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Discussion Papers on Business and Economics
No. 8/2014

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The positive core for games with precedence constraints*

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April 30, 2014

Abstract

We generalize the characterizations of the positive core and the positive prekernel to TU games with precedence constraints and show that the positive core is characterized by non-emptiness (NE), boundedness (BOUND), covariance under strategic equivalence, closedness (CLOS), the reduced game property (RGP), the reconfirmation property (RCP) for suitably generalized Davis-Maschler reduced games, and the possibility of nondiscrimination. The bounded positive core, i.e., the union of all bounded faces of the positive core, is characterized similarly. Just RCP has to be replaced by a suitable weaker axiom, a weak version of CRGP (the converse RGP) has to be added, and CLOS can be deleted. For classical games the prenucleolus is the unique further solution that satisfies the axioms, but for games with precedence constraints it violates NE as well as the prekernel. The positive prekernel, however, is axiomatized by NE, anonymity, reasonableness, the weak RGP, CRGP, and weak unanimity for two-person games (WUTPG), and the bounded positive prekernel is axiomatized similarly by requiring WUTPG only for classical two-person games and adding BOUND.

Keywords: TU games, restricted cooperation, game with precedence constraints, positive core, bounded core, positive prekernel, prenucleolus

JEL Classification: C71

1 Introduction

Since the seminal papers of Myerson (1977) and Faigle (1989) introducing the idea of restricted cooperation, that is, considering the possibility to have unfeasible or forbidden coalitions, many investigations have been done in order to study or adapt the main solution concepts of classical TU games, for various structures of the set of feasible coalitions.

Considering a finite set of players N and a collection of feasible coalitions $\mathcal{F} \subseteq 2^N$, many structures borrowed from combinatorial optimization and partially ordered sets have been proposed for $\mathcal{F}$: feasible coalitions induced by a communication graph (Myerson 1977), distributive lattices (Faigle and Kern 1992), convex geometries (Bilbao, Lebrón, and Jiménez 1998), union-stable systems (Algaba, Bilbao, Borm, and López 2000), antimatroids (Algaba, Bilbao, van den Brink, and Jiménez-Losada 2004), regular set systems

*The first author thanks the Agence Nationale de la Recherche for financial support under contract ANR-13-BSHS1-0010. The second author acknowledges support from the Danish Council for Independent Research | Social Sciences under the FINQ project and from the Spanish Ministerio de Ciencia e Innovación under project ECO2012-33618, co-funded by the ERDF.

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(Lange and Grabisch 2009), to cite a few (see a survey in Grabisch 2012). The case of distributive lattices appears to be of special interest since it corresponds by Birkhoff's theorem to a situation which is often encountered in practice: the presence of a hierarchy, or any kind of partial order $\preceq$, on the set of players N. In this case, feasible coalitions are those satisfying the following rule: if player i is present, then any of its subordinates $j \prec i$ must be present (this is called by Faigle and Kern "precedence constraints"). Moreover, the set of feasible coalitions is closed under union and intersection. For these reasons, we precisely focus in this paper on this type of structure.

Considering that $\mathcal{F}$ is a distributive lattice induced by some partial order on N, the core of games on such structures has already deserved much attention, and many results valid for classical games have been extended to this case, in particular, the Shapley-Ichiishi theorem (Shapley 1971, Ichiishi 1981) characterizing convex TU games (Derks and Gilles 1995, Grabisch and Sudhölter 2014). The essential difference with classical games is that a nonempty core is always (unless $\mathcal{F} = 2^N$) unbounded, however it is pointed. Extremal rays of the core have been found by Tomizawa (1983), and bounded facets have been studied by Grabisch (2011). The set of all bounded facets of the core, called the *bounded core*, has been studied and axiomatized by Grabisch and Sudhölter (2012). From the interpretation point of view, it is the set of core elements for which any player takes the maximum from its subordinates.

Other solution concepts like the nucleolus, the kernel, etc. have been much less studied in the context of restricted cooperation. It is the main purpose of this paper to fill this gap for games with precedence constraints. Our first aim is to study the positive core (Orshan and Sudhölter 2010), which is closely related to the prenucleolus. We find that the positive core can be axiomatized in a way which is very close to the classical case, up to a suitable generalization of the axioms, namely by non-emptiness (NE), reasonableness (REAS), covariance (COV), the reduced game property (RGP), the reconfirmation property (RCP), nondiscrimination (ND), and closedness (CLOS), the latter permitting to eliminate the relative interior of the positive core as a candidate for the solution. The positive core being unbounded unless $\mathcal{F} = 2^N$, we propose likewise the *bounded positive core*, which has the same intuitive interpretation as for the bounded core. We find that it can be axiomatized by NE, COV, RGP, RCP restricted to classical games, ND, boundedness (BOUND), and a variant of the converse reduced game property, called RCRGP. The bounded positive core contains a particular point, which can be considered as the prenucleolus of the game, since it coincides with the usual prenucleolus when $\mathcal{F} = 2^N$. It lexicographically minimizes the excesses of all coalitions in $\mathcal{F}$ the complements of which are also in $\mathcal{F}$, and then lexicographically maximizes the remaining excesses so that they are non-positive (thereby keeping the idea that players should take the maximum of their subordinates, while guaranteeing minimal losses if any). Lastly, we study the prekernel. A simple consideration shows that the classical definition of the prekernel leads to an empty set as soon as $\mathcal{F} \neq 2^N$. We propose therefore to study the positive prekernel instead (Sudhölter and Peleg 2000), which contains the positive core. We show that it is characterized by NE, anonymity (AN), REAS, a weak RGP property, CRGP, and weak unanimity for 2-persons games (WUTPG).

The paper is organized as follows. Section 2 gives the basic material on partially ordered sets, the core, and the bounded core of games with precedence constraints. Section 3 introduces the positive core, which is axiomatized in Section 4. Section 5 is devoted to the bounded positive core and introduces the prenucleolus, for which a Kohlberg-like criterion is given. The positive prekernel and its axiomatization are addressed in Section 6, and Section 7 studies the logical independence of all the axioms introduced in the various axiomatizations.

2 Notation, Definitions, and Preliminaries

A *partially ordered set* (poset) is a pair $(P, \preceq)$ such that P is a nonempty finite set and $\preceq$ is a partial order on P, i.e., a reflexive, antisymmetric, and transitive binary relation on P. As usual, we write $x \preceq y$ for $(x, y) \in \preceq$ and use $x \prec y$ if $x \preceq y$ and $x \neq y$. If $x \prec y$ and there is no $z \in P$ such that $x \prec z \prec y$ then y *covers* x, denoted by $x \prec\!\cdot\, y$. A *chain* in $(P, \preceq)$ is a sequence $(x_0, \ldots, x_q)$ such that $x_0 \prec \cdots \prec x_q$ where q is called the *length* of the chain. The *height* of a poset is the length of a longest chain. The *height* of $x \in P$, denoted by $h(x)$, is the maximal length of a chain from a minimal element to x.

Let U, $|U| \geqslant 3$, be a set, the universe of players. A *coalition* is a finite nonempty subset of U. Let N be a coalition and $(N, \preceq)$ be a poset. Then $S \subseteq N$ is a *downset* of $(N, \preceq)$ if $i \in S$ and $j \preceq i$ implies $j \in S$. Denote by $\mathcal{O}(N, \preceq)$ the set of downsets of $(N, \preceq)$. Note that $(\mathcal{O}(N, \preceq), \subseteq)$ is a *distributive lattice* of height[1] $|N|$. By Birkhoff's representation theorem the opposite statement is also true: If $\mathcal{F} \subseteq 2^N$ and $(\mathcal{F}, \subseteq)$ is a distributive lattice of height $|N|$, then there exists a poset $(N, \preceq)$ such that $\mathcal{F} = \mathcal{O}(N, \preceq)$.

A (cooperative TU) *game* with *precedence constraints* (Faigle and Kern 1992) is a triple $(N, \preceq, v)$ such that N is a coalition, $(N, \preceq)$ is a poset, and $v : \mathcal{O}(N, \preceq) \to \mathbb{R}$, $v(\emptyset) = 0$. Note that a classical TU game is a pair (N, v) such that $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Hence, we may identify a game (N, v) with $(N, \preceq, v)$ where $(N, \preceq)$ is the poset of height 0. Let Γ denote the set of TU games with precedence constraints.

Throughout this section let $(N, \preceq, v)$ be a game with precedence constraints and denote $\mathcal{F} = \mathcal{O}(N, \preceq)$. Let

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leqslant v(N)\} \text{ and } X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

denote the set of *feasible* and *Pareto efficient* feasible payoffs (*preimputations*), respectively. We use $x(S) = \sum_{i \in S} x_i$ ($x(\emptyset) = 0$) for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$, and we write $x = (x_S, x_{N \setminus S})$.

