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Discussion Papers on Business and Economics
No. 13/2015

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Abstract

A highway problem is a cost sharing problem that arises if the common resource is an ordered set of sections with fixed costs such that each agent demands consecutive sections. We show that the core, the prenucleolus, and the Shapley value on the class of TU games associated with highway problems possess characterizations related to traditional axiomatizations of the solutions on certain classes of games. However, in the formulation of the employed simple and intuitive properties the associated games do not occur. The main axioms for the core and the nucleolus are consistency properties based on the reduced highway problem that arises from the original highway problem by eliminating any agent of a specific type and using her charge to maintain a certain part of her sections. The Shapley value is characterized with the help of individual independence of outside changes, a property that requires the fee of an agent only depending on the highway problem when truncated to the sections she demands. An alternative characterization is based on the new contraction property. Finally it is shown that the games that are associated with generalized highway problems in which agents may demand non-connected parts are the positive cost games, i.e., nonnegative linear combinations of dual unanimity games.

Keywords: TU game · airport problem · highway problem · core · nucleolus · Shapley value
JEL codes: C71

1 Introduction

In this paper we analyze a particular kind of cost allocation problem in which some agents jointly produce and finance a common resource or facility. The peculiarity is that this resource can be separated into a number of ordered sections. Moreover, each agent requires some consecutive sections, and each section has a fixed cost. The issue of our present study is how to share the total cost of all sections among the users in an efficient and fair way. A simple example that

*Support from the Spanish Ministerio de Ciencia e Innovación under project ECO2012-33618, co-funded by the ERDF, is acknowledged, and the first author acknowledges support from the Danish Council for Independent Research | Social Sciences (Grant-id: DFF – 1327-00097)

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illustrates this situation is a linear highway, where the sections are delimited by the entry and exit points, and each car only needs the highway sections between its entry and exit point.[1] This example motivates why these problems are called highway problems (Kuipers, Mosquera, and Zarzuelo 2013). The well-known airport problems can be seen as special highway problems, in which all agents' entries coincide.

Cooperative game theory has proved to be very useful to solve cost allocation problems, since transferable utility (TU) games are highly appropriate to model these kinds of situations. Moreover, the solution concepts of these games embody a criterion of fairness in their definition and satisfy certain properties or axioms which make them particularly suitable for these problems. With every highway problem a cooperative TU game is associated by assigning to every coalition the cost of the highway sections that accommodates all the members in that coalition. Such a cooperative game is called a highway game. Subsequently some cooperative solution concepts are applied to the highway game to solve the original problem. In this paper we will focus on three solution concepts on the class of highway problems: the core, the (pre)nucleolus and the Shapley value. We axiomatize these three solutions for the class of highway problems.

The axiomatizations of the core and the (pre)nucleolus of highway problems are based on the consistency principle. According to this principle, if a group of agents pays its share and leaves the others in a renegotiation, then the shares of the remaining agents do not change in the subsequent reduced situation. The consistency property has proved to be very powerful in characterizing some of the most important solutions concepts in cooperative game theory: the prenucleolus (Sobolev 1975); the core (Peleg 1985, Peleg 1986, Tadenuma 1992, Hwang and Sudhölter 2001); the Nash bargaining solution (Lensberg 1988); the Shapley value of TU games and the egalitarian NTU value (Hart and Mas-Colell 1989); the Harsanyi NTU value (Hinojosa, Romero, and Zarzuelo 2012). Consistency has also played a prominent role in other contexts: for instance, in bankruptcy problems (Aumann 1985), airport problems (Potters and Sudhölter 1999), and other allocation problems. The aforementioned articles employ different definitions of consistency because of the context's diversity. In general, the crucial issue is to identify the available alternatives for intermediate coalitions in a reduced situation. As a consequence there is not a canonical way of modeling the reduced problem, and many different kinds of reduced problems have been proposed in the literature. In the case of a highway problem, we require the consistency property only for an agent i, whose needs are minimal in the sense that they do not cover those of any other agent (up to an exception the explanation of which is postponed to Section 3). We assume that i's intention is to pay only for the part he uses, so it seems natural that her payment before leaving should be subtracted from the cost of her sections. So every agent sharing some of the segments used by i might benefit from this reduction, but it cannot harm her.

On the other hand one major axiomatization of the Shapley value of a highway problem is based on a monotonicity principle. This principle has already been used in the context of cooperative games (Young 1985a) and in cost allocation problems as well (Young 1985b). In the case of highway problems this principle states that the share payed by agent i may only depend on the

[1] This example is a simplification of a highway problem because there are other issues of primary importance, as congestion. In our context these issues will not appear explicitly, but at least some of them might be taken into account implicitly by the cost of each section.

highway problem when restricted to the sections of the highway used by that agent.

The paper is organized as follows. In Section 2, we introduce two representations of highway problems and their corresponding highway games. Section 3 is devoted to the axioms employed in the subsequent characterizations of the core and the (pre)nucleolus. The game associated with the *reduced highway problem* coincides with the Davis–Maschler reduced game of the original highway game. However, neither the definition of the reduced highway problem nor of the corresponding *reduced highway problem property* (RHP) and its converse (CRHP) refer to the associated highway games. In Section 4, resembling a result of Peleg (1986), we show that the core is the unique solution for highway problems that satisfies *individual rationality*, *unanimity for 2-person highway problems* (UTPH), RHP, and CRHP. Moreover, if a stronger version of CRHP is employed, then UTPH may be replaced by *non-emptiness.* In Section 5 we prove that the (pre)nucleolus is characterized by *single-valuedness*, the *equal treatment property*, *covariance under exclusive prolongations*, and RHP. In Section 6 we characterize the Shapley value (a) with the help of *individual independence of outside changes*, which is a suitable translation of Young's (1985a) strong monotonicity, and, alternatively, (b) with the help of the *contraction property* which is some kind of consistency property and entirely new. Indeed, the game associated with a contracted highway problem, the definition of which requires to compute the solution of a certain truncated highway problem, is not the Hart–Mas-Colell (1989) "reduced" game of the original highway problem. In Section 7 we show that a generalized highway game in which the sections used by an agent may not be connected is a positive cost game, i.e., a nonnegative linear combination of dual unanimity games, and vice versa. Finally, Section 8 closes the paper with some discussions.

2 Preliminaries

Let U be a set ($|U| \geqslant 5$ is needed in Example 8.1, and we always assume that $\{1,\dots,\ell\} \subseteq U$ if $|U| \geqslant \ell$), called the *universe of agents.* A finite nonempty subset of U is called a *coalition.*

Definition 2.1 *A* highway problem *is a pair (N,I) such that the following conditions are satisfied:*

(1) N is a coalition.

(2) I is a mapping that assigns to each $i \in N$ a compact nonempty interval $I_i \subseteq \mathbb{R}_+$.

(3) $I_N = [0,b]$ for some $b \in \mathbb{R}_+$, where $I_S = \bigcup_{i\in S} I_i$ for $S \subseteq N$.

Denote by $\mathcal{H}$ the set of highway problems. For the generic element $(N,I) \in \mathcal{H}$, we typically write $I_i = [a_i, b_i]$. As $I_i \neq \emptyset$, we have $a_i \leqslant b_i$.

The elements in N represent the agents involved in the problem. For each $i \in N$, the interval I_i associated with agent i is an interval representing the (connected) parts of the common facility that is used by agent i. This common facility is symbolized by I_N. Condition (3) says that the first part starts at 0, the last one finishes at certain real number b, and there are no gaps between the parts used by N. The length of an interval represents its cost. Thus the cost of serving agent i is $b_i - a_i$, and accordingly the total cost of the common facility is b that is the amount to be shared between all the agents.

In order to define the sections of the highway problem (N, I) let $(\beta_0, \dots, \beta_m)$ be a real sequence of minimal length that satisfies the following properties:

- $0 = \beta_0 \leqslant \cdots \leqslant \beta_m$.
- For each $i \in N$ there exist r, r', $0 \leqslant r < r' \leqslant m$, such that $I_i = [\beta_r, \beta_{r'}]$.

Note that m and $(\beta_0, \dots, \beta_m)$ are uniquely determined by the foregoing properties and minimality. Then

$$M^I = \{[\beta_r, \beta_{r+1}] \mid r = 0, \dots, m\} \tag{2.1}$$

is the set of *sections* of (N, I). Note also that, if $\beta_r = \beta_{r+1}$ (i.e., if the $r+1$-th section is a singleton), then there exists $i \in N$ with $I_i = [\beta_r, \beta_{r+1}]$, and if $r + 1 < m$ in addition, then $\beta_{r+2} > \beta_{r+1}$. Finally note that M^I is totally ordered by $[0, \beta_1] \prec \cdots \prec [\beta_{m-1}, \beta_m]$, where $\prec = \prec^I$ is the strict total order relation.

