(u^2)\} \\ K_2(u^2) \phi^\omega K_1(u^1) &= \{t \phi^\omega z : z \in K_1(u^1), t \in K_2(u^2)\} \end{aligned}$$

Lemma 2.5. *Suppose that Assumption 2.3 holds. Let $\omega \in \Omega$, $u^1, u^2 \in U$. If either of the following two conditions hold*

- (a) $x(u^2) \phi^\omega K_1(u^1) = \{0\}$ for every solution $x(u^2)$ of the system of equations (2).
- (b) $K_2(u^2) \phi^\omega y(u^1) = \{0\}$ for every solution $y(u^1)$ of the system of equations (1).

then, $K_2(u^2) \phi^\omega K_1(u^1) = \{0\}$.

Proof. Suppose that condition (a) holds. Let $t \in K_2(u^2)$. Recall that we may write t as $t = x_1 - x_2$, where x_1, x_2 are two solutions of the system of equations (2). Let $z \in K_1(u^1)$. Then, $t \phi^\omega z = x_1 \phi^\omega z - x_2 \phi^\omega z = \alpha d_n z - \alpha d_n z = 0$. Hence, it follows that $K_2(u^2) \phi^\omega K_1(u^1) = \{0\}$. Similarly, condition (b) implies that $K_2(u^2) \phi^\omega K_1(u^1) = \{0\}$. \square

Proposition 2.6. *Suppose that Assumption 2.3 holds. Let $u^1, u^2 \in U$. The set of QE of the game defined by $u^1(\phi)$ and $u^2(\phi)$ induce finitely many quasi-distributions on Ω if and only if the following two conditions hold.*

- (a) $x(u^2) \phi^\omega K_1(u^1) = \{0\}$ for every $\omega \in \Omega$ and every solution $x(u^2)$ of the system of equations (2).
- (b) $K_2(u^2) \phi^\omega y(u^1) = \{0\}$ for every $\omega \in \Omega$ and for every solution $y(u^1)$ of the system of equations (1).

Proof. Since, the set of QE's of the game $(u^1(\phi), u^2(\phi))$ is convex we have that the set

$$\{z \in \mathbb{R}^\Omega : z \text{ is a quasi-distribution on } \Omega \text{ for the game } (u^1(\phi), u^2(\phi))\}$$

is connected. Hence, the QEs induce finitely many quasi-distributions on outcomes if and only if they induce a unique quasi-distribution on outcomes. We prove next part (a). Assume there is a unique quasi-distribution induced on outcomes by the QEs of the game. Let $x = x(u^2)$ be a solution of (2), and

$z = z(u^1) \in K_1(u^1)$. We write $z = y_1 - y_2$, where y_1, y_2 are two solutions of (1). Then, for each $\omega \in \Omega$, we have that $x\phi^\omega y_1 = x\phi^\omega y_2$. So,

$$x\phi^\omega z = x\phi^\omega y_1 - x\phi^\omega y_2 = 0$$

Thus, $x\phi^\omega K_1(u^1) = \{0\}$ and (a) follows. Similarly, we can prove (b).

Conversely, suppose that conditions (a) and (b) hold. Let $u^1, u^2 \in U$ and $\omega \in \Omega$. By Lemma 2.5 we have that $K_2(u^2)\phi^\omega K_1(u^1) = \{0\}$. Therefore, in (7), for every QE $(x^p(u^2) + x^h(u^2, w), y^p(u^1) + y^h(u^1, v))$, we have that

$$(x^p(u^2) + x^h(u^2, w))\phi^\omega (y^p(u^1) + y^h(u^1, v)) = x^p(u^2)\phi^\omega y^p(u^1)$$

And the QEs of the game defined by $u^1(\phi)$ and $u^2(\phi)$ induce a unique quasi-distribution on Ω . \square

Corollary 2.7. *Let Assumption 2.3 hold. Suppose there are two sets of vectors $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ such that for every $u \in U$ and $i = 1, 2$ we have that V_i generates $K_i(u)$. Then, for any $u^1, u^2 \in U$ the set of CMNE of the game $(u^1(\phi), u^2(\phi))$ induce finitely many probability distributions on outcomes.*

Proof. Let $z \in V_1$ and $t \in V_2$. For every $u \in U$, we have that $tu(\phi) = u(\phi)z = 0$. Since, the entries of $u(\phi)$ are linear in u and U contains an open subset of $\mathbb{R}^{|\Omega|}$, we see that $t\phi^\omega = \phi^\omega z = 0$ for every $\omega \in \Omega$. Hence, $\phi^\omega K_1(u) = K_2(u)\phi^\omega = \{0\}$ for every $u \in U$ and $\omega \in \Omega$. The result follows now from Proposition 2.6. \square

The following result provides necessary and sufficient conditions for the existence of finitely many probability distributions on outcomes in games induced by outcome game forms.