The *core* of $(N, \preceq, v)$, denoted by $C(N, \preceq, v)$, is defined by

$$C(N, \preceq, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geqslant v(S) \text{ for all } S \in \mathcal{F}\}. \tag{2.1}$$

By its definition, the core of $(N, \preceq, v)$ is a convex polyhedral set. It is well known (Derks and Gilles 1995) that it does not contain lines. More precisely,

$$C(N, \preceq, v) = \operatorname{conv}(\operatorname{ext}(C(N, \preceq, v))) + C(N, \preceq, 0), \tag{2.2}$$

where "conv" means "convex hull", "ext" means "set of extreme points", and "+" denotes "Minkowski sum". For any $S \subseteq N$, let ${}^N\chi^S = \chi^S \in \mathbb{R}^N$ be the indicator function of S, i,e. $\chi^S_i = 1$ for $i \in S$ and $\chi^S_j = 0$

[1] A poset $(P, \preceq)$ is a *lattice* if for any $x, y \in P$ their supremum, denoted $x \wedge y$, and infimum, denoted $x \vee y$, exist. A lattice is *distributive* if $\wedge$ and $\vee$ satisfy distributivity.

for $j \in N \setminus S$. If $(N, \preceq, v)$ is a classical game, i.e., if the height of $(N, \preceq)$ is 0, then $C(N, \preceq, 0) = \{0\}$. Otherwise, i.e., if there exists a pair $(i, j) \in N$ such that $i \prec j$, then (Derks and Gilles 1995)

$$C(N, \preceq, 0) = \text{cone}(\{\chi^{\{i\}} - \chi^{\{j\}} \mid i, j \in N, i \prec j\}), \tag{2.3}$$

where "cone" denotes "convex cone generated by". For $i \prec j$, say $i = i_0 \prec\!\!\cdot \cdots \prec\!\!\cdot\, i_k = j$,

$$\chi^{\{i\}} - \chi^{\{j\}} = \sum_{\ell=0}^{k-1} \chi^{\{i_\ell\}} - \chi^{\{i_{\ell+1}\}}$$

so that

$$C(N, \preceq, 0) = \text{cone}(\{\chi^{\{i\}} - \chi^{\{j\}} \mid i, j \in N, i \prec\!\!\cdot\, j\}) \tag{2.4}$$

(also shown by Tomizawa (1983), see also Fujishige (2005, Th. 3.26)).

The *bounded core* of $(N, \preceq, v)$, denoted by $C^b(N, \preceq, v)$, is the set of all elements $x \in C(N, \preceq, v)$ that satisfy the following condition for any $i, j \in N$ with $i \prec\!\!\cdot\, j$: There is no $\varepsilon > 0$ such that $x + \varepsilon\left(\chi^{\{j\}} - \chi^{\{i\}}\right) \in C(N, \preceq, v)$. Hence,

$$C^b(N, \preceq, v) = \{x \in C(N, \preceq, v) \mid (\{x\} - C(N, \preceq, 0)) \cap C(N, \preceq, v) = \{x\}\}.$$

Therefore, if $(N, \preceq, v)$ is a classical game, the bounded core coincides with the classical core.

Remark 2.1 According to Rockafellar (1970, Section 18) a closed convex set is the disjoint union of the relative interiors of its faces. Hence, any element of $C^b(N, \preceq, v)$ is in the interior of some face of $C(N, \preceq, v)$. We conclude that $C^b(N, \preceq, v)$ is the disjoint union of the relative interiors of the bounded faces of $C(N, \preceq, v)$, i.e., $C^b(N, \preceq, v)$ is the union of all bounded faces of $C(N, \preceq, v)$. Thus, the bounded core is connected.

The bounded core may be non-convex and, hence, a proper subset of the convex hull of the extreme points of the core (called "convex part of the core") even if the poset is connected as Example 2.2 shows. We say that $i, j \in S \subseteq N$ are *connected* in $(S, \preceq)$ if there is a path in S that connects i and j, that is, if there exist $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in N$ such that $i = i_1$, $j = i_k$, and, for each $\ell = 1, \ldots, k-1$, either $i_\ell \prec i_{\ell+1}$ or $i_{\ell+1} \prec i_\ell$. Any $\emptyset \neq S \subseteq N$ may be partitioned into its connected components, and $S \subseteq N$ is *connected* if $S = \emptyset$ or S consists of a single component.

Example 2.2 Let $N = \{1, 2, 3, 4\}$ and $(N, \preceq, v)$ be defined by $i \prec 4$ for $i \in T = \{1, 2, 3\}$ and, for $S \in \mathcal{F}$, $v(S) = 6$ if $|S \cap T| = 2$, $v(N) = 12$, and $v(S) = 0$, otherwise. Then $(0, 6, 6, 0), (6, 0, 6, 0), (6, 6, 0, 0) \in C^b(N, \preceq, v)$, but the convex midpoint of these points $x = (4, 4, 4, 0) \notin C^b(N, \preceq, v)$ because $(2, 4, 4, 2) \in C(N, \preceq, v)$, which can be obtained from $(4, 4, 4, 0)$ by a transfer $(-2, 0, 0, 2)$.

One can show that the vertices of $C^b(N, \preceq, v)$ are $(0, 6, 6, 0), (6, 0, 6, 0), (6, 6, 0, 0)$ and $(3, 3, 3, 3)$. Then, the bounded core is the union of the three segments between $(3, 3, 3, 3)$ and each of the vertices $(0, 6, 6, 0)$, $(6, 0, 6, 0)$, and $(6, 6, 0, 0)$. Indeed, any point in the segment between $(3, 3, 3, 3)$ and $(0, 6, 6, 0)$ has the form $(3\alpha, -3\alpha+6, -3\alpha+6, 3\alpha)$, and any transfer $\varepsilon(\chi^{\{4\}} - \chi^{\{i\}})$ for $i \prec 4$ would lead to a point outside the core.

The same conclusion holds for the two other segments by symmetry. Now, consider the point $(3,3,5,1)$, which is the midpoint of $(0,6,6,0),(6,0,6,0)$ and $(3,3,3,3)$. Observe that a transfer $(0,0,-2,2)$ on this point would give $(3,3,3,3) \in C(N,\preceq,v)$, hence it does not belong to the bounded core. By symmetry again, any point in the relative interior of the convex hull of $(3,3,3,3)$ and any two of the vertices $(0,6,6,0),(6,0,6,0)$, and $(6,6,0,0)$ is outside the bounded core.

For any $S \in \mathcal{F}$ and $x \in \mathbb{R}^N$, let $e(S,x,v) = v(S) - x(S)$ be the *excess* at x.

For any $\alpha \in \mathbb{R}$ the *α-core* of $(N,\preceq,v)$, denoted by $C_\alpha(N,\preceq,v)$, is the set

$$C_\alpha(N,\preceq,v) = \{x \in X(N,v) \mid e(S,x,v) \leqslant \alpha \forall S \in \mathcal{F} \setminus \{\emptyset, N\}\}.$$

The following lemma follows from Lemma 3.2 of Grabisch and Sudhölter (2012).

Lemma 2.3 *If $(N,\preceq)$ is connected (i.e., N consists of a unique connected component) and $\alpha \in \mathbb{R}$, then $C_\alpha(N,\preceq,v) \neq \emptyset$ and $C_\alpha(N,\preceq,v) = \operatorname{conv}(\operatorname{ext}(C_\alpha(N,\preceq,v))) + C(N,\preceq,0)$.*

Proof: Let w differ from v only inasmuch as $w(S) = v(S)+\alpha$ for all $S \in \mathcal{F}\setminus\{\emptyset, N\}$. By the aforementioned lemma, $\mathcal{C}(N,\preceq,w) \neq \emptyset$. As $\mathcal{C}(N,\preceq,w) = C_\alpha(N,\preceq,v)$, the proof is finished by (2.2). **q.e.d.**

Let $(N,\preceq^*)$ be the reverse partially ordered set (i.e., $i \preceq^* j$ iff $j \preceq i$). Note that $\mathcal{O}(N,\preceq^*) = \{S \subseteq N \mid N \setminus S \in \mathcal{F}\}$. Hence, $(N,\preceq)$ is connected iff $(N,\preceq^*)$ is connected. The *dual game* of $(N,\preceq,v)$ is the game $(N,\preceq^*,v^*)$ defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \in \mathcal{O}(N,\preceq^*)$. As for classical games, if $x \in X(N,v)$ and $S \in \mathcal{O}(N,\preceq^*)$, then

$$e(S,x,v^*) = v^*(S) - x(S) = v(N) - v(N \setminus S) - x(N) + x(N \setminus S) = -e(N \setminus S, x, v). \tag{2.5}$$

Hence

$$\mathcal{C}_{-\alpha}(N,\preceq^*,v^*) = \{x \in X(N,v) \mid e(S,x,v) \geqslant \alpha \forall S \in \mathcal{F} \setminus \{\emptyset, N\}\} \text{ for all } \alpha \in \mathbb{R}. \tag{2.6}$$

In order to generalize Lemma 2.3 to games that do not necessarily have a connected hierarchy, we define the intermediate game (see Owen (1977) for a similar construction) of $(N,\preceq,v)$ as follows. Denote by $\mathcal{R}^{(N,\preceq)} = \mathcal{R}$ the partition of N whose elements are the connected components of N. In the present context $\mathcal{R}$ is considered as natural set of "a priori unions" and not a "coalition structure" à la Aumann and Drèze (1974), i.e., we do not consider "component feasible" payoffs. The *intermediate game* of $(N,\preceq,v)$ is the classical TU game $(\mathcal{R}, v_\mathcal{R})$ defined by $v_\mathcal{R}(\mathcal{T}) = v\left(\bigcup \mathcal{T}\right)$ for all $\mathcal{T} \subseteq \mathcal{R}$. Let $\mathcal{O}_0(N,\preceq) = \mathcal{F}_0$ be the subset of all elements of $\mathcal{F}$ that are not unions of connected components. Hence, $\mathcal{F} \setminus \mathcal{F}_0 = \{\bigcup \mathcal{T} | \mathcal{T} \subseteq \mathcal{R}\}$. Now, for any $y \in X(\mathcal{R}, v_\mathcal{R})$ and $\alpha \in \mathbb{R}$ define the *α-core w.r.t. y*, $C_{\alpha,y}(N,\preceq,v)$, by

$$C_{\alpha,y}(N,\preceq,v) = \{x \in X(N,v) \mid x(T) = y_T \forall T \in \mathcal{R} \text{ and } e(S,x,v) \leqslant \alpha \forall S \in \mathcal{F}_0\}.$$

By slightly abusing notation, for any $S \subseteq N$, $S \neq \emptyset$, the sub-poset of $(N,\preceq)$ on S, i.e., the intersection of $\preceq$ and $S \times S$, is denoted by $(S,\preceq)$. Note that $\mathcal{R}^{(S,\preceq)} = \{T \cap S \mid T \in \mathcal{R}\} \setminus \{\emptyset\}$, which we denote by $\mathcal{R}(S)$ for simplicity. Similarly we denote for any $\emptyset \neq S \subseteq N$, the sublattice $\mathcal{O}(S,\preceq) = \{T \cap S \mid T \in \mathcal{F}\}$ simply by $\mathcal{F}(S)$.