The (cost) TU game associated with the highway problem (N, I) is the game (N, c^I) defined by

$$c^I(S) = \lambda^*(I_S) \text{ for all } S \subseteq N, \tag{2.2}$$

where λ^* denotes the Lebesgue measure on $\mathbb{R}$. A TU game is a *highway game* if it the TU game associated with a highway problem. Note that highway games are concave.[2]

That is, the real number $c^I(S)$ is the cost of serving the agents in S.

Remark 2.2 Kuipers, Mosquera, and Zarzuelo (2013) define a "highway problem" to be a quadruple (N, M, C, T) that satisfies the following properties: N is a coalition, M is a finite nonempty set strictly ordered by $\prec$ (the set of sections), $C : M \to \mathbb{R}_+$ is a mapping that represents the cost of each section, and $T : N \to 2^M \setminus \{\emptyset\}$ is a mapping, where $T(i)$ represents the set of sections used by agent i, satisfying

$$T(i) \neq \emptyset \text{ for all } i \in N; \tag{2.3}$$

$$\text{if } i \in N, j, k \in T(i), \ell \in M, \text{ and } j \prec \ell \prec k, \text{ then } \ell \in T(i); \tag{2.4}$$

$$\text{for } k, \ell \in M, k \neq \ell \text{ there exists } i \in N \text{ such that } k \notin T(i) \ni \ell; \tag{2.5}$$

$$\text{for } k \in C^{-1}(0) \text{ there exists } i \in N \text{ such that } T(i) = \{k\}; \tag{2.6}$$

$$\text{if } k \in C^{-1}(0) \text{ and } k < m, \text{ then } C(k+1) > 0. \tag{2.7}$$

Though (2.5) – (2.7) do formally not appear in the aforementioned paper, these conditions may be added without loss of generality.

[2]For $(N, I) \in \mathcal{H}$, $c^I(S) + c^I(T) \geqslant c^I(S \cap T) + c^I(S \cup T)$ for all $S, T \subseteq N$.

The coalition function $c^{M,C,T}$ of the corresponding "highway problem" (N,c) is defined for each $S \subseteq N$ by

$$c^{M,C,T}(S) = \sum_{t \in T(S)} C(t) \text{ for all } S \subseteq N, \tag{2.8}$$

where $T(S) = \bigcup_{i \in S} T(i)$.

Although "highway problems" are formally different from those in Definition 2.1, they are conceptually equivalent. Indeed, if (N,M,C,T) is a "highway problem", then it can be associated with an element of $\mathcal{H}$ as follows. Define $\alpha_k = \sum_{j=1}^{k} C(j)$ for $k = 0,\dots,m$, and $I_i = \bigcup_{k \in T(i)} [\alpha_{k-1}, \alpha_k]$ for all $i \in N$, so it is a closed interval. Then it is straightforward to show that $(N,M,C,T) \mapsto (N,I)$ defines a bijective mapping from "highway problems" to $\mathcal{H}$ and that $c^{M,C,T} = c^I$. Hence, we may say that a highway problem may be represented by (N,I) as well as by (N,M,C,T). In what follows we shall use the representation (N,I) except in Section 7 where the representation (N,M,C,T) is more convenient.

3 Axioms

We now introduce the axioms employed in the subsequent characterizations and their proofs of the core and the (pre)nucleolus. Denote the set of *feasible cost allocations* and the set of *efficient feasible cost allocations (preimputations)* by $X^*(N,I)$ and $X(N,I)$ respectively, i.e.,

$$X^*(N,I) = \left\{x \in \mathbb{R}^N \mid x(N) \geqslant c^I(N)\right\} \quad \text{and} \quad X(N,I) = \left\{x \in \mathbb{R}^N \mid x(N) = c^I(N)\right\},$$

where $x(S) = \sum_{i \in S} x_i$ for all $S \subseteq N$ and $x \in \mathbb{R}^N$.

Moreover, let $\mathbb{1}^S \in \mathbb{R}^N$ denote the *indicator* vector of S, i.e., $\mathbb{1}^S_j = \begin{cases} 1, \text{ if } j \in S, \\ 0, \text{ if } j \in N \setminus S. \end{cases}$

A *solution* σ assigns to each highway problem (N,I) a subset of $X^*(N,I)$. Its restriction to a set $\mathcal{H}' \subseteq \mathcal{H}$ is again denoted by σ. A solution on $\mathcal{H}'$ is the restriction to $\mathcal{H}'$ of a solution. A solution σ on $\mathcal{H}'$ satisfies

(1) *non-emptiness* (NEM) if for all $(N,I) \in \mathcal{H}'$: $\sigma(N,I) \neq \emptyset$;

(2) *Pareto optimality* (PO) if for all $(N,I) \in \mathcal{H}'$: $\sigma(N,I) \subseteq X(N,I)$.

(3) *single-valuedness* (SIVA) if for all $(N,I) \in \mathcal{H}'$: $|\sigma(N,I)| = 1$;

(4) the *equal treatment property* (ETP) if for all $(N,I) \in \mathcal{H}'$, all $i,j \in N$, and all $x \in \sigma(N,I)$: $I_i = I_j$ implies $x_i = x_j$;

(5) *individual rationality* (IR) if for all $(N,I) \in \mathcal{H}'$, all $i \in N$, and all $x \in \sigma(N,I)$: $x_i \leqslant \lambda^*(I_i)$;

(6) *reasonableness from below* (REASB) if for all $(N,I) \in \mathcal{H}'$, all $x \in \sigma(N,I)$, and all $i \in N$: $x_i \geqslant \lambda^*\left(I_i \setminus I_{N\setminus\{i\}}\right)$;

(7) *covariance under exclusive prolongations* (PCOV) if for all $(N,I),(N,I') \in \mathcal{H}'$: If there exist $i \in N$, $a \in I_i \setminus \operatorname{int} I_{N\setminus\{i\}}$, and $x > 0$ such that,[3] with $I_j = [a_j, b_j]$ for $j \in N$,

$$I'_j = \begin{cases} [a_i, b_i + x], & \text{if } j = i, \\ [a_j, b_j], & \text{if } b_j \leqslant a_i, \\ [a_j + x, b_j + x], & \text{if } a_j \geqslant a_i, \end{cases}$$

then $\sigma(N,I) = \sigma(N,I') + x\mathbb{1}^{\{i\}}$;

NEM, PO, SIVA, ETP, and IR are standard in the literature and do not need further explanation. It should be remarked that by the concavity of highway games, IR is equivalent to *reasonableness from above* in the sense that each agent has to pay at most his maximal marginal contribution to the cost of the entire highway. Hence, IR and REASB together may be called *reasonableness.* The interpretation of PCOV is simple: If an agent asks for prolonging the highway just for herself, then the cost for this modification is added to her payment, whereas the payments of the other agents is not changed.

Remark 3.1 Recall that a *solution* σ on a set Γ of TU cost games assigns a subset $\sigma(N,c)$ of $X^*(N,c) = \{x \in \mathbb{R}^N \mid x(N) \geqslant c(N)\}$ to each $(N,c) \in \Gamma$. The axioms NEM, PO (x is *Pareto optimal* if $x(N) = c(N)$), SIVA, ETP (players $i,j \in N$ are *equals* if $c(S \cup \{i\}) = c(N \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$), IR ($x \in \mathbb{R}^N$ is *individually rational* if $x_i \geqslant c(\{i\})$ for all $i \in N$), REASB ($x \in \mathbb{R}^N$ is *reasonable from below* if $x_i \geqslant \min_{S\subseteq N\setminus\{i\}}(c(S \cup \{i\}) - c(S))$ for all $i \in N$), and COV ($\sigma(N, \alpha c + \beta) = \alpha\sigma(N,c) + \beta$ for $\alpha > 0, \beta \in \mathbb{R}^N$ whenever $(N,c),(N,\alpha c + \beta) \in \Gamma$) on the set Γ of all games associated with elements of $\mathcal{H}$ imply (1) – (8) for the corresponding solution on $\mathcal{H}$ that is, by a slight abuse of notation, again denoted by σ (i.e., $\sigma(N,I) = \sigma(N,c^I)$ for all $(N,I) \in \mathcal{H}$). Here, COV implies PCOV as well as *scale covariance*, a property that, though satisfied by all our solutions, is not needed in our characterization results.

We now define reduced highway problems. Let $(N,I) \in \mathcal{H}, i \in N$, and $k \in \mathbb{N} \cup \{0\}$. Say that i is of type k if

$$\left|\{j \in N \mid I_j \subseteq \operatorname{int} I_i\}\right| = k.$$

Moreover, let $[a_j, b_j] = I_j$ for all $j \in N$. We call i a *left* (resp. *right*) agent, if for all $j \in N$, $a_j < b_i \leqslant b_j$ implies $a_j \leqslant a_i$ (resp. $a_j \leqslant a_i < b_j$ implies $b_i \leqslant b_j$). Agent i is *oriented* if i is a left or a right agent.