Theorem 2.8. *Let Assumption 2.3 hold, $u^1, u^2 \in U$ and suppose there is, at least, one CMNE of the game $(u^1(\phi), u^2(\phi))$. Then, the set of all the CMNE of that game induces finitely many probability distributions on outcomes iff for every $\omega \in \Omega$*

$$\text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix} = 2k \quad \text{and} \quad \text{rank} \begin{pmatrix} u^2(\phi) & \phi^\omega & 0 \\ 0 & u^1(\phi) & d_m \end{pmatrix} = 2k$$

The proof is based on techniques developed in Litan and Marhuenda (2012). A review of these and the details of the proof are provided in the appendix.

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A The proof of Theorem 2.8

Lemma A.1. *Suppose that Assumption 2.3 holds. Let $u^1, u^2 \in U$. The set of QE of the game $(u^1(\phi), u^2(\phi))$ induce finitely many quasi-distributions on outcomes if and only if for every $\omega \in \Omega$ and every QE $(x(u^2), y(u^1))$ of that game the following two conditions hold.*

- (a) $x(u^2)\phi^\omega$ is in the image of $u^1(\phi)^t$
- (b) $\phi^\omega y(u^1)$ is in the image of $u^2(\phi)$.

Proof. Let $u^1, u^2 \in U$. By Proposition 2.6, the set of QE of the game $(u^1(\phi), u^2(\phi))$ induce finitely many quasi-distributions on outcomes if and only if for every $\omega \in \Omega$, every solution $x(u^2)$ of the system of equations (2) and every solution $y(u^1)$ of the system of equations (1) we have that $x(u^2)\phi^\omega K_1(u^1) = \{0\}$ and $K_2(u^2)\phi^\omega y(u^1) = \{0\}$. This is equivalent to the statement that $x(u^2)\phi^\omega$ is orthogonal to $K_1(u^1)$ and $\phi^\omega y(u^1)$ is orthogonal to $K_2(u^2)$, which occurs if and only if $x(u^2)\phi^\omega$ is in the image of $u^1(\phi)^t$ and $\phi^\omega y(u^1)$ is in the image of $u^2(\phi)$. \square

Lemma A.2. *Let $\omega \in \Omega$ and $u^1, u^2 \in U$. Let Assumption 2.3 hold and suppose that $K_2(u^2)\phi^\omega K_1(u^1) = \{0\}$. Then,*

$$\text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \end{pmatrix} = 2k$$

Proof. We use the notation

$$\phi^\omega = \begin{pmatrix} B^\omega & C^\omega \\ D^\omega & E^\omega \end{pmatrix}$$

to denote the decomposition of the matrix $u(\phi)$ in (3) applied to the matrix ϕ^ω . Let

$$F = \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \end{pmatrix}$$

We can write now

$$F = \begin{pmatrix} B(u^1) & C(u^1) & 0 & 0 \\ D(u^1) & E(u^1) & 0 & 0 \\ B^\omega & C^\omega & B(u^2) & C(u^2) \\ D^\omega & E^\omega & D(u^2) & E(u^2) \end{pmatrix}$$

By elementary row and column operations,

$$\text{rank } F = \text{rank} \begin{pmatrix} B(u^1) & C(u^1) & 0 \\ B^\omega & C^\omega & B(u^2) \\ D^\omega & E^\omega & D(u^2) \end{pmatrix} = \text{rank} \begin{pmatrix} B(u^1) & C(u^1) & 0 \\ B^\omega & C^\omega & B(u^2) \\ D_1^\omega & E_1^\omega & 0 \end{pmatrix}$$

where

$$D_1^\omega = D^\omega - D(u^2) B^{-1}(u^2) B^\omega \quad (8)$$

$$E_1^\omega = E^\omega - D(u^2) B^{-1}(u^2) C^\omega \quad (9)$$

Finally,

$$\text{rank } F = \text{rank} \begin{pmatrix} B(u^1) & 0 & 0 \\ B^\omega & C_2^\omega & B(u^2) \\ D_1^\omega & E_2^\omega & 0 \end{pmatrix}$$