Proposition 2.4 *For any game* $(N, \preceq, v)$ *with precedence constraints, any preimputation* y *of the intermediate game* $(\mathcal{R}, v_{\mathcal{R}})$*, and any* $\alpha \in \mathbb{R}$,

$$C_{\alpha,y}(N, \preceq, v) \neq \emptyset \text{ and } C_{\alpha,y}(N, \preceq, v) = \operatorname{conv}(\operatorname{ext}(C_{\alpha,y}(N, \preceq, v))) + C(N, \preceq, 0).$$

Proof: Let $\mathcal{F}_0 = \mathcal{O}_0(N, \preceq)$ and $\mathcal{R} = \mathcal{R}^{(\mathcal{N}, \preceq)}$. To show non-emptiness we construct the components of some $x \in C_{\alpha,y}(N, \preceq, v)$ as follows. Let $\beta \geqslant (-\alpha) + \max\{v(S) \mid S \in \mathcal{F}_0\} - \min\{y(\mathcal{T}) \mid \mathcal{T} \subseteq \mathcal{R}\}$ and $\beta \geqslant 0$. For $Q \in \mathcal{R}$ define $(Q, \preceq, v^Q)$ by

$$v^Q(S) = \begin{cases} 0 & \text{, if } S = \emptyset, \\ y_Q & \text{, if } S = Q, \\ \beta & \text{, if } S \in \mathcal{F}(Q) \setminus \{\emptyset, Q\}. \end{cases}$$

As $(Q, \preceq)$ is connected there exists $x_Q \in C(Q, \preceq, v^Q)$ by Lemma 2.3. Hence, $x(Q) = y_Q$ for all $Q \in \mathcal{R}$. Let $S \in \mathcal{F}_0$ and $\widehat{\mathcal{T}} = \{Q \in \mathcal{R} \mid Q \subseteq S\}$. Then there exists $\widehat{Q} \in \mathcal{R}$ such that $\emptyset \neq \widehat{Q} \cap S \neq S$. As $\beta \geqslant 0$,

$$x(S) = x(S \cap \widehat{Q}) + x(S \setminus \widehat{Q}) \geqslant \beta + y(\widehat{\mathcal{T}}) \geqslant -\alpha + v(S) + \max\{-y(\mathcal{T}) \mid \mathcal{T} \subseteq \mathcal{R}\} + y(\widehat{\mathcal{T}}) \geqslant -\alpha + v(S).$$

Hence $v(S) - x(S) \leqslant \alpha$.

The observation that $z(Q) = 0$ for any $z \in C(N, \preceq, 0)$ and any $Q \in \mathcal{R}$ implies the second statement of the proposition. **q.e.d.**

Note that for any $y \in X\,(\mathcal{R}, v_{\mathcal{R}})$,

$$\mathcal{C}_{-\alpha,y}(N, \preceq^*, v^*) = \{x \in X(N, v) \mid x(T) = y_T \forall T \in \mathcal{R}, e(S, x, v) \geqslant \alpha \forall S \in \mathcal{F}_0\} \text{ for all } \alpha \in \mathbb{R}. \quad (2.7)$$

3 The positive core

In order to expand the definition of the positive core (Orshan and Sudhölter 2010) to TU games with precedence constraints, we employ and recall Justman's (1977) notion of a "general nucleolus" (Schmeidler 1969).

Let D be a finite nonempty set, X be a set, let $h : X \to \mathbb{R}^D$, and denote $d := |D|$. Define $\theta : X \to \mathbb{R}^d$ by

$$\theta_t(x) = \max_{T \subseteq D, |T| = t} \min_{i \in T} h_i(x) \text{ for all } x \in X \text{ and all } t = 1, \ldots, d,$$

that is, for any $x \in X$, $\theta(x)$ is the vector, whose components are the numbers $h_i(x)$, $i \in D$, arranged in non-increasing order. Let $\geq_{lex}$ denote the lexicographical order of $\mathbb{R}^d$. The *nucleolus* of h w.r.t. X, $\mathcal{NUC}(h, X)$, is defined by

$$\mathcal{NUC}(h, X) = \{x \in X \mid \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in X\}.$$

Let $(N, \preceq, v) \in \Gamma$, $\mathcal{F} = \mathcal{O}(N, \preceq), \mathcal{R} = \mathcal{R}^{(N, \preceq)}$, and recall that $\mathcal{F} \setminus \mathcal{F}_0 = \{\bigcup \mathcal{T} | \mathcal{T} \subseteq \mathcal{R}\}$. For $x \in \mathbb{R}^N$ and $S \in \mathcal{F}$ denote $e(S, x, v) = v(S) - x(S)$ (the *excess* of v at x). Then the *positive core* of $(N, \preceq, v)$, denoted by $C_+(N, \preceq, v)$, is defined by

$$C_+(N, \preceq, v) = \mathcal{NUC}((e(S, \cdot, v)_+)_{S \in \mathcal{F}}, X^*(N, v))$$

where $t_+ = \max\{0,t\}$ for $t \in \mathbb{R}$. Let $x \in C_+(N,\preceq,v)$. Then $x \in X(N,v)$ (otherwise there are positive excesses and all of them could be diminished) and, $e(S,x,v) \leqslant 0$ for all $S \in \mathcal{F}_0$ (otherwise, if $e(S,x,v) > 0$ for some $S \in \mathcal{F}_0$, then there exist $\ell \notin S \ni k$, $k \prec\cdot\, \ell$, and one could diminish this excess and the excesses of all $S' \in \mathcal{F}_0$, $\ell \notin S' \ni k$, by a transfer $t(\chi^{\{k\}} - \chi^{\{\ell\}})$, $t > 0$, without changing the excesses of further coalitions). We conclude that

$$C_+(N,\preceq,v) = \{x \in X(N,v) \mid (x(Q))_{Q\in\mathcal{R}} \in C_+(\mathcal{R},v_{\mathcal{R}}) \text{ and } e(S,x,v) \leqslant 0 \text{ for all } S \in \mathcal{F}_0\}. \tag{3.1}$$

It is well-known that for a classical TU game (N,v) the *prenucleolus*, i.e., the set

$$\mathcal{NUC}((e(S,\cdot,v))_{S\subseteq N}, X^*(N,v)),$$

consists of a single element $\nu(N,v)$ (the prenucleolus point) and that

$$C_+(N,v) = \{x \in X(N,v) \mid e(S,x,v)_+ = e(S,\nu(N,v),v)_+ \text{ for all } S \subseteq N\}. \tag{3.2}$$

Now, let ν be the prenucleolus point of the intermediate game $(\mathcal{R},v_{\mathcal{R}})$ and define $(N,\preceq,w)$ by $w\left(\bigcup\mathcal{T}\right) = \nu(\mathcal{T})$ for all $\mathcal{T} \subseteq \mathcal{R}$ such that $e(\mathcal{T},\nu,v_{\mathcal{R}}) > 0$ and $w(S) = v(S)$ for all other $S \in \mathcal{F}$. By (3.2), $C_+(N,\preceq,v) = C(N,\preceq,w)$ so that, by (2.2),

$$C_+(N,\preceq,v) = \operatorname{conv}(\operatorname{ext}(C(N,\preceq,w))) + C(N,\preceq,0)$$

and we may define the *bounded* positive core by

$$C_+^b(N,\preceq,v) = C^b(N,\preceq,w). \tag{3.3}$$

Remark 3.1 The variant of the "Kohlberg (1971) criterion" for the positive core of classical games is still valid. For $(N,\preceq,v) \in \Gamma$ and $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ let $\mathcal{D}(\alpha,x,v) = \{N\} \cup \{S \in \mathcal{F} \mid e(S,x,v) \geqslant \alpha\}$ where $\mathcal{F} = \mathcal{O}(N,\preceq,v)$. Recall that $\mathcal{D} \subseteq 2^N$ is *balanced* if there are *balancing coefficients* for $\mathcal{D}$, i.e., there are $\delta_S > 0, S \in \mathcal{D}$ such that $\sum_{S\in\mathcal{D}} \delta_S \chi^S = \chi^N$. If $x \in X(N,v)$, then following statements are equivalent (Orshan and Sudhölter 2010, Theorem 3.4):