Definition 3.2 *Let $(N,I) \in \mathcal{H}$, $I_j = [a_j, b_j]$ for all $j \in N$, such that $|N| \geqslant 2$. An agent $i \in N$ is called* feasible *(w.r.t. reduction) if i is of type 0 or if i is an oriented agent of type 1.*

[3]The symbol int A stands for the interior of A, for every $A \subseteq \mathbb{R}$.

Let $i \in N$ be feasible and $x \in \mathbb{R}^N$. The reduced problem $(N \setminus \{i\}, I^{-i,x})$ *is defined as follows, where $I_j^{-i,x} = [a'_j, b'_j]$ for all $j \in N \setminus \{i\}$ and, in case that i is of type 1, $k \in N$ is the unique agent such that $a_i < a_k \leqslant b_k < b_i$:*

If i is of type 0, then

$$a'_j = \begin{cases} a_j, & \text{if } a_j \leqslant a_i, \\ \max\{a_i, a_j - x_i\}, & \text{if } a_i < a_j, \end{cases} \tag{3.1}$$

$$b'_j = \begin{cases} \min\{b_j, b_i - x_i\}, & \text{if } b_j < b_i, \\ b_j - x_i, & \text{if } b_j \geqslant b_i, \end{cases} \tag{3.2}$$

The definition of a'_j and b'_j in the case that i is an oriented agent of type 1 may differ from (3.1) *and* (3.2) *only inasmuch as*

$$a'_k = \max\big\{a_i, \min\{a_k, b_i - x_i - b_k + a_k\}\big\}, \text{ if } i \text{ is a left agent, and} \tag{3.3}$$
$$b'_k = \min\big\{b_i - x_i, \max\{b_k - x_i, b_k - a_k + a_i\}\big\}, \text{ if } i \text{ is a right but not left agent,} \tag{3.4}$$

i.e., if i is both a left and a right agent, then we regard her as a left agent.

Note that reduced problems are not necessarily highway problems. Indeed, if x_i in Definition 3.2 is small enough, then the "highway" may receive a gap, i.e. I_N is not an interval; and if x_i is large enough, then some "intervals" may have a negative length, i.e., are empty because $a'_j > b'_j$ may occur.

Lemma 3.3 *For any highway problem (N, I) with $|N| > 1$ there exist at least two distinct feasible agents.*

Proof: Call $i \in N$ minimal if for all $j \in N \setminus \{i\}$ such that $I_j \neq I_i$ it holds $I_j \setminus I_i \neq \emptyset$. There exists at least one minimal agent $i \in N$, and moreover a minimal agent is of type 0, hence feasible. Now assume that i is the unique minimal agent. Then $I_i \subsetneqq I_j$ for all $j \in N \setminus \{i\}$. Choose $j \in N \setminus \{i\}$ such that $a_j = \max_{k \in N \setminus \{i\}} a_k$, where $I_i = [a_i, b_i]$ for all $i \in N$. Then j is a left agent of type 1 or 0, i.e., the second feasible agent has been found. **q.e.d.**

We recall the definition of the Davis-Maschler reduced cost game now. Let (N, c) be a TU cost game (i.e., N is a coalition and $c : 2^N \to \mathbb{R}, c(\emptyset) = 0$), $x \in X^*(N, c)$, and $\emptyset \neq S \subsetneqq N$. The *reduced game* with respect to (w.r.t.) S and x is the TU game $(S, c_{S,x})$ defined by

$$c_{S,x}(T) = \begin{cases} c(N) - x(N \setminus S), \text{ if } T = S, \\ \min_{P \subseteq N \setminus S} \big(c(T \cup P) - x(P)\big), \text{ if } \emptyset \neq T \subsetneqq S. \end{cases}$$

Lemma 3.4 *Let $(N,I) \in \mathcal{H}$ with $|N| \geqslant 2$, $i \in N$ be a feasible agent, and $x \in X(N,I)$. If $\lambda^*\left(I_i \setminus I_{N\setminus\{i\}}\right) \leqslant x_i \leqslant \lambda^*(I_i)$ (i.e., x_i is reasonable from below and individually rational for i), then $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}$, $x_{N\setminus\{i\}} \in X(N \setminus \{i\}, I^{-i,x})$, and $c^{I^{-i,x}} = (c^I)_{N\setminus\{i\},x}$.*

Proof: Let $I_j = [a_j, b_j]$ for all $j \in N$, denote $I' = I^{-i,x}$, and let $I'_j = [a'_j, b'_j]$ for all $j \in N \setminus \{i\}$. From $\lambda^*\left(I_i \setminus I_{N\setminus\{i\}}\right) \leqslant x_i \leqslant \lambda^*(I_i)$ it follows that $(N \setminus \{i\}, I') \in \mathcal{H}$ and, moreover, that $\max_{\ell\in N} b_\ell - x_i = \max_{j\in N\setminus\{i\}} b'_j$, hence $x_{N\setminus\{i\}} \in X(N\setminus\{i\}, I')$ and $c^{I'}(N\setminus\{i\}) = c_{N\setminus\{i\},x}(N\setminus\{i\})$. Now, let $\emptyset \neq T \subsetneqq N \setminus \{i\}$. Denote $T_1 = \{j \in T \mid a_j \leqslant a_i < b_j\}$ and $T_2 = \{j \in T \mid a_j < b_i \leqslant b_j\}$. If i is an agent of type 0, then

$$c^I(T \cup \{i\}) - c^I(T) = \max\Big\{0, \min\left(\{a_j \mid j \in T_2\} \cup \{b_i\}\right) - \max\left(\{b_j \mid j \in T_1\} \cup \{a_i\}\right)\Big\}.$$

A careful inspection of (3.1) and (3.2) finishes the proof in this case. If i is a left agent of type 1 and k is the unique agent such that $a_i < a_k \leqslant b_k < b_i$, then the case $k \notin T$ can be treated as before. If $k \in T$, then the cases $T_2 \neq \emptyset$ or ($T_1 \neq \emptyset$ and $\max\{b_j \mid j \in T_1\} \leqslant b_i$) are straightforward as well as the case that $x_i \geqslant (b_i - a_i) - (b_k - a_k)$. In the remaining case, $I'_k = I_k - \varepsilon$, where $\varepsilon = b_k + x_i - b_i$, again a careful inspection (3.1) and (3.2) completes the proof. Finally, if i is a right agent of type 1, but not a left agent, then we may argue similarly. **q.e.d.**

We now recall explicitly the definitions by Peleg (1986) of the reduced game property and its converse. A solution σ on a set Γ of TU games satisfies

(8') the *reduced game property* (RGP) if for all $(N,c) \in \Gamma$, $\emptyset \neq S \subsetneqq N$, and $x \in \sigma(N,c)$: $(S, c_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, c_{S,x})$;

(9') the *converse reduced game property* (CRGP) if the following condition is satisfied for $(N,c) \in \Gamma$ with $|N| \geqslant 3$, $x \in X(N,c) = \{x \in \mathbb{R}^N \mid x(N) = c(N)\}$: If, for any $S \subseteq N$ with $|S| = 2$, $(S, c_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, c_{S,x})$, then $x \in \sigma(N,c)$.

Unfortunately, in general the reduced game of a highway game even w.r.t. imputations that are reasonable from below may not be highway games (see Example 5.6). However, according to Lemma 3.4, if only feasible agents may successively be removed, then reducing yields highway games. Hence, the following definitions are motivated. A solution σ on $\mathcal{H}'$ satisfies

(8) the *reduced highway problem property* (RHP) if for any $(N,I) \in \mathcal{H}'$ with $|N| > 1$, any feasible agent $i \in N$, and any $x \in \sigma(N,I)$: $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}'$ and $x_{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x})$;

(9) the *converse reduced highway problem property* (CRHP) if for any $(N,I) \in \mathcal{H}'$ with $|N| \geqslant 3$ and any $x \in X(N,I)$ the following condition holds: If, for each feasible agent $i \in N$, $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}'$ and $x_{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x})$, then $x \in \sigma(N,I)$.

Now, for a solution on $\mathcal{H}$ we suitably restate Peleg's notion (see also Sudhölter and Peleg 2002) of "unanimity for two-person games" and present a strong version of CRHP that is similar to a modification of CRGP that has been employed by Serrano and Volij (1998) in an axiomatization of the core and by Sudhölter and Potters (2001) in the axiomatization of the semi-reactive prebargaining set.