with $C_2^\omega = C^\omega - B^\omega B^{-1}(u^1) C(u^1)$ and

$$\begin{aligned} E_2^\omega &= E_1^\omega - D_1^\omega B^{-1}(u^1) C(u^1) \\ &= E^\omega - D(u^2) B^{-1}(u^2) C^\omega - D^\omega B^{-1}(u^1) C(u^1) + D(u^2) B^{-1}(u^2) B^\omega B^{-1}(u^1) C(u^1) \\ &= \begin{pmatrix} -D(u^2) B^{-1}(u^2) & I_{m-k} \end{pmatrix} \begin{pmatrix} B^\omega & C^\omega \\ D^\omega & E^\omega \end{pmatrix} \begin{pmatrix} -B^{-1}(u^1) C(u^1) \\ I_{n-k} \end{pmatrix} \\ &= \begin{pmatrix} -D(u^2) B^{-1}(u^2) & I_{m-k} \end{pmatrix} \phi^\omega \begin{pmatrix} -B^{-1}(u^1) C(u^1) \\ I_{n-k} \end{pmatrix} \end{aligned}$$

It follows from Lemma 2.4 that

$$K_1(u^1) = \{y^h(u^1, v) : v \in \mathbb{R}^{n-k}\} = \left\{ \begin{pmatrix} -B^{-1}(u^1) C(u^1) \\ I_{n-k} \end{pmatrix} v : v \in \mathbb{R}^{n-k} \right\}$$

and

$$K_2(u^2) = \{x^h(u^2, w) : w \in \mathbb{R}^{m-k}\} = \left\{ w \begin{pmatrix} -D(u^2) B^{-1}(u^2) & I_{m-k} \end{pmatrix} : w \in \mathbb{R}^{m-k} \right\}$$

Since $K_2(u^2)\phi^\omega K_1(u^1) = \{0\}$, for any $v \in \mathbb{R}^{n-k}$ and $w \in \mathbb{R}^{m-k}$ we have that

$$w E_2^\omega v = w \begin{pmatrix} -D(u^2) B^{-1}(u^2) & I_{m-k} \end{pmatrix} \phi^\omega \begin{pmatrix} -B^{-1}(u^1) C(u^1) \\ I_{n-k} \end{pmatrix} v = 0$$

Therefore, $E_2^\omega = 0$ and

$$\text{rank } F = \text{rank} \begin{pmatrix} B(u^1) & 0 & 0 \\ B^\omega & C_2^\omega & B(u^2) \\ D_1^\omega & 0 & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} B(u^1) & 0 & 0 \\ B^\omega & C_2^\omega & B(u^2) \end{pmatrix} = 2k$$

because, since $\text{rank } B(u^1) = \text{rank } B(u^2) = k$, the rows of D_1^ω are a linear combination of the rows of $B(u^1)$. \square

Lemma A.3. *Suppose that Assumption 2.3 holds. Let $u^1, u^2 \in U$, $\omega \in \Omega$. Then,*

(a) $x(u^2)\phi^\omega K_1(u^1) = \{0\}$ for every solution $x(u^2)$ of the system of equations (2) if and only if

$$\text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix} = 2k$$

(b) $K_2(u^2)\phi^\omega y(u^1) = 0$ for every solution $y(u^1)$ of the system of equations (1) if and only if

$$\text{rank} \begin{pmatrix} u^2(\phi) & \phi^\omega & 0 \\ 0 & u^1(\phi) & d_m \end{pmatrix} = 2k$$

Proof. We prove only part (a). The proof of part (b) is similar. Fix a solution $x = x(u^2)$ of the system of equations (2). Let

$$F = \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix}$$

Since, $d_n = \frac{1}{\alpha(u^2)} x u^2(\phi)$, by elementary row operations we have that

$$\text{rank } F = \text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ \frac{1}{\alpha(u^2)} x \phi^\omega & 0 \end{pmatrix}$$

Assume that $x \phi^\omega K_1(u^1) = 0$. Then, by Lemma A.1, $x \phi^\omega$ is in the image of $u^1(\phi)^t$ and hence $x \phi^\omega$ is a linear combination of the rows of $u^1(\phi)$. Therefore,

$$\text{rank } F = \text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \end{pmatrix}$$

and, by Lemmas 2.5 and A.2 we have that $\text{rank } F = 2k$.