$$x \in C_+(N,\preceq,v) \tag{3.4}$$

$$\alpha > 0, y \in \mathbb{R}^N, y(N) = 0, y(S) \geqslant 0 \forall S \in \mathcal{D}(\alpha,x,v) \implies y(S) = 0 \forall S \in \mathcal{D}(\alpha,x,v) \tag{3.5}$$

$$\alpha > 0 \implies \mathcal{D}(\alpha,x,v) \text{ is balanced} \tag{3.6}$$

We also note that the definitions of the classical bargaining set and its variants use the notion of "individual" objections. In a TU game $(N,\preceq,v)$ with precedence constraints a player $\ell \in N$ has no objection against any of her subordinates $k \prec \ell$ so that the Aumann-Davis-Maschler pre-bargaining set (Aumann and Maschler 1964, Davis and Maschler 1967), the reactive pre-bargaining set (Granot 1994), and the semireactive pre-bargaining set (Sudhölter and Potters 2001) consist of all preimputations x so that $(x(Q))_{Q\in\mathcal{R}}$ belongs to the respective pre-bargaining set of the intermediate game $(\mathcal{R},v_{\mathcal{R}})$ and, moreover, $e(S,x,v) \leqslant 0$ for all $S \in \mathcal{F}_0$. For the Mas-Colell pre-bargaining (Mas-Colell 1989) of $(N,\preceq,v)$ we only have one inclusion: If $(y(Q))_{Q\in\mathcal{R}}$ belongs to the Mas-Colell pre-bargaining set of the intermediate game and if $e(S,x,v) \leqslant 0$ for all $S \in \mathcal{F}_0$, then y belongs to the Mas-Colell pre-bargaining set of $(N,\preceq,v)$. However, the positive core is a subset of any of the mentioned pre-bargaining sets.

4 Characterizing the positive core

On classical games the prenucleolus and the positive core are the unique solutions that satisfy a collection of plausible properties. In this section we show that the positive core on the set of TU games with precedence constraints is characterized by these axioms.

We first recall the mentioned axioms of a solution. Let $\mathcal{F} = \mathcal{O}(N, \preceq)$ wherever it occurs in this section. Let $(N, \preceq, v) \in \Gamma$, $\emptyset \neq S \subseteq N$, $x \in \mathbb{R}^N$, and let $\pi : N \to N$ be a bijection (a permutation of N). The *reduced game* of $(N, \preceq, v)$ w.r.t. S and x is the game $(S, \preceq, v_{S,x})$ defined by

$$v_{S,x}(T) = \begin{cases} 0 & \text{, if } T = \emptyset, \\ v(N) - x(N \setminus S) & \text{, if } T = S, \\ \max\{v(P) - x(P \setminus T) \mid P \in \mathcal{F}, P \cap S = T\} & \text{, if } T \in \mathcal{F}(S) \setminus \{\emptyset, S\}. \end{cases} \tag{4.1}$$

The permutation π is a *symmetry* of $(N, \preceq, v)$ if $\{\pi(S) \mid S \in \mathcal{F}\} = \mathcal{F}$ and $v(\pi(S)) = v(S)$ for all $S \in \mathcal{F}$. Let $SYM(N, \preceq, v)$ denote the set of symmetries.

A *solution* is a mapping σ that assigns a subset $\sigma(N, \preceq, v)$ of $X^*(N, v)$ to any $(N, \preceq, v) \in \Gamma$. Its restriction to a set $\Gamma' \subseteq \Gamma$ is again denoted by σ. Moreover, a solution on Γ' is the restriction to Γ' of some solution. A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- *non-emptiness* (NE) if $\sigma(N, \preceq, v) \neq \emptyset$ for all $(N, \preceq, v) \in \Gamma'$;
- *Pareto optimality* (PO) if $\sigma(N, \preceq, v) \subseteq X(N, v)$ for all $(N, \preceq, v) \in \Gamma'$;
- *Covariance under strategic equivalence* (COV) if, for all $(N, \preceq, v), (N, \preceq, w) \in \Gamma', \alpha > 0$, and $\beta \in \mathbb{R}^N$, the following holds: If $w(S) = \alpha v(S) + \beta(S)$ for all $S \in \mathcal{F}$, then $\sigma(N, \preceq, w) = \alpha\sigma(N, \preceq, v) + \beta$;
- *Anonymity* (AN) if, for all $(N, \preceq, v) \in \Gamma'$ and all injective mappings $\pi : N \to U$ the following holds: If $(\pi(N), \preceq', \pi v) \in \Gamma'$, where $\pi(i) \preceq' \pi(j)$ iff $i \preceq j$, $(\pi v)(\pi(S)) = v(S)$ for all $S \subseteq \mathcal{F}$, and $\pi(x) = y \in \mathbb{R}^{\pi(N)}$ is defined by $y_{\pi(i)} = x_i \forall x \in \mathbb{R}^N, \forall i \in N$, then $\sigma(\pi(N), \preceq', \pi v) = \pi(\sigma(N, \preceq, v))$;
- *Boundedness* (BOUND) if $\sigma(N, \preceq, v)$ is a bounded set for all $(N, \preceq, v) \in \Gamma'$;
- *reasonableness* (REAS) if, for all $(N, \preceq, v) \in \Gamma'$ and all $x \in \sigma(N, \preceq, v)$ the following property holds for each minimal element k and each maximal element ℓ in N:

  $$x_k \geqslant \min\{v(S \cup \{k\}) - v(S) \mid S \in \mathcal{F}, k \notin S\} \text{ and } x_\ell \leqslant \max\{v(S) - v(S \setminus \{\ell\}) \mid S \in \mathcal{F}, \ell \in S\};$$

- *closedness* (CLOS) if $\sigma(N, \preceq, v)$ is a closed set for all $(N, \preceq, v) \in \Gamma'$;
- *reduced game property* (RGP) if the following condition holds: If $(N, \preceq, v) \in \Gamma', \emptyset \neq S \subseteq N$, and $x \in \sigma(N, \preceq, v)$, then $(S, \preceq, v_{S,x}) \in \Gamma'$ and $x_S \in \sigma(S, \preceq, v_{S,x})$;
- *reconfirmation property* (RCP) if the following condition holds for every $(N, \preceq, v) \in \Gamma'$, every $x \in \sigma(N, \preceq, v)$, and every $\emptyset \neq S \subseteq N$: If $(S, \preceq, v_{S,x}) \in \Gamma'$ and $y_S \in \sigma(S, \preceq, v_{S,x})$, then $(y_S, x_{N \setminus S}) \in \sigma(N, \preceq, v)$;

- the possibility of *nondiscrimination* (ND) if, for every $(N, \preceq, v) \in \Gamma'$ with $\sigma(N, \preceq, v) \neq \emptyset$, there exists $x \in \sigma(N, \preceq, v)$ such that $\pi(x) = x$ for all $\pi \in SYM(N, \preceq, v)$, i.e., x is invariant under all symmetries.

NE, PO, BOUND and CLOS are self-explanatory. COV means that the solution should not depend on a change of unit and an initial endowment to the players. AN means that the solution should be independent of the labeling of the players. REAS means that any player k corresponding to a minimal element in the hierarchy should receive at least her minimal marginal contribution (in particular, at least $v(\{k\})$), while a player ℓ being a maximal element should not receive more than her maximal marginal contribution (in particular, not more than $v(N) - v(N \setminus \{\ell\})$). RGP means that if x is a solution element, then its restriction to a subset S should be an element of the solution of the corresponding reduced game. RCP means that from a solution element x and a solution element y of the reduced game on S, one can build another solution element by concatenating y and $x_{N \setminus S}$. Lastly, ND means that it is possible to find a solution element invariant under all symmetries of the game, and thus not favoring any player.

Remark 4.1 (1) Let Γ^{free} be the set of classical TU games[2]. We recall (Orshan and Sudhölter 2010, Theorem 4.1) that, if $|U| = \infty$, there there are precisely three solutions on Γ^{free} that satisfy NE, REAS, COV, RGP, RCP, and ND, namely the prenucleolus, the positive core C_+, or its relative interior rint C_+. Here, for $(N, \preceq, v) \in \Gamma$ and with $\mathcal{R} = \mathcal{R}^{(N, \preceq)}$,

$$\text{rint } C_+(N, \preceq, v) = \left\{ x \in C_+(N, \preceq v) \;\middle|\; \begin{array}{c} e(S, x, v) < 0 \text{ for } S \in \mathcal{F}_0 \text{ or } S = \bigcup \mathcal{T} \text{ for some} \\ \mathcal{T} \subseteq \mathcal{R} \text{ such that } e(\mathcal{T}, \nu(\mathcal{R}, v_{\mathcal{R}}), v_{\mathcal{R}}) < 0 \end{array} \right\}. \quad (4.2)$$

For classical TU games, the foregoing set is nonempty because it contains the prenucleous point. Hence, by Proposition 2.4 it is also nonempty in the general case so that it indeed coincides with the relative interior of the positive core.