The solution σ on $\mathcal{H}'$ satisfies

(10) *unanimity of two-person highway problems* (UTPH) if, for any $(N, I) \in \mathcal{H}'$ with $|N| = 2$: $\sigma(N, I) = \{x \in X(N, I) \mid x_i \leqslant c^I(\{i\}) \text{ for all } i \in N\}$;

(11) the *strong converse reduced highway problem property* (SCRHP) if for any $(N, I) \in \mathcal{H}'$ with $|N| \geqslant 2$, and any $x \in X(N, I)$ the following condition holds: If, for each feasible agent $i \in N$, $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}'$ and $x_{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x})$, then $x \in \sigma(N, I)$.

4 Characterization of the core of highway problems

Recall that the *core* of (N, I), denoted $\mathcal{C}(N, I)$, is the set

$$\mathcal{C}(N, I) = \{x \in X^*(N, I) \mid x(S) \leqslant \lambda^*(I_S) \text{ for all } S \subseteq N\}.$$

Lemma 4.1 *The core on $\mathcal{H}$ satisfies* NEM, PO, IR, REASB, PCOV, RHP, CRHP, SCRHP, *and* UTPH.

Proof: NEM follows from the concavity of highway games, PO, IR, REASB, and UTPH are immediate consequences of the definition of the core, and PCOV follows from the well-known scale covariance and translation covariance of the core on any set of games. As the core is reasonable and satisfies RGP on the set of games with nonempty cores (Peleg 1986), Lemma 3.4 shows RHP.

In order to show CRHP and SCRHP, let $(N, I) \in \mathcal{H}$ such that $|N| \geqslant 2$. Let $x \in X(N, I)$ such that $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}$ and $x_{N\setminus\{i\}} \in \mathcal{C}(N \setminus \{i\}, I^{-i,x})$ for each feasible agent $i \in N$. Assume that $x \notin \mathcal{C}(N, I)$ and let $\emptyset \neq S \subsetneqq N$ such that $x(S) > c^I(S)$. Two cases may occur:
(a) If $|N| > 2$, by Lemma 3.3 one of the following subcases must occur: (a1) There exists a feasible $i \in S$ and $|S| \geqslant 2$. In this case $x(S \setminus \{i\}) > c^I(S) - x_i \geqslant c^{I^{-i,x}}(S \setminus \{i\})$. (a2) There exists a feasible $i \in N \setminus S$ and $|S| \leqslant |N| - 2$. In this case $x(S) > c^I(S) \geqslant c^{I^{-i,x}}(S)$. Hence, in both subcases $x_{N\setminus\{i\}} \notin \mathcal{C}(N \setminus \{i\}, I^{-i,x})$ and the desired contradiction has been obtained. Thus, CRHP has been verified.
(b) If $|N| = 2$, $I_j = [a_j, b_j]$ for $j \in N, N = \{k, \ell\}$, then both agents are feasible by Lemma 3.3. If $I_k \setminus I_\ell \neq \emptyset \neq I_\ell \setminus I_k$, then we may assume that $a_k = 0$. We conclude that $x_\ell \geqslant b_\ell - b_k$ (otherwise $x_k \notin X(\{k\}, I^{-\ell,x})$). Moreover, $x_k \geqslant a_\ell - a_k$ (otherwise $(\{\ell\}, I^{-k,x}) \notin \mathcal{H}$ because $a'_\ell > 0$, where $I^{-k,x}_\ell = [a'_\ell, b'_\ell]$) so that $x \in \mathcal{C}(N, I)$. In the remaining case, we may assume that $I_\ell \subseteq I_k$, i.e., $a_k = 0 \leqslant a_\ell \leqslant b_\ell \leqslant b_k$. Then $x_k \leqslant b_k$ (otherwise $b'_\ell < 0$). If $x_k < b_k + a_\ell - b_\ell$, then $a_\ell = 0$ (otherwise $a'_\ell > 0$) and $b_\ell = b_k$ (otherwise $x_\ell < \lambda^*(I^{-k,x}_\ell)$). If, however, $I_k = I_\ell$,

then $x_k < b_k + a_\ell - b_\ell = 0$ would imply $x_\ell > b_\ell$, hence $b''_k < 0$, where $I_k^{-\ell,x} = [a''_k, b''_k]$ which is impossible. Thus, $x \in \mathcal{C}(N, I)$ and SCRHP has been verified. **q.e.d.**

Theorem 4.2 *The core is the unique solution that satisfies* NEM, IR, RHP*, and* SCRHP*.*

Proof: By Lemma 4.1 the core satisfies NEM, IR, RHP, and CRHP. In order to show the opposite implication, let σ be a solution that satisfies the desired properties. Let $(N, I) \in \mathcal{H}$. If $|N| = 1$, then $\sigma(N, I) = \mathcal{C}(N, I)$ by NEM and IR. Assume that $\sigma(N, I) = \mathcal{C}(N, I)$ whenever $|N| < k$ for some $k > 1$. If $|N| = k$ and $x \in \mathcal{C}(N, I)$, then, by RHP of the core, $x_{N\setminus\{i\}} \in \mathcal{C}(N \setminus \{i\}, I^{-i,x}) = \sigma(N \setminus \{i\}, I^{-i,x})$ for each feasible $i \in N$ so that by CRHP of σ, $x \in \sigma(N, I)$. The other inclusion follows by exchanging the roles of σ and $\mathcal{C}$. **q.e.d.**

The following alternative characterization of the core resembles Peleg's (1986) one for TU games.

Theorem 4.3 *The core is the unique solution that satisfies* IR, UTPH, RHP*, and* CRHP*.*

Proof: By Lemma 4.1 the core satisfies the required axioms. In order to show uniqueness, let σ be a solution that satisfies IR, UTPH, RHP, and CRHP. Let $(N, \mathcal{I}) \in \mathcal{H}$. If $|N| \leqslant 2$, then by IR, UTPH, and RHP, $\sigma(N, \mathcal{I}) = \mathcal{C}(N, \mathcal{I})$. We proceed by induction on $|N|$ and assume that $\sigma(N, \mathcal{I}) = \mathcal{C}(N, \mathcal{I})$ whenever $|N| < t$ for some $t > 2$. Now, if $|N| = t$, let $x \in \sigma(N, \mathcal{I})$ and $y \in \mathcal{C}(N, \mathcal{I})$. By RHP of σ and CRHP of $\mathcal{C}$, $x \in \mathcal{C}(N, \mathcal{I})$. By RHP of $\mathcal{C}$ and CRHP of σ, $y \in \sigma(N, \mathcal{I})$. **q.e.d.**

Remark 4.4 As it is well known the core does not satisfy SIVA nor ETP on the general class of TU games, and the same happens for highway problems.

5 Characterization of the nucleolus of a highway game

We now recall the definition of the prenucleolus (Schmeidler 1969) and the prekernel (Maschler, Peleg, and Shapley 1972).

Let (N, c) be a cost game, $x \in \mathbb{R}^N$, $S \subseteq N$, and $i, j \in N$, $i \neq j$. The *excess* of S at x is $e(S, x, c) = x(S) - c(S)$, and the *maximum surplus* of i over j at x is $s_{ij}(x, c) = \max\{e(S, x, c) \mid i \in S \subseteq N \setminus \{j\}\}$. With $X(N, c) = \{x \in \mathbb{R}^N \mid x(N) = c(N)\}$, the *prekernel* of (N, c), denoted by $\mathcal{P}K(N, c)$, is the set

$$\mathcal{P}K(N, c) = \big\{x \in X(N, c) \mid s_{ij}(x, c) = s_{ji}(x, c) \text{ for all } i \in N, j \in N \setminus \{i\}\big\}.$$

The *prenucleolus* of (N, c) is the subset of elements of $X(N, c)$ that lexicographically minimize the non-increasingly ordered vector $(e(S, x, c))_{S \subseteq N}$ of excesses. According to Schmeidler (1969), the prenucleolus of (N, c) is a singleton whose unique element is denoted by $\nu(N, c)$. For a zero-antitonic game (N, c)[4] – a concave game is zero-antitonic – the prekernel coincides with the kernel (Maschler, Peleg, and Shapley 1972), i.e., prenucleolus is individually rational, it is the *nucleolus*.

Remark 5.1 According to Maschler, Peleg, and Shapley (1972), the prekernel of a concave cost game consists of a single point, namely of the nucleolus.

We define the *nucleolus* of a highway problem (N, I) to be the (pre)nucleolus of the associated cost game (N, c^I) and denote $\nu(N, I) = \nu(N, c^I)$.

The following technical lemma is useful.

Lemma 5.2 *For any* $(N, I) \in \mathcal{H}$ *that has exactly two distinct feasible agents* k *and* ℓ*,* $(I_k \cup I_\ell) \subseteq \text{int } I_i$ *for all* $i \in N \setminus \{k, \ell\}$.