Conversely, suppose now that $\text{rank } F = 2k$. We proceed now as in Lemma A.2 and write F as

$$F = \begin{pmatrix} u^1(\phi) & 0 & 0 \\ B^\omega & C^\omega & B(u^2) & C(u^2) \\ D^\omega & E^\omega & D(u^2) & E(u^2) \\ 0 & 0 & d_k & d_{n-k} \end{pmatrix}$$

The above argument shows that

$$\text{rank } F = \text{rank} \begin{pmatrix} u^1(\phi) & 0 & 0 \\ B^\omega & C^\omega & B(u^2) & C(u^2) \\ D^\omega & E^\omega & D(u^2) & E(u^2) \\ \frac{1}{\alpha(u^2)}x(u^2)\phi^\omega & 0 & 0 \end{pmatrix}$$

Since $\text{rank } u^2(\phi) = \text{rank } B(u^2) = k$, we get that

$$\text{rank } F = \text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ B_1^\omega & C_1^\omega & B(u^2) \\ D_1^\omega & E_1^\omega & 0 \\ \frac{1}{\alpha(u^2)}x\phi^\omega & 0 \end{pmatrix} = 2k$$

where D_1^ω, E_1^ω are defined in (8) and (9). Since, $\text{rank } u^1(\phi) = \text{rank } B(u^2) = k$, the rows of the matrix $\begin{pmatrix} D_1^\omega & E_1^\omega \end{pmatrix}$ and $x\phi^\omega$ are a linear combination of the rows of $u^1(\phi)$. It follows that $x\phi^\omega$ is orthogonal to $K_1(u^1)$ and (a) follows. \square

The following result follows now immediately from Proposition 2.6 and Lemma A.3.

Theorem A.4. *Suppose that Assumption 2.3 holds. Let $u^1, u^2 \in U$. Then, the set of all the QE of the game $(u^1(\phi), u^2(\phi))$ induces finitely many quasi-distributions on outcomes iff for every $\omega \in \Omega$*

$$\text{rank} \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix} = 2k \quad \text{and} \quad \text{rank} \begin{pmatrix} u^2(\phi) & \phi^\omega & 0 \\ 0 & u^1(\phi) & d_m \end{pmatrix} = 2k$$

We address now the proof of Theorem 2.8. We show first the ‘if’ part. If the rank conditions in Theorem 2.8 hold, then by Theorem A.4 the set of QE induce a unique quasi-distribution on outcomes. Since the set of CMNE is a subset of the set of QE, the set of CMNE also induces, at most, a unique distribution on outcomes. Thus, the ‘if’ part of Theorem 2.8 holds.

Conversely, suppose that the set of CMNE of the game $(u^1(\phi), u^2(\phi))$ induces finitely many distributions on outcomes and that the game $(u^1(\phi), u^2(\phi))$ has, at least, a CMNE, say $\bar{x} = x^p(u^2) + x^h(u^2, w_0) \in \Delta_+(S^1)$ and $\bar{y} = y^p(u^1) + y^h(u^1, v_0) \in \Delta_+(S^2)$ with $w_0 \in \mathbb{R}^{m-k}$, $v_0 \in \mathbb{R}^{n-k}$. By continuity, there are open sets $H_1 \subset \mathbb{R}^{n-k}$ and $H_2 \subset \mathbb{R}^{m-k}$ such that for $v \in H_1$, $w \in H_2$ we have that $x = x^p(u^2) + x^h(u^2, w) \in \Delta_+(S^1)$ and $y = y^p(u^1) + y^h(u^1, v) \in \Delta_+(S^2)$ is a CMNE of the game $(u^1(\phi), u^2(\phi))$.

If the rank conditions in Theorem 2.8 do not hold, then, by Theorem A.4 for some outcome $\omega \in \Omega$ the polynomial $q_\omega(v, w)$ in (7) is not constant and, hence, it takes a continuum of values as the variables (v, w) vary on the open set $H_1 \times H_2$. It follows that the set CMNE of the game $(u^1(\phi), u^2(\phi))$ induce infinitely many distributions on outcomes, which contradicts our assumption. And the ‘only if’ part of Theorem 2.8 follows.