(2) Note that C_+ satisfies REAS. Indeed, let $(N, \preceq, v) \in \Gamma$, $x \in X(N, v)$, and $k, \ell \in N$ such that k is minimal and ℓ is maximal. If $x_k < t := \min_{S \in \mathcal{F}, k \notin S} v(S \cup \{k\}) - v(S)$, then choose $\varepsilon > 0$ such that $x_k + (|N| - 1)\varepsilon < t$, define $y \in X(N, v)$ by $y_k = x_k + (|N| - 1)\varepsilon$ and $y_i = x_i - \varepsilon$ for all $i \in N \setminus \{k\}$ and let $S \in \mathcal{F} \setminus \{\emptyset, N\}$. If $k \in S$, then $e(S, y, v) \leq e(S, x, v) - \varepsilon$ and if $k \notin S$, then $e(S, y, v) < e(S \cup \{k\}, x, v)$. As $e(\{k\}, x, v) > 0$, $x \notin C_+(N, \preceq, v)$. Similarly it is shown that $x_\ell > \max_{S \in \mathcal{F}, \ell \in S} v(S) - v(S \setminus \{\ell\}$ implies that $x \notin C_+(N, \preceq, v)$.

(3) It is straightforward to verify that C_+ on any $\Gamma' \subseteq \Gamma$ satisfies NE, PO, COV, AN, and CLOS.

We say that $\Gamma' \subseteq \Gamma$ is *closed under reduction* if, for all $(N, \preceq, v) \in \Gamma', \emptyset \neq S \subseteq N$, and $x \in X(N, v)$, $(S, \preceq, v_{S,x}) \in \Gamma'$.

Lemma 4.2 *Let* $\Gamma' \subseteq \Gamma$.

(1) If Γ' *is closed under reduction, then both* C_+ *and rint* C_+ *on* Γ' *satisfy* RGP.

[2]When cooperation is not restricted, i.e., all coalitions are feasible, we call a game *unrestricted* (free).

(2) Both C_+ and rint C_+ on Γ' satisfy RCP.

Proof: For C_+, Remark 3.1 (3.5) implies (see Theorem 6.3.14 of Peleg and Sudhölter 2007) RGP, and in order to show RCP the relevant part of the proof of Theorem 6.3.14 of RCP may be literally copied. In the case of rint C_+ we may proceed similarly as before to show RGP. The "Kohlberg criterion" has to be modified only inasmuch as $\alpha > 0$ has to be replaced by $\alpha \geqslant 0$ in (3.5) and (3.6).

In order to show that rint C_+ satisfies RCP, let $(N, \preceq, v) \in \Gamma$, $x \in$ rint $C_+(N, \preceq, v)$, $\emptyset \neq S \subseteq N$, $y \in$ rint $C_+(S, \preceq, v_{S,x})$, $T \in \mathcal{F} = \mathcal{O}(N, \preceq)$, and put $z = (y, x_{N\setminus S})$. We have to show that $e(T, z, v) \leqslant e(T, x, v)_+$ and that $e(T, x, v) < 0$ implies $e(T, z, v) < 0$. If $x(T \cap S) = y(T \cap S)$, then $e(T, x, v) = e(T, z, v)$. Hence, we may assume that $y(T \cap S) \neq x(T \cap S)$. Therefore $T \cap S \in \mathcal{F}(S) \setminus \{\emptyset, S\}$. Let $P \subseteq N \setminus S$ such that $(T \cap S) \cup P \in \mathcal{F}$ and $v_{S,x}(T \cap S) = v((T \cap S) \cup P) - x(P)$. By definition of the reduced game, $e((T \cap S) \cup P, z, v) \geqslant e(T, z, v)$. If $e((T \cap S) \cup P, z, v) > 0$ or $e((T \cap S) \cup P, x, v) > 0$, then by RCP of C_+, $e((T \cap S) \cup P, z, v) = e((T \cap S) \cup P, x, v)$ so that $x(T \cap S) = y(T \cap S)$ which was excluded. If $e((T \cap S) \cup P, z, v) = 0$, then $e((T \cap S) \cup P, x, v) < 0$ so that $e(S \cap T, x_S, v_{S,x}) < 0 = e(S \cap T, y, v_{S,x})$ so that $y \notin$ rint $C_+(S, \preceq, v_{S,x})$ by RGP of rint C_+. **q.e.d.**

Lemma 4.3 *On any $\Gamma' \subseteq \Gamma$ the positive core satisfies* ND.

Proof: For any $(N, \preceq, v) \in \Gamma', x \in C_+(N, \preceq, v)$, and $\pi \in$ SYM$(N \preceq, v)$, by AN also $\pi(x) \in C_+(N, \preceq, v)$. By convexity of the positive core, $y = \sum_{\pi \in \text{ SYM}(N,\preceq,v)} \frac{\pi(x)}{|\text{SYM}(N,\preceq,v)|} \in C_+(N, \preceq, v)$. Finally, y is invariant under symmetries. **q.e.d.**

Lemma 4.4 *If σ is a solution that satisfies* REAS *and* RGP*, then $e(S, x, v) \leqslant 0$ for all $(N, \preceq, v) \in \Gamma, x \in \sigma(N, \preceq, v)$, and all $S \in \mathcal{O}_0(N, \preceq)$.*

Proof: Let $x \in \sigma(N, \preceq, v)$, $T \in \mathcal{O}_0(N, \preceq)$, and assume, on the contrary, that $e(T, x, v) > 0$. Then there exist $k \in T$ and $\ell \in N \setminus T$ such that $k \prec \ell$. Now, $v_{\{k,\ell\},x}(\{k\}) \geqslant v(T) - x(T \setminus \{k\})$ and, by RGP, $x_{\{k,\ell\}} \in \sigma(\{k, \ell\}, \preceq, v_{\{k,\ell\},x})$ so that REAS is violated. **q.e.d.**

Lemma 4.5 *Let σ be a solution that satisfies* RGP*. If, for any $(N, v) \in \Gamma^{\text{free}}$, $\sigma(N, v) \subseteq C_+(N, v)$, and if, for any $(N, \preceq, v) \in \Gamma$ and any $x \in \sigma(N, \preceq, v)$, $e(T, x, v) \leqslant 0$ for all $T \in \mathcal{O}_0(N, \preceq)$, then σ is a subsolution of the positive core, i.e., $\sigma(N, \preceq, v) \subseteq C_+(N, \preceq, v)$ for $(N, \preceq, v) \in \Gamma$.*

Proof: Let $S \subseteq N$ such that $|Q \cap S| = 1$ for all $Q \in \mathcal{R}$. For any $i \in S$ denote Q_i the connected component that contains i. Hence, the mapping $S \to \mathcal{R}$, $i \mapsto Q_i$ for all $i \in S$, is a bijection. As $\mathcal{F}(S) = 2^S$, by RGP and our assumption, $x_S \in C_+(S, \preceq, v_{S,x})$. Now, let $y \in \mathbb{R}^{\mathcal{R}}$ be defined by $y_Q = x\left(\bigcup Q\right)$ for all $Q \in \mathcal{R}$. Then, for any $T \subseteq S$, $e(T, x_S, v_{S,x})_+ = e(\{Q_i \mid i \in T\}, y, v_{\mathcal{R}})_+$. Indeed, $v_{S,x}(T) = v(T \cup P) - x(P)$ for some $P \subseteq N \setminus S$ such that $T \cup P \in \mathcal{F}$. If $e(T, x_S, v_{S,x}) > 0$ then $T \cup P$ must be a union of connected components, hence $T \cup P = \bigcup_{i \in T} Q_i$ so that the foregoing equation is valid. Conversely, if $e(\{Q_i \mid i \in T\}, y, v_{\mathcal{R}}) > 0$, then $e(\{Q_i \mid i \in T\}, y, v_{\mathcal{R}}) = v(\bigcup_{i \in T} Q_i) - x(\bigcup_{i \in T} Q_i) \leqslant e(T, x_S, v_{S,x})$.

By Remark 3.1 (3.5), $y \in C_+(\mathcal{R}, v_{\mathcal{R}})$. The proof is finished by (3.1). **q.e.d.**

Theorem 4.6 *Assume that* $|U| = \infty$. *There is a unique solution that satisfies* NE, REAS, COV, CLOS, RGP, RCP, *and* ND, *and it is the positive core.*

Proof: The positive core satisfies the desired axioms by Remark 4.1, Lemma 4.2, and Lemma 4.3.

In order to show the remaining direction, let σ be a solution that satisfies the desired axioms. On Γ^{free}, σ must be one of the following three solutions (Orshan and Sudhölter 2010, Theorem 4.1): The prenucleolus, the positive core, or its relative interior. Hence, σ is a subsolution of the positive core by Lemmas 4.4 and 4.5.