Proof: Let $I_j = [a_j, b_j]$ for $j \in N$, $a = \min\{a_k, a_\ell\}$, say $a_k = a$, and $b = \max\{b_k, b_\ell\}$. Let $i_1, i_2 \in N \setminus \{k, \ell\}$ such that $a_{i_1} = \max\{a_i \mid i \in N \setminus \{k, \ell\}\}$ and $b_{i_2} = \min\{b_i \mid i \in N \setminus \{k, \ell\}\}$. Note that w.r.t. the highway subproblem $\left(N \setminus \{k, \ell\}, (I_j)_{j \in N \setminus \{k, \ell\}}\right)$, agent i_1 is a left agent of type 0 and i_2 is a right agent of type 0. Assume that $a_{i_1} \geqslant a$. As i_1 is not feasible, she is not an agent of type 0, i.e., $I_\ell \subseteq \text{ int } I_{i_1}$. But then i_1 is a left agent of type 1, i.e., still feasible, which was excluded. Similarly it is seen that $b_{i_2} > b$: Assuming that, on the contrary, $b_{i_2} \leqslant b$ yields a contradiction because on the on hand side i_2 cannot be of type 0 and on the other hand she cannot be a right agent of type 1. **q.e.d.**

Lemma 5.3 *The nucleolus (on* $\mathcal{H}$*) satisfies* NEM, PO, SIVA, ETP, IR, REASB, PCOV, RHP *and* CRHP.

Proof: The prenucleolus on cost games satisfies the properties corresponding to SIVA (hence NEM), and ETP so that these properties are also satisfied by the nucleolus of highway problems. Moreover, it satisfies translation covariance which implies PCOV, and, by definition, if satisfies PO. The prenucleolus always selects a core element if the core is nonempty. By Lemma 3.4 the reduced problems w.r.t. feasible agents are highway problems, the associated games of which are Davis-Maschler reduced games. According to Sobolev (1975) the prenucleolus satisfies RGP which implies that our nucleolus satisfies RHP. In order to show CRHP, let $(N, I) \in \mathcal{H}$ with $|N| \geqslant 3$ and let $x \in X(N, I)$ such that $x_{N \setminus \{i\}} = \nu(N \setminus \{i\}, I^{-i,x})$ for all feasible agents $i \in N$. Let $k, \ell \in N, k \neq \ell$. By Remark 5.1 it suffices to show that $s_{k\ell}(x, c^I) = s_{\ell k}(x, c^I)$. If there is a feasible agent $i \in N \setminus \{k, \ell\}$, then the game $(N \setminus \{i\}, c)$ associated with the reduced highway problem $(N \setminus \{i\}, I^{-i,x})$ is the Davis-Maschler reduced game of (N, c^I) so that $s_{k\ell}(x, c^I) = s_{k\ell}(x_{N \setminus \{i\}}, c) = s_{\ell k}(x_{N \setminus \{i\}}, c) = s_{\ell k}(x, c^I)$. Otherwise, k and ℓ are the unique feasible agents and we know that

[4] (N, c) is *zero-antitonic* if $c(S \cup \{i\}) - c(S) \leqslant c(\{i\})$ for all $i \in N$ and $S \subseteq N \setminus \{i\}$.

$s_{ij}(x, c^I) = s_{ji}(x, c^I)$ for all $\{i, j\} \subseteq N$ with $i \neq j$ except $\{k, \ell\}$. As the nucleolus selects a member of the core, $x \in \mathcal{C}(N, I)$ by CRHP of the core. Let $\mu = \max\{e(S, x, c^I) \mid \emptyset \neq S \subsetneqq N\}$ and define $\mathcal{D} = \{S \subsetneqq N \mid S \neq \emptyset, e(S, x, c^I) = \mu\}$. It suffices to show that $s_{k\ell}(x, c^I) = \mu$. Assume the contrary. We claim that $\mathcal{D} = \{N \setminus \{k\}\}$. Let $S \in \mathcal{D}$. If $S \cap (N \setminus \{k, \ell\}) \neq \emptyset$, choose $i \in S \cap (N \setminus \{k, \ell\})$. As $x_k \geqslant 0$ and as $I_k \subseteq I_i$ by Lemma 5.2, $e(S \cup \{k\}, x, c^I) \geqslant \mu$ so that $\ell \in S$ by our assumption. If there exists $j \in N \setminus (S \cup \{k, \ell\})$, then $\mu = s_{\ell j}(x, c^I) = s_{j\ell}(x, c^I)$ so that there exists $S' \in \mathcal{D}$ with $\ell \notin S' \ni j$ which cannot be true by the former argument. Hence, $S = N \setminus \{k\}$ in this case. If $S \cap N \setminus \{k, \ell\} = \emptyset$, then $\ell \in S$ because $e(\{k\}, x, c^I) < \mu$. Therefore, for $i \in N \setminus \{k, \ell\}$, $s_{\ell i}(x, c^I) = \mu = s_{i\ell}(x, c^I)$, and hence there exists $S' \in \mathcal{D}$ with $\ell \notin S' \ni i$ which is impossible by the former argument. Now the proof can be finished. By our claim, $s_{ik}(x, c^I) = \mu = s_{ki}(x, c^I)$ for $i \in N \setminus \{k, \ell\}$ so that we have derived a contradiction to our claim that $N \setminus \{k\}$ is the unique coalition in $\mathcal{D}$. **q.e.d.**

Theorem 5.4 *The nucleolus on $\mathcal{H}$ is the unique solution that satisfies* SIVA, ETP, PCOV, *and* RHP *provided* $|U \geqslant 2$.

Proof: By Lemma 5.3 the nucleolus satisfies these properties. In order to show the opposite implication, let σ be a solution that satisfies the desired axioms. Let $(N, I) \in \mathcal{H}$ and let x be the unique element of $\sigma(N, I)$. We have to show that $x = \nu(N, c^I)$. If $|N| = 1$, say $N = \{i\}$, then by PCOV we may assume that $I_i = [0, 0]$. Choose $j \in U \setminus \{i\}$, define $I_j = I_i$, and let $y = \sigma(\{i, j\}, I)$. By ETP, $y_i = y_j$. By RHP, $y_j = 0$ because otherwise $I_i^{-j,y} = \emptyset$. Hence, $x = y_i = 0$. If $|N| = 2$, then by PCOV we may assume that $I_i = I_j$ for $i, j \in N$, and, hence, $x_i = x_j$ by ETP. By RHP, $x \in X(N, I)$, hence $x = \nu(N, I)$. Now we proceed by induction on $|N|$ and assume that the unique element of $\sigma(N, I)$ coincides with $\nu(N, I)$ whenever $|N| < r$ for some $r > 2$. If $|N| = r$, then by RHP, $x_{N \setminus \{i\}} = \nu(N \setminus \{i\}, c^{I^{i,x}})$ for each feasible agent so that, by CRHP of ν, $x = \nu(N, I)$. **q.e.d.**

Remark 5.5 (1) A careful inspection of its proof shows that the axiom SIVA in Theorem 5.4 may be replaced by NEM and PO.

(2) The nucleolus does neither satisfy UTPH nor SCRHP.

By means of the following examples we show that a reduced game w.r.t. the nucleolus of a highway game may not result in a highway game if (1) a non-oriented agent of type 1 is removed or if (2) an oriented agent of type 2 is removed (provided $|U| \geqslant 4$). This is the reason for the present definition of RHP, where only agents of type 0 or oriented agents of type 1 may be removed.

Example 5.6 (1) Let (N, I) the highway problem with $N = \{1, \ldots, 4\} \subseteq U$, $I_1 = [0, 2]$, $I_2 = [1, 3]$, $I_3 = [2, 4]$, and $I_4 = [0, 4]$. For $x = (1, 1, 1, 1)$, $\min\{c^I(S) - x(S) \mid \emptyset \neq S \subsetneqq N\} = 1$ is attained by all 3-person coalitions the indicator functions of which span $\mathbb{R}^4$ so that the characterization by a balancedness condition due to Kohlberg (1971) shows that $\nu(N, c^I) = x$. Let $N' = \{1,2,3\}$ and $c = c^I_{N',x}$. Then, for any $\emptyset \neq S \subseteq M$,

$$c(S) = \begin{cases} 2, & \text{if } |S| = 1, \\ 3, & \text{if } |S| \geqslant 2. \end{cases}$$

We now show that (N', c) is not strategically equivalent to a highway game. Assume the contrary. As each positive multiple of a highway game is a highway game, there exist $(N', I') \in \mathcal{H}$ and $y \in \mathbb{R}^{N'}$ such that $c + y = c^{I'}$. Let $I'_i = [a'_i, b'_i]$ for $i \in N'$, choose $j, k, \ell \in N'$ such that $a'_j = \min_{i \in N'} a'_i$, and choose $k \in N' \setminus \{j\}$ such that $b_k = \max_{i \in N' \setminus \{j\}} b_i$, and $N' = \{j, k, \ell\}$. As $c(\{j\}) = 2$, $b'_j = a'_j + 2 + y_j$. As $a'_i \geqslant a'_j$ and $c(\{j, i\}) = 3$ for $i \in N' \setminus \{k\}$, $b'_i = a'_j + 3 + y_j + y_i$. Moreover, as $c(\{i\}) = 2$, $a'_i = a'_j + 1 + y_j$. Finally, as $c(\{k, \ell\}) = 3$, $a'_k = a'_\ell$, and $b'_k \geqslant b'_\ell$,

$$b'_k = a'_k + 3 + y_k + y_\ell = a'_j + 4 + y(N') = a'_j + 3 + y_j + y_k$$

so that $y_\ell = -1$, i.e., $c^{I'}(N) = c^{I'}(N' \setminus \{\ell\}) - 1$, and the desired contradiction has been obtained. Note that agent 4 is of type 1, but not oriented.

(2) Now we consider $(N, I'') \in \mathcal{H}$ defined by $N = \{1, \ldots, 4\} \subseteq U$, $I''_1 = [0,2]$, $I''_2 = [1,3]$, $I''_3 = [2,4]$, and $I''_4 = [0,5]$. Then $\nu(N, I'') = (1,1,1,2)$ and the reduced game of $(N, c^{I''})$ w.r.t. N' and $\nu(N, I'')$ is, again, (N', c). Moreover, agent 4 is a left agent of type 2.

6 The Shapley value of highway problems

In order to recall the definition of the Shapley value (Shapley 1953) of a TU game (N, c), recall that for any $\emptyset \neq S \subseteq N$, the *dual unanimity game* (N, u^*_S) is defined by

$$u^*_S(T) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for all $T \subseteq S$. Note that $\{(N, u^*_S) \mid \emptyset \neq S \subseteq N\}$ is a vector space basis of the Euclidean space of all TU games with player set N, i.e., of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$. Hence, there are unique real coefficients λ_S, $\emptyset \neq S \subseteq N$, such that $c = \sum_{\emptyset \neq S \subseteq N} \lambda_S u^*_S$. Now, the Shapley value of (N, c) is the vector

$$\phi(N, c) = \sum_{\emptyset \neq S \subseteq N} \frac{\lambda_S}{|S|} \mathbb{1}^S \tag{6.1}$$

so that ϕ is additive.

The Shapley value of a highway problem $(N, I) \in \mathcal{H}$, denoted by $\phi(N, I)$, is the Shapley value of the associated game (N, c^I). By slightly abusing notation, for a single-valued solution σ on $\mathcal{H}$ the unique element of the singleton $\sigma(N, I)$ is also denoted by $\sigma(N, I)$ and, conversely, we use $\phi(N, I)$ for $\{\phi(N, v)\}$ so that ϕ becomes a solution. In order to obtain an explicit formula for $\phi(N,I)$, let $T^I : N \to M^I$ (see (2.1) for the definition of M^I) be the function that assigns to each agent the set of section she uses, i.e., $T^I(i) = \{j \in M \mid j \subseteq I_i\}$ for all $i \in N$, and denote $(T^I)^{-1}(j) = \{i \in N \mid j \in T^I(i)\}$ for all $j \in M$. Then we obtain $c^I = \sum_{j \in M^I} \lambda^*(j) u^*_{(T^I)^{-i}(j)}$ so that, by (6.1),

$$\phi_i(N,I) = \sum_{j \in T^I(i)} \frac{\lambda^*(j)}{|(T^I)^{-1}(j)|} \text{ for all } i \in N. \tag{6.2}$$

Let $(N,I) \in \mathcal{H}$, $I_i = [a_i, b_i]$ for all $i \in N$, and $b = \max_{i \in N} b_i$. In order to resemble Young's (1985a) characterization of ϕ with the help of strong monotonicity, some preparation is useful. For $\alpha, \beta \in [0,b], \alpha \leqslant \beta$, the $[\alpha,\beta]$-*truncated* highway problem $(N, I^{[\alpha,\beta]})$ is defined by $I_i^{[\alpha,\beta]} = [\min\{(a_i - \alpha)_+, \beta - \alpha\}, \min\{(b_i - \alpha)_+, \beta - \alpha\}]$ for all $i \in N$, where $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Hence, $(N, I^{[\alpha,\beta]})$ is the highway problem that arises from (N,I) if the highway is restricted to the interval $[\alpha,\beta]$. We say that a single-valued solution σ (on $\mathcal{H}' \subseteq \mathcal{H}$) satisfies

(12) *individual independence of outside changes* (IIOC) if for all $(N,I), (N,I') \in \mathcal{H}'$ and all $i \in N$: If $I^{I_i} = I'^{I'_i}$, then $\sigma_i(N,I) = \sigma_i(N,I')$.

IIOC means that the charge of an agent i may only depend on the highway problem truncated to her used interval. Note that *strong monotonicity* of a single-valued solution σ on the set of games requires that, for $(N,c), (N,c') \in \Gamma, i \in N$, if $c(S \cup \{i\}) - c(S) \geqslant c'(S \cup \{i\}) - c'(S)$ for all $S \subseteq N$, then $\sigma_i(N,c) \geqslant \sigma_i(N,c')$. By (6.2), strong monotonicity of σ implies IIOC of the corresponding solution on $\mathcal{H}$.

Theorem 6.1 *The Shapley value on $\mathcal{H}$ is the only solution that satisfies* SIVA, PO, ETP, *and* IIOC.

Proof: The Shapley value satisfies the four axioms by (6.2). In order to prove the other implication, let σ be a solution that satisfies SIVA, PO, ETP, and IIOC. Let $(N,I) \in \mathcal{H}$ and $b = \max I_N$. By induction on $\left|M^I\right|$ we prove that $\sigma(N,I) = \phi(N,I)$. If $\left|M^I\right| = 1$, then $I_i = I_j$ for every $i, j \in N$, and the result follows from SIVA, PO, and ETP. Assume that $\sigma(N,I) = \phi(N,I)$ whenever $\left|M^I\right| < k$ for some $k \geqslant 2$. If $\left|M^I\right| = k$, let α, β be determined by $\alpha \neq b, \beta \neq 0$, and $[0,\beta], [\alpha, b] \in M^I$. Define

$$P = \{i \in N \mid [0,\beta[\ \cap I_i = \emptyset\} \text{ and } Q = \{i \in N \mid\ ]\alpha, b] \cap I_i = \emptyset\}.$$

Note that, for any $i \in N \setminus (P \cup Q)$, $I_i = I_N$. Hence, by SIVA, PO, and ETP it suffices to show that $\sigma_{P \cup Q}(N,I) = \phi_{P \cup Q}(N,I)$. With $I' = I^{[\beta,b]}$ we have $\left|M^{I'}\right| < k$ and $I'_i = I_i$ for all $i \in P$, and with $I' = I^{[0,\alpha]}$ we have $\left|M^{I'}\right| < k$ and $I'_j = I_j$ for all $j \in Q$ so that the inductive hypothesis finishes the proof. **q.e.d.**

6.1 The Shapley value of airport problems

A highway problem $(N, I) \in \mathcal{H}$, $I_i = [a_i, b_i]$ for $i \in N$, is an *airport problem* if $a_i = 0$ for all $i \in N$. Let $\mathcal{A}$ denote the set of airport problems. Let $(N, I) \in \mathcal{A}$ and let $N = \{i_1, \ldots, i_n\}$ so that $b_{i_1} \leqslant \cdots \leqslant b_{i_n}$. By (6.2), the Shapley value can be recursively computed as

$$\phi_{i_1}(N, I) = \frac{b_{i_1}}{n} \text{ and } \phi_{i_{j+1}}(N, I) = \phi_{i_j} + \frac{b_{i_{j+1}} - b_{i_j}}{n - j} \quad \text{for all } j = 1, \ldots, n-1. \tag{6.3}$$

Let $|N| \geqslant 2$, $i \in N$ and $x \in \mathbb{R}^N$. The *contracted problem* w.r.t. i and x, denoted $(N \setminus \{i\}, I^{-i,x,\text{ctr}})$, is defined as follows. For $j \in N \setminus \{i\}$, $I_j^{-i,x,\text{ctr}} = [0, b_j - \min\{x_j, x_i\}]$.

The contracted problem can be interpreted as a kind of reduced problem in the following way. Assume that a payoff vector, say x is at stake, and everybody accepts the payoff assigned to a certain agent, say i. That is, agent i pays x_i and the remaining agents renegotiate in a new airport problem, the contracted problem, in which the cost of the runway that is used by every agent $j \in N \setminus \{i\}$ has to be updated taking into account the payoff made by i. In the contracted problem, we are assuming that the cost of the runway used by agent $j \neq i$ is decreased by x_i, unless this discount were higher than x_j, in which case the discount would be x_j.