Step 1: We claim that, on Γ^{free}, σ is the positive core. In order to show this statement it suffices to construct a single TU game $(M,u) \in \Gamma^{\text{free}}$ such that $\sigma(M,u) \setminus \text{ rint } C_+(M,u) \neq \emptyset$. For this purpose, choose any set M' of three players, say without loss of generality, $M' = \{1,2,3\}$, and let $\preceq'$ be defined by $1 \prec' 2$. Moreover, let $(M', \preceq', u') \in \Gamma$ be defined by $u'(M) = 2$ and $u'(S) = 0$ for $S \in \mathcal{F}' \setminus \{M'\}$, where $\mathcal{F}' = \mathcal{O}(M', \preceq')$. Let $P = \{1,3\}$, $Q = \{2,3\}$, and $R = \{1,2\}$. By NE, there exists $x \in \sigma(M', \preceq', u')$. As $x \in C_+(M', \preceq', u')$, $x_1, x_3 \geqslant 0, x_3 \leqslant 2$, and $x(M') = 2$. Note that all reduced games of $(M', \preceq', u')$ w.r.t. P or Q belong to Γ^{free}. By RGP, $(x_2, x_3) \in \sigma(Q, u'_{Q,x})$. As $u'_{Q,x}(\{2\}) = -x_1, u'_{Q,x}(\{3\}) = 0$, $(1 - x_1, 1)$ is the prenucleolus of $(Q, u'_{Q,x})$. As $\sigma(Q, u'_{Q,x})$ contains the prenucleolus, RCP implies that $y = (x_1, 1 - x_1, 1) \in \sigma(M', \preceq', u')$. Now, if $x_1 = 0$, then by RGP $(0,1) \in \sigma(P, u'_{P,y})$. However, $(0,1) \in C_+(P, u'_{P,y}) \setminus \text{ rint } C_+(P, u'_{P,y})$. If $x_1 > 0$, then by RGP, $y_R \in \sigma(R, \preceq', u'_{R,y})$. As $u'_{R,y}(\{1\}) = 0$ and $u'_{R,y}(R) = 1$, by (translation) COV $(x_1, -x_1) \in \sigma(R, \preceq', 0)$ and by (scale) COV, $(t, -t) \in \sigma(R, \preceq', 0)$ for all $t > 0$ (choose $\alpha = \frac{t}{x_1}$ in the definition of scale COV). By CLOS, $(0,0) \in \sigma(R, \preceq', 0)$ so that by translation COV and RCP, $z = (0,1,1) \in \sigma(M', \preceq', u')$. Finally, RGP yields that $(0,1) \in \sigma(P, u'_{P,z})$ and $(0,1) \in C_+(P, u'_{P,z}) \setminus \text{ rint } C_+(P, u'_{P,z})$.

Let $(N, \preceq, v) \in \Gamma$. It remains to show that

$$C_+(N, \preceq, v) \subseteq \sigma(N, \preceq, v). \tag{4.3}$$

Let $\mathcal{F} = \mathcal{O}(N, \preceq, v)$. By Step 1 we may assume that $\mathcal{F} \neq 2^N$.

Step 2: We consider the case $|N| = 2$ first and assume without loss of generality that $N = \{1,2\}$ and $1 \prec 2$. By COV we may assume that $v(S) = 0$ for all $S \in \mathcal{F}$. With $(M', \preceq', u')$ defined in Step 1, by NE there exists $y \in \sigma(M', \preceq', u')$. As in Step 1 we may assume that $y_3 = 1$. By RGP and RCP applied to R, we may assume that $y_2 \geqslant 0$. Furthermore, by RGP and RCP applied to P, we may replace (y_1, y_3) by $(y_1 + y_3, 0)$ (or by $(0, y_1 + y_3)$ if necessary because $\sigma(P, u'_{P,y}) = C_+(P, u'_{P,y})$). Thus, there exists $z, z' \in \sigma(M', \preceq', u')$ with $z_1 > 0$ and $z'_1 = 0$. By RGP and COV, $(z_1, -z_1), (0,0) \in \sigma(N, \preceq, v)$ so that by scale covariance, $(t, -t) \in \sigma(N, \preceq, v)$ for all $t \geqslant 0$, i.e., $\sigma(N, \preceq, v) = C_+(N, \preceq, v)$.

Now the proof of (4.3) can be finished by induction on $|N|$. For $|N| = 1$, $C_+(N, \preceq, v)$ is a singleton so that (4.3) holds by NE and Lemmas 4.4 and 4.5. For $|N| = 2$, Step 2 shows (4.3). Assume that (4.3) is proved whenever $|N| \leqslant k$ for some $t \in \mathbb{N}$ with $t > 1$. If $|N| = t + 1$, we may assume by Step 1 that $\mathcal{R} = \mathcal{R}^{(N,\preceq)}$ contains a non-singleton Q. Let $k \in Q$ be minimal. Let $A = \{x_k \mid x \in C_+(N, \preceq, v)\}$. Then $A = [\alpha, \infty[$ for some $\alpha \in \mathbb{R}$. Indeed, by REAS, A has a lower bound and by closedness, there is a largest lower bound

α. Moreover, let $\ell \in Q$ with $k \prec \ell$. Replacing x_k by $x_k + t$ and x_ℓ by $x_\ell - t$ for any $t \geqslant 0$ yields another element of $C_+(N, \preceq, v)$. Similarly, if $y \in \sigma(N, \preceq, v)$, then by RCP applied to $\{k, \ell\}$, we may replace y_k by $y_k + t$ and y_ℓ by $y_\ell - t$ and receive another element of $\sigma(N, \preceq, v)$. Also, by applying RCP to the coalition $N' = N \setminus \{k\}$, by the the inductive hypothesis, $y' = (y_k, z_{N'}) \in \sigma(N, \preceq, v)$ for any $z_{N'} \in C_+(N', \preceq, v_{N',y})$. By RGP and RCP of C_+, $C_+(N', \preceq, v_{N',y}) = \{z_{N'} \in \mathbb{R}^{N'} \mid (y_k, z_{N'}) \in C_+(N, \preceq, v)\}$. Hence, it remains to show that $\sigma(N, \preceq, v)$ contains an element x such that $x_k = \alpha$. Let x' arise from x by replacing x_k by y_k and x_ℓ by $x_\ell + x_k - y_k$. Then $x' \in \sigma(N, \preceq, v)$ as we have shown. Now, applying RCP to $\{k, \ell\}$ allows us to reverse this modification so that $x \in \sigma(N, \preceq, v)$. **q.e.d.**

5 Characterizing the bounded positive core

This section is devoted to the characterization of the bounded positive core. Let $(N, \preceq, v) \in \Gamma$, $\mathcal{F} = \mathcal{O}(N, \preceq)$, and $\mathcal{R} = \mathcal{R}^{(N,\preceq)}$. We may rewrite (3.3) as

$$C_+^b(N, \preceq, v) = \{x \in C_+(N, \preceq, v) \mid \max\{e(S, x, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} = 0 \text{ for all } k, \ell \in N, k \prec\!\cdot\, \ell\}. \quad (5.1)$$

Note that C_+^b inherits the following properties from C_+: NE, PO, COV, AN, REAS, and CLOS. We now define a particular subsolution of C_+^b that is single-valued and satisfies ND. Let $\mathcal{F}^* = \mathcal{O}(N, \preceq^*)$ where $(N, \preceq^*)$ is the reverse order of $(N, \preceq)$. Moreover, denote by ν the prenucleolus of the intermediate game $(\mathcal{R}, v_{\mathcal{R}})$. The *prenucleolus* of $(N, \preceq, v)$ is the set

$$\mathcal{N}(N, \preceq, v) = \mathcal{NUC}\left((e(S, \cdot, v^*))_{S \in \mathcal{F}^*}, \{y \in C_+(N, \preceq, v) \mid y(Q) = \nu_Q \text{ for } Q \in \mathcal{R}\}\right). \quad (5.2)$$

Proposition 5.1 *For any $(N, \preceq, v) \in \Gamma$, $\mathcal{N}(N, \preceq, v)$ is a singleton that is contained in $C_+^b(N, \preceq, v)$.*

Proof: Let $y \in C_+(N, \preceq, v)$ such that $y(Q) = \nu_Q$ for all $Q \in \mathcal{R}$ and define $\mu^* = \max\{e(S, y, v^*) \mid S \in \mathcal{F}^*\}$, $\mu = \max\{e(S, y, v) \mid S \in \mathcal{F}\}$, and

$$X = \{x \in C_+(N, \preceq, v) \mid x(Q) = \nu_Q \text{ for all } Q \in \mathcal{R} \text{ and } e(S, x, v^*) \leqslant \mu^* \text{ for all } S \in \mathcal{F}^*\}.$$

Then $X \neq \emptyset$ because $y \in X$. Therefore

$$\mathcal{N}(N, \preceq, v) = \mathcal{NUC}\left((e(S, \cdot, v^*))_{S \in \mathcal{F}^*}, X\right).$$

Moreover, X is closed and convex. We now show that X is bounded. For $i \in N$ let $h(i)$ the height of i w.r.t. $(N, \preceq)$. Let $x \in X$. If $h(i) = 0$, then $i \in \mathcal{F}$ and, hence, $x_i \geqslant v(\{i\}) - \mu$. Moreover, $N \setminus \{i\} \in \mathcal{F}^*$ so that $x(N \setminus \{i\}) \geqslant v^*(N \setminus \{i\}) - \mu^*$, i.e., $x_i \leqslant \mu^* + v(\{i\})$. Assume now that we have shown already that all x_i with $h(i) < k$ for some $k \in \mathbb{N}$ are bounded. If, now, $h(i) = k$, then let $S = \{j \in N \mid j \preceq i\}$. On the one hand, $x_i \geqslant v(S) - \mu - x(S \setminus \{i\})$, hence x_i has a lower bound because $h(j) < k$ for all $j \in S \setminus \{i\}$ so that the inductive hypothesis can be applied. On the other hand $x_i \leqslant \mu^* + v(S) - x(S \setminus \{i\})$ so that x_i is bounded from above.

As the excess functions are continuous and convex, and as X is nonempty, compact, and convex, the prenucleolus $\mathcal{NUC}\left((e(S, \cdot, v^*))_{S \in \mathcal{F}^*}, X\right)$ is a nonempty convex set such that $e(S, x, v^*) = e(S, x', v^*)$ for all $S \in \mathcal{F}^*$ and (Justman 1977) all $x, x' \in \mathcal{N}(N, \preceq, v)$. By induction on $h^*(i)$, the height of i w.r.t. $(N, \preceq^*)$ we show that $x_i = x'_i$. Indeed, if $h^*(i) = 0$, then $\{i\} \in \mathcal{F}^*$ so that $e(\{i\}, x, v^*) = e(\{i\}, x', v^*)$ implies $x_i = x'_i$. If $h^*(i) > 0$, then, with $S = \{j \in N \mid j \preceq^* i\}$,

$$e(S, x, v^*) = v^*(S) - x(S \setminus \{i\}) - x_i = v^*(S) - x'(S \setminus \{i\}) - x'_i = e(S, x', v^*)$$

so that $x_i = x'_i$ by the inductive hypothesis applied to $j \in S \setminus \{i\}$. Let $\nu(N, \preceq, v) = \widehat{\nu}$ denote the unique element of $\mathcal{N}(N, \preceq, v)$. Moreover, let $k, \ell \in N$ such that $k \prec\cdot \ell$. Assume that $t = \max\{e(S, \widehat{\nu}, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} < 0$. Let $0 < \varepsilon < -t$ and let $x \in \mathbb{R}^N$ only differ from $\widehat{\nu}$ inasmuch as $x_k = \widehat{\nu}_k - \varepsilon$ and $x_\ell = \widehat{\nu}_\ell + \varepsilon$. Then $x \in X$ and $e(T, x, v^*) < e(T, \widehat{\nu}, v^*)$ for all $T \in \mathcal{F}^*$ such that $k \notin T \ni \ell$ and $e(T, x, v^*) = e(T, \widehat{\nu}, v^*)$ for all other $T \in \mathcal{F}^*$ which is a contradiction. By (5.1), $\widehat{\nu} \in C^b_+(N, \preceq, v)$. **q.e.d.**

Remark 5.2 The solution $\mathcal{N}(\cdot)$ satisfies ND because it is single-valued and satisfies AN.

We provide now a combinatorial characterization of the prenucleolus of $(N, \preceq, v)$ by a Kohlberg-like criterion. To this end, we introduce the following collections, for any $Q \in \mathcal{R}$:

$$\mathcal{D}'(\alpha, x, v^*, Q) = \{S \cap Q \mid S \in \mathcal{F}^*_0, e(S, x, v^*) \geqslant \alpha\}$$
$$\mathcal{E}(x, v, Q) = \{T \cap Q \mid T \in \mathcal{F}_0, x(T) = v(T)\} \cup \{Q\},$$

where $\mathcal{F}^*_0 = \mathcal{O}_0(N, \preceq^*)$. For two collections $\mathcal{B}, \mathcal{B}'$ in 2^N such that $\mathcal{B} \subseteq \mathcal{B}'$, we say that $\mathcal{B}$ is *balanced within* $\mathcal{B}'$ if there exists a collection $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{B}'$ which is balanced.

Proposition 5.3 *Let $(N, \preceq, v) \in \Gamma$, and consider $x \in C_+(N, \preceq, v)$ s.t. $x(Q) = \nu_Q$ for $Q \in \mathcal{R}$, where ν is the prenucleolus of $(\mathcal{R}, v_{\mathcal{R}})$. The following are equivalent:*

(1) $x = \mathcal{N}(N, \preceq, v)$

(2) For $\alpha > 0$, $Q \in \mathcal{R}$, $y \in \mathbb{R}^Q$, with $y(Q) = 0$ and $y(S) \leqslant 0$ for all $S \in \mathcal{D}'(\alpha, x, v^*, Q) \cup \mathcal{E}(x, v, Q)$, we have $y(S) = 0$ for all $S \in \mathcal{D}'(\alpha, x, v^*, Q)$.

(3) For $\alpha > 0$ and $Q \in \mathcal{R}$, $\mathcal{D}' = \mathcal{D}'(\alpha, x, v^*, Q)$ is balanced within $\mathcal{D}' \cup \mathcal{E}(x, v, Q)$.

Proposition 5.3 generalizes Kohlberg's (1971) characterization of the nucleolus. The equivalence between (1) and (2) is proved similarly as in the classical case. The equivalence between (2) and (3) which is an immediate consequence of Farkas' lemma occurs as a special case of Lemma 2.1 of Derks, Peters, and Sudhölter (2014) and is, e.g., explicitly proved by Derks and Peters (1998).

By Remark 5.2 and Proposition 5.1 the bounded positive core satisfies ND. Moreover, a careful inspection of the definition of the reduced game together with (5.1) show that, by Lemma 4.2, C^b_+ also inherits RGP from C_+. Thus, C^b_+ does not satisfy RCP if $|U| = \infty$ by Theorem 4.6.

In order to characterize the bounded positive core, the following properties of a solution σ on a set $\Gamma' \subseteq \Gamma$ are useful. The solution σ satisfies

- the *converse reduced game property* (CRGP) if for $(N, \preceq, v) \in \Gamma'$ with $|N| \geqslant 2$ and $x \in X(N, v)$ the following condition holds: If, for every $S \subseteq N$ with $|S| = 2$, $(S, \preceq, v_{S,x}) \in \Gamma'$ and $x_S \in \sigma(S, \preceq, v_{S,x})$, then $x \in \sigma(N, \preceq, v)$.

- the *restricted converse reduced game property* (RCRGP) if for $(N, \preceq, v) \in \Gamma'$ and $x \in X(N, v)$, with $\mathcal{R} = \mathcal{R}^{(N,\preceq)}$ the following condition holds: If there exists $S \subseteq N$ such that $|S \cap Q| = 1$ for all $Q \in \mathcal{R}$, $(S, \preceq, v_{S,x}) \in \Gamma$, and $x_S \in \sigma(S, v_{S,x})$, and if $(T \preceq, v) \in \Gamma'$ and $x_T \in \sigma(T, \preceq, v_{T,x})$, for any $T = \{k, \ell\} \subseteq N$ with $k \prec\!\cdot\, \ell$, then $x \in \sigma(N, \preceq, v)$;

- the *unrestricted reconfirmation property* (RCP^{free}) if the restriction of σ to $\Gamma' \cap \Gamma^{\text{free}}$ satisfies RCP.

RCRGP has not been used in the literature. It may be interpreted as follows. If every "block", i.e., every connected component, may select a representative so that the coalition of representatives S is satisfied with the preimputation x (i.e., $x_S \in \sigma(S, v_{S,x})$) and if every pair of players consisting of a player ℓ and her immediate subordinate k is also satisfied (i.e., $x_T \in \sigma(T, \preceq, v_{T,x})$, where $T = \{k, \ell\}$), then no player in the grand coalition has an objection against x (i.e., $x \in \sigma(N, \preceq, v)$).

As C_+ satisfied RCP, C^b_+ satisfies RCP on Γ^{free}, i.e., C^b_+ satisfies RCP^{free}. By (3.3), it also satisfies BOUND.

The following result may be proved similarly to Lemma 4.5.

Lemma 5.4 *The bounded positive core satisfies RCRGP.*

Proof: Let $(N, \preceq, v) \in \Gamma$, $\mathcal{F} = \mathcal{O}(N, \preceq, v)$, $\mathcal{R} = \mathcal{R}^{(N,\preceq)}$, and let $x \in X(N, v)$ such that $x_{\{k,\ell\}} \in C^b_+(\{k, \ell\}, \preceq, v_{\{k,\ell\},x})$ for all $k, \ell \in N$ such that $k \prec\!\cdot\, \ell$ and such that there exists $S \subseteq N$ with $|Q \cap S| = 1$ for all $Q \in \mathcal{R}$ and $x_S \in C^b_+(S, v_{S,x})$. For any $i \in S$ denote Q_i the connected component that contains i. Hence, the mapping $S \to \mathcal{R}$, $i \mapsto Q_i$ for all $i \in S$, is a bijection.

Let $k, \ell \in N$ such that $k \prec\!\cdot\, \ell$. Now, $v_{\{k,\ell\},x}(\{k\}) = v(T) - x(T \setminus \{k\})$ for some $T \in \mathcal{F}$ with $\ell \notin T \ni k$. As $x_{\{k,\ell\}} \in C^b_+(\{k, \ell\}, \preceq, v_{\{k,\ell\},x})$, $0 = v(T) - x(T) = \max\{v(P) - x(P) \mid \ell \notin P \ni k, P \in \mathcal{F}\}$.

By (3.1) and (5.1) it suffices to show that $y = (x(Q))_{Q \in \mathcal{R}} \in C_+(\mathcal{R}, v_{\mathcal{R}})$. Now, $x_S \in C_+(S, v_{S,x})$. As in the proof of Lemma 4.5, $e(T, x_S, v_{S,x})_+ = e(\{Q_i \mid i \in T\}, y, v_{\mathcal{R}})_+$ for all $T \subseteq \mathcal{F}(S)$ so that the proof is finished by (3.5). **q.e.d.**

Remark 5.5 In the proof of Theorem 5.6 we shall employ the following stronger version Theorem 4.1 of Orshan and Sudhölter (2010) in which REAS is replaced by BOUND: If $|U| = \infty$, then the unique solutions on Γ^{free} that satisfy NE, BOUND, COV, RGP, and ND, are (a) the prenucleolus, (b) the positive core, and (c) the relative interior of the positive core. This result is Theorem 4.9 of Orshan and Sudhölter (2001). The article from 2010 is a modification of the discussion paper from 2001. Theorem 4.9 relies on the lengthy and technically sophisticated proof of Lemma 4.2, whereas when replacing BOUND by REAS, the proof of the corresponding lemma (Lemma 4.6) takes just a few lines so that the authors used the well-accepted REAS in their 2010 article basically to offer an easier reading. In the present context, when restrictions are possible, REAS is not implying BOUND.