Note that $(N, I^{-i,x,\text{ctr}}) \in \mathcal{A}$ if and only if $b_j - \min\{x_j, x_i\} \geqslant 0$ for all $j \in N \setminus \{i\}$.

We say that a solution σ on $\mathcal{A}$ satisfies the

(13') *contraction property* (CONTR) if it is consistent w.r.t. contracted problems, i.e., if, for all $(N, I) \in \mathcal{A}$ with $|N| > 1$, all $x \in \sigma(N, I)$, and all $i \in N$: $(N \setminus \{i\}, I^{-i,x,\text{ctr}}) \in \mathcal{A}$ and $x_{N\setminus\{i\}} \in \sigma(N, I^{-i,x,\text{ctr}})$.

Theorem 6.2 *On $\mathcal{A}$ the Shapley value is the unique solution that satisfies* NEM, PO*, and* CONTR*.*

Proof: By definition the Shapley value is a singleton, hence satisfies NEM. By (6.3) it satisfies PO and CONTR as well. In order to show the uniqueness part, let σ be a solution on $\mathcal{A}$ that satisfies NEM, PO, and CONTR. Let $(N, I) \in \mathcal{A}$, $I_j = [a_j, b_j]$ for $j \in N$, and $x \in \sigma(N, I)$. By NEM it suffices to show that $x = \phi(N, I)$. We proceed by induction on $|N|$. If $|N| = 1$, then $x = \phi(N, I)$ by NEM and PO. Now assume that $x = \phi(N, I)$ whenever $|N| < k$ for some $k \geqslant 2$. If $|N| = k$, then choose $N = \{i_1, \ldots, i_n\}$ so that $b_{i_1} \leqslant \cdots \leqslant b_{i_n}$.
Claim: $x_j \leqslant x_{i_n}$ for all $j \in N$. Indeed, assume on the contrary that there exists $j \in N$ such that $x_j > x_{i_n}$. Then $c^{I^{-j,x,\text{ctr}}}(N \setminus \{i\}) \geqslant b_{i_n} - x_{i_n} = x(N) - x_{i_n} > x(N \setminus \{j\})$ so that $x_{N\setminus\{j\}}$ is not feasible for the reduced airport problem $(N \setminus \{j\}, I^{-j,x,\text{ctr}})$.
Now let $I' = I^{-i_n,x,\text{ctr}}$, $I'_\ell = [a'_\ell, b'_\ell]$ for $\ell \in N \setminus \{i\}$. By our claim, $b'_j = b_j - x_j$. By the inductive hypothesis, $x_{N\setminus\{i_n\}} = \phi(N \setminus \{i_n\}, I')$ so that we conclude from (6.3) that $x_i \leqslant x_j$ if and only if $b'_i \leqslant b'_j$ for all $i, j \in N \setminus \{i_n\}$. Hence, $b'_{i_1} \leqslant \cdots \leqslant b'_{i_{n-1}}$. By (6.3), $x_{i_1} = \frac{b'_{i_1}}{n-1} = \frac{b_{i_1} - x_{i_1}}{n-1}$ so that

$x_{i_1} = \frac{b_{i_1}}{n} = \phi_{i_1}(N, I)$. We proceed recursively and assume that $x_{i_j} = \phi_{i_j}(N, I)$ for $j = 1, \ldots, t$. If $t < n - 1$, then, by (6.3),

$$x_{i_{t+1}} = x_{i_t} + \frac{b'_{i_{t+1}} - b'_{i_t}}{n - i_t} = x_{i_t} + \frac{b_{i_{t+1}} - x_{i_{t+1}} - (b_{i_t} - x_{i_t})}{n - i_t},$$

hence $x_{i_{t+1}} = x_{i_t} + \frac{b_{i_{t+1}} - b_{i_t}}{n+1-i_t} = \phi_{i_{t+1}}(N, I)$. Finally, by PO, $x_{i_n} = \phi_{i_n}(N, I)$. **q.e.d**

We now generalize CONTR to highway problems.

6.2 The contraction property on highway games

Let $(N, I) \in \mathcal{H}$, $I_j = [a_j, b_j]$ for all $j \in N$, and assume $|N| \geqslant 2$.

Now assume that $|N| \geqslant 2$. For any left agent $i \in N$ of type 0 (i.e., $a_j > a_i$ implies $a_j \geqslant b_i$) and any $y \in \mathbb{R}^N$ we define the *contracted problem* w.r.t. i and y, $(N \setminus \{i\}, I^{-i,y,\text{ctr}})$, for any $j \in N \setminus \{i\}$, by $I_j^{-i,y,\text{ctr}} = [a'_j, b'_j]$, where

$$a'_j = \begin{cases} a_j & \text{, if } a_j \leqslant a_i, \\ a_j - y_i & \text{, if } a_j > a_i, \end{cases} \quad \text{and } b'_j = \begin{cases} b_j & \text{, if } b_j < a_i, \\ b_j - \min\{y_j, y_i\} & \text{, if } b_j \geqslant a_i \geqslant a_j, \\ b_j - y_i & \text{, if } a_j > a_i. \end{cases}$$

Note that a contracted problem may not be a highway problem.

We say that a solution σ on $\mathcal{H}$ satisfies the

(13) *contraction property* (CONTR) if, for all $(N, I) \in \mathcal{H}$ with $|N| \geqslant 2$, all left agents $i \in N$ of type 0, with $I_i = [a_i, b_i]$, and $x \in \sigma(N, I)$: $(N, I^{-i,y,\text{ctr}}) \in \mathcal{H}$ and $x_{N\setminus\{i\}} \in \sigma(N, I^{-i,y,\text{ctr}})$ for all $y \in \sigma(N, I^{I_N \setminus [0, a_i[})$.

Hence, CONTR requires that σ is consistent w.r.t. any contraction of a highway problem according to the solution applied to the truncated highway problem the highway of which starts at the interval used by a left agent of type 0.

In an airport problem each agent i is a left agent of type 0 and $a_i = 0$ so that the current CONTR coincides with the former CONTR on $\mathcal{A}$ – the only further requirement that must be satisfied on $\mathcal{H}$ is that consistency must be satisfied w.r.t. contracted highway problems defined with the help of any element of the solution applied to the truncated highway problem. Of course we may also use this slightly stronger contraction property in Theorem 6.2.

It should be noted that this kind of "reduction" that depends on the solution applied to certain derived problems (here certain truncated highways) is not new for axiomatizations of the Shapley value – Hart and Mas-Colell (1989) also define their consistency property only for solutions that satisfy SIVA so that their "reduced game" is defined with the help of the solution applied to subgames. Note, however, that the TU game corresponding to a contracted highway problem w.r.t. the Shapley value does typically not coincide with the corresponding Hart–Mas-Colell "reduced game" of the initial highway game (which may be illustrated by any 4-person airport problem with equal positive demands).

Of course, we may also reverse the start and the endpoints of the landing strip of the airport and call a highway problem (N, I) and airport problem if $b_i = b_j$ for all $i, j \in N$. This would lead to a contraction property requiring that the solution is consistent according to a certain kind of contraction w.r.t. any right agent of type 0.

Theorem 6.3 *On $\mathcal{H}$ the Shapley value is the unique solution that satisfies* NEM, PO*, and* CONTR.

Proof: The Shapley value satisfies NEM and PO. In order to show CONTR let $(N, I) \in \mathcal{H}, I_j = [a_j, b_j]$ for $j \in N$, $|N| \geqslant 2$, and i a left agent of type 0. Let $I' = I^{[0,a_i]}, I'' = I^{I_i}$, and $I''' = I^{[b_i, \max I_N]}$, i.e., I' represents the first part of the highway from 0 to a_i, I'' is the middle part from a_i to b_i, and I''' represents the rest, namely the part from b_i to $\max I_N$. Moreover, let y', y'', y'' be the Shapley values of $(N, I'), (N, I''), (N, I''')$ respectively. As $c^I = c^{I'} + c^{I''} + c^{I'''}$, by the well-known additivity of ϕ (see (6.2)), $x := \phi(N, I) = y' + y'' + y'''$. Moreover, agent i is a null-player of $(N, c^{I'})$ and $(N, c^{I'''})$ (an agent $k \in N$ is a *null-player* of a TU game (N, c) if $c(S \cup \{k\}) = c(S)$ for all $S \subseteq N$) so that $x'_i = x'''_i = 0$ by definition of ϕ. Now, $(N, I'') \in \mathcal{A}$ so that $y''_{N \setminus \{i\}} = \phi(N \setminus \{i\}, I''^{-i,y'',\text{ctr}})$ by Theorem 6.2. Also, it is well-known that the Shapley value satisfies the strong null-player property, i.e., $\phi_{N \setminus \{i\}}(N, c) = \phi(N \setminus \{i\}, c)$ (where $(N \setminus \{i\}, c)$ denotes the subgame of (N, c) with player set $N \setminus \{i\}$). Let $(N \setminus \{i\}, I'^{-i})$ and $(N \setminus \{i\}, I'''^{-i})$ denote the corresponding highway subproblems of (N, I') and (N, I'''). As $c^{I^{-i,x,\text{ctr}}} = c^{I'^{-i}} + c^{I''^{-i,y',\text{ctr}}} + c^{I'''^{-i}}$, additivity of the Shapley value yields $\phi(N \setminus \{i\}, I^{-i,x,\text{ctr}}) = y'_{N \setminus \{i\}} + y''_{N \setminus \{i\}} + y'''_{N \setminus \{i\}} = x_{N \setminus \{i\}}$.

To prove uniqueness, let σ be a solution that satisfies NEM, PO, and CONTR. Let $(N, I) \in \mathcal{H}$, $I_j = [a_j, b_j]$ for all $j \in N$, $x \in \sigma(N, I)$. It remains to show that $x = \phi(N, I)$. We proceed by induction on $|N|$. If $|N| = 1$, then $x = \phi(N, I)$ by NEM and PO. Now assume that $x = \phi(N, I)$ whenever $|N| < k$ for some $k \geqslant 2$. If $|N| = k$, choose $i \in N$ such that $a_i \geqslant a_j$ for all $j \in N$. Then i is not only a left agent of type 0, but the truncated highway problem $(N, I^{[a_i, \max I_N]})$ is an airport problem so that, by Theorem 6.2, $\sigma\left(N, I^{[a_i, \max I_N]}\right) = \phi\left(N, I^{[a_i, \max I_N]}\right)$. CONTR of ϕ and the inductive hypothesis complete the proof. **q.e.d.**

Note that the empty solution satisfies PO and CONTR, but violates NEM, and that the nucleolus satisfies NEM and PO, but violates CONTR provided that $|U| \geqslant 3$. The solution σ that differs from the Shapley value only in as much as $\sigma\big(\{i\}, I\big) = \big\{\phi(\{i\}, I), \phi(\{i\}, I) + 1\big\}$ for one-person highway problems $(\{i\}, I)$ satisfies NEM and CONTR, but violates PO. Hence, each of the axioms employed in Theorem 6.2 as well as in Theorem 6.3 is logically independent of the remaining axioms.

7 Generalized highway problems

The definition of a *generalized highway problem* (N, I) differs from Definition 2.1 only inasmuch as (2) is weakened to "I is a mapping that assigns to each $i \in N$ a finite union of compact nonempty intervals in $\mathbb{R}_+$." Hence, in a generalized highway problem the customers may use disconnected sections of the highway. The associated TU cost game (N, c^I) is still defined by (2.2). For a generalized highway problem it is convenient to use the representation of Kuipers, Mosquera, and Zarzuelo (2013): A tuple (N, M, C, T) is a generalized highway problem if it satisfies (2.3), (2.5), (2.6), and (2.7) of Remark 2.2 so that the cost function $c^{M,C,T}$ is defined by (2.8). In particular we do not need a strict ordering $\prec$ of M. Hence, we denote by $\mathcal{GH}$ the set of generalized highway games (N, M, C, T).

Let $(N, M, C, T) \in \mathcal{GH}$. We now show that $(N, c^{M,C,T})$ is a positive game. A game (N, c) is called *positive* if $c = \sum_{\emptyset \neq S \subseteq N} \lambda_S u_S^*$, where the unique coefficients $\lambda_S, \emptyset \neq S \subseteq N$, are nonnegative. For each section $j \in M$ let $T^{-1}(j) = \{i \in N \mid j \in T(i)\}$, i.e., the set of users of j. Therefore, we have

$$c^{M,C,T} = \sum_{j \in M} C(j) u_{T^{-1}(j)}^*. \tag{7.4}$$

As $C(j) \geqslant 0$ for all $j \in M$, $(N, c^{M,C,T})$ is a positive game. The following theorem shows that the converse is also true.

Theorem 7.1 *A TU cost game is a positive game if and only if it is a highway game.*

Proof: By (7.4) we only have to show the only-if part. Let N be a coalition, $\lambda_S \geqslant 0$ for all $\emptyset \neq S \subseteq N$, and let $c = \sum_{\emptyset \neq S \subseteq N} \lambda_S u_S^*$. We define the *direct* generalized highway problem (N, M, C, T) (corresponding to (N, c)) by

$$M = 2^N \setminus \{\emptyset\}, T(i) = \{S \in M \mid i \in S\} \text{ for all } i \in N, \text{ and } C(S) = \lambda_S \text{ for all } S \in M.$$

Then $T^{-1}(S) = S$ and, by (7.4), $C^{M,C,T} = c$. **q.e.d.**

It should be remarked that the game (N', c) defined in Example 5.6 is a generalized highway game. Indeed, with $I_1 = [0, 2], I_2 = [1, 3]$, and $I_3 = [0, 1] \cup [2, 3]$, $c^I = c$. Note also, that this example is not pathological. Thus, the set of generalized highway game strictly contains the set of highway games.

8 Discussion

The first part of Theorem 5.5 of Potters and Sudhölter (1999) provides a characterization of the nucleolus on airport problems that employs properties similar to those that occur in our Theorem 5.4. However, our properties are defined without mentioning the games associated with the corresponding cost sharing problems (here highway problems) whereas in the mentioned paper, e.g., the covariance property refers to the associated games rather than directly to airport problems.

The first part of the aforementioned Theorem 5.5 characterizes the modiclus (Sudhölter 1996) on airport problems. However, for an airport game (N, c) the modiclus coincides with the prenucleolus of the dual game (N, c^*) (defined by $c^*(S) = c(S) - c(N \setminus S)$ for all $S \subseteq N$) and it is a member of $\mathcal{C}(N, c)$. By means of the following 5-person example we show that the prenucleolus of the dual of a highway game (N, c) may not be a member of the core of this game. In fact, we show that the *least core*[5] of the dual of the highway game does not intersect the core. (Another 5-person example of a general convex game the modiclus of which does not belong to the least core of the dual game was already found by Sudhölter (1997, Example 3.2(iii)).)

Example 8.1 Let $(N, I) \in \mathcal{H}$ be defined by $N = \{1, \ldots, 5\}$, $I_1 = [0, 6], I_2 = [0, 4], I_3 = [3, 9], I_4 = [5, 10]$, and $I_5 = [6, 10]$, and let $c = c^I$. With $x = (4, 3, 1, 1, 1)$, $\mu = \max_{S \subseteq N}(c(S) - x(S)) = 5$. Hence, for any $y \in \mathcal{LC}(N, c^*)$, $c(S) - y(S) \leqslant 5$ for all $S \subseteq N$.
Claim 1: $y_1 \geqslant 3$. Assume, on the contrary, that $y_1 = 3 - \varepsilon$ for some $\varepsilon > 0$. As $c(\{1, 4\}) = c(\{1, 5\}) = 10$, $y_4, y_5 \geqslant 2 + \varepsilon$. As $c(\{1, 3\}) = 9, x_3 \geqslant 1 + \varepsilon$. By Pareto optimality of y, $y_2 \leqslant 2 - 2\varepsilon$. We conclude that $x_3 \geqslant 2 + 2\varepsilon$, and a contradiction to Pareto optimality has been obtained.
Claim 2: $y_2 \geqslant 3$. Assume, on the contrary, $y_2 = 3 - \varepsilon$ for some $\varepsilon > 0$. Then $y_3, y_4 \geqslant 1 + \varepsilon, y_4 \geqslant \varepsilon$, and, hence, $y_1 \leqslant 5 - 2\varepsilon$. Thus, $y_5 \geqslant 2\varepsilon$, and the desired contradiction has been obtained.
Claim 3: $y_1 + y_2 > 6 = c(\{1, 2\})$. Assume the contrary. By Claims 1 and 2, $y_1 = y_2 = 3$. Then $y_3 \geqslant 1, y_4, y_5 \geqslant 2$, which is in contradiction to Pareto optimality. Thus, $\mathcal{LC}(N, c^*) \cap \mathcal{C}(I, c) = \emptyset$.

We may define the Shapley value for generalized highway problems and characterize it similarly to Theorem 6.1 by conveniently adapting the proof. The other characterizations on highway games proposed in this paper do not possess straightforward generalizations on generalized highway games.

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[5]The *least core* of a TU game (N, c), $\mathcal{LC}(N, c)$ is the intersection of all ε-cores $\{x \in X(N, c) \mid e(S, x, c) \leqslant \varepsilon$ for all $S \subseteq N\}$ that are nonempty.

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