Theorem 5.6 *The bounded positive core is the unique solution that satisfies* NE, BOUND, COV, RGP, RCP$^{\text{free}}$, ND, *and* RCRGP*, provided* $|U| = \infty$.

We postpone the proof and first show a useful variant of Lemma 4.4.

Lemma 5.7 *If* σ *is a solution that satisfies* BOUND, COV, *and* RGP*, then, for any* $k, \ell \in N$ *such that* $k \prec\!\cdot\, \ell$ *with* $P = \{k, \ell\}$*,* $\sigma(P, \preceq, v_{P,x}) = C^b(P, \preceq, v_{P,x})$ *for any* $x \in \sigma(N, \preceq, v)$*, which yields* $\max\{e(S, x, v) \mid l \notin S \ni k, S \in \mathcal{F}\} = 0$.

Proof: Let $u = v_{P,x}$. Define $y \in \mathbb{R}^P$ by $y_k = x_k - u(\{k\})$ and $y_\ell = x_\ell + u(\{k\}) - u(P)$. By RGP and translation COV, $y \in \sigma(P, \preceq, 0)$. By scale COV, $ty \in \sigma(P, \preceq, 0)$ for $t > 0$ so that, by BOUND, $y = 0$, which yields $x_k = \max\{v(S \cup \{k\}) - x(S) \mid k \notin S \in \mathcal{F}\}$. **q.e.d.**

Proof of Theorem 5.6: It remains to show the uniqueness part. Let σ be a solution that satisfies the desired properties. By Lemmas 5.7 and 4.5, σ is a subsolution of C^b_+. Thus, it remains to show that $C^b_+(N, \preceq, v) \subseteq \sigma(N, \preceq, v)$.

Claim: The solution σ coincides with $C_+ = C^b_+$ on Γ^{free}. Recall that $(M', \preceq', u') \in \Gamma$ of the proof of Theorem 4.6 is defined by $M' = \{1, 2, 3\}$, $u'(M) = 2$ and $u'(S) = 0$ for $S \in \mathcal{F}' \setminus \{M'\}$, where $\mathcal{F}' = \mathcal{O}(M', \preceq')$. Also, let $P = \{1, 3\}$, $Q = \{2, 3\}$, and $R = \{1, 2\}$. By NE, there exists $x \in \sigma(M', \preceq', u')$. As $x \in C^b_+(M', \preceq', u')$, $x_1 = 0$ (by (5.1)) and $0 \leqslant x_3 \leqslant 2$. If $x_3 > 0$, then $(x_1, x_3) \in C_+(P, u'_{P,x}) \setminus \operatorname{rint} C_+(P, u'_{P,x})$, and if $x_3 = 0$, then $(x_2, x_3) \in C_+(Q, u'_{Q,x}) \setminus \operatorname{rint} C_+(Q, u'_{Q,x})$. Hence, our claim is valid by Remark 5.5 in any case.

Now we can finish the proof. Let $x \in C^b_+(N, \preceq, v)$ and choose any $S \subseteq N$ such that $|S \cap Q| = 1$ for any $Q \in \mathcal{R}^{(N,\preceq)}$. Then $(S, \preceq)$ has height 0 so that the reduced game w.r.t. S belongs to Γ^{free}. By RGP of C^b_+, $x_S \in C_+(S, v_{S,x})$. Hence, by our claim $x_S \in \sigma(S, v_{S,x})$. Moreover, if $k, \ell \in N$ satisfy $k \prec\!\cdot\, \ell$, then $(\{k, \ell\}, \preceq, v_{\{k,\ell\},x})$ has a single-valued bounded (positive) core by (5.1). By RGP of C^b_+, the unique element is (x_k, x_ℓ), i.e., $x_k = v_{\{k,\ell\},x}(\{k\})$. Hence, $(x_k, x_\ell) \in \sigma(\{k, \ell\}, \preceq, v_{\{k,\ell\},x})$ by NE of σ. By RCRGP, $x \in \sigma(N, \preceq, v)$. **q.e.d.**

Remark 5.8 Assume that $|U| = \infty$. According to Sobolev (1975), the prenucleolus is the unique solution on Γ^{free} that satisfies SIVA, AN, COV, and RGP. For a single-valued solution, however, RGP is equivalent to RCP. Moreover, Orshan (1993) showed that AN may be replaced by the equal treatment property, and Orshan and Sudhölter (2003) proved that the four axioms may be replaced by NE, ETP, COV, and RCP. However, a nonempty solution on Γ that contains the prenucleolus for unrestricted games and satisfies RGP or RCP and COV, cannot coincide with the prenucleolus on Γ^{free} as an easy analysis of the reduced games of $(M', \preceq', u')$ (the game used in the proofs of the characterizations) shows. Hence, in the foregoing sense there is no "prenucleolus" on Γ.

6 Characterizing the positive prekernel and its bounded variant

This section is devoted to generalize the definition and characterization (Sudhölter and Peleg 2000) of the positive prekernel to games with precedence constraints. Let $(N, \preceq, v) \in \Gamma$, $\mathcal{F} = \mathcal{O}(N, \preceq)$, and $\mathcal{R} = \mathcal{R}^{(N,\preceq)}$. For $x \in \mathbb{R}^N$ and $k, \ell \in N, k \neq \ell$, denote by $s_{k\ell}(x, v)$ the *maximum surplus* of k over ℓ at x, defined by

$$s_{k\ell}(x, v) = \sup\{e(S, x, v) \mid \ell \notin S \ni k\}.$$

The *positive prekernel* of $(N, \preceq, v)$ is the set

$$PK_+(N, \preceq, v) = \{x \in X(N, v) \mid s_{k\ell}(x, v) \leqslant s_{\ell k}(x, v)_+ \text{ for all } k, \ell \in N, k \neq \ell\}. \tag{6.1}$$

If the height of $(N, \preceq)$ is not 0, i.e., if restrictions are present, then the prekernel (an element x of which, similarly as in (6.1), has to satisfy $s_{k\ell}(x, v) = s_{\ell k}(x, v)$) is empty. Indeed, if $k \prec \ell$, then $s_{k\ell}(x, v) \in \mathbb{R}$, but $s_{\ell k}(x, v) = \sup \emptyset = -\infty$. Hence, if $x \in PK_+(N, \preceq, v)$, then $s_{k\ell}(x, v) \leqslant 0$. We conclude that

$$PK_+(N, \preceq, v) = \{x \in X(N, v) \mid (x(Q))_{Q \in \mathcal{R}} \in PK_+(\mathcal{R}, v_{\mathcal{R}}) \text{ and } e(S, x, v) \leqslant 0 \text{ for all } S \in \mathcal{F}_0\}. \tag{6.2}$$

Thus, we define the *bounded positive prekernel* by

$$PK_+^b(N, \preceq, v) = \{x \in PK_+(N, \preceq v) \mid \max\{e(S, x, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} = 0 \text{ for all } k, \ell \in N, k \prec\cdot\, \ell\}. \tag{6.3}$$

Remark 6.1 On Γ^{free} the positive prekernel is characterized by NE, AN, REAS, the *weak reduced game property* (WRGP) the definition of which differs from RGP only inasmuch as only reduced games w.r.t. coalitions $S \subseteq N$ with $|S| \leqslant 2$ are considered, CRGP, and *weak unanimity for 2-person games* (WUTPG) requiring that, applied to any 2-person game (N, v), the solution contains $\{x \in X(N, v) \mid x_i \geqslant v(\{i\}) \text{ for } i \in N\}$. (Sudhölter and Peleg 2000, Theorem 7.1)

We now show that the result mentioned in Remark 6.1 may be generalized to Γ. A solution σ on $\Gamma' \subseteq \Gamma$ satisfies WUTPG if for all $(N, \preceq, v) \in \Gamma'$ with $|N| = 2$, $\sigma(N, \preceq, v) \supseteq \{x \in X(N, v) \mid x_i \geqslant v(\{i\}) \text{ for all minimal } i \in N\}$.

Proposition 6.2 *The positive prekernel is characterized by* NE, AN, REAS, WRGP, CRGP, *and* WUTPG.

Proof: By Proposition 2.4 and non-emptiness of PK_+ on Γ^{free}, PK_+ satisfies NE, and clearly it satisfies AN and WUTPG. Moreover, REAS, RGP, and CRGP follow from (6.2) and the corresponding properties on Γ^{free}. In order to show the uniqueness part, let σ be a solution that satisfies the desired properties. By WRGP applied to 1-person reduced games and REAS, σ satisfies PO. Morever, for any $(N, v) \in \Gamma^{\text{free}}$, $\sigma(N, v) = PK_+(N, v)$ by the mentioned classical result. If $(N, \preceq, v) \in \Gamma \setminus \Gamma^{\text{free}}$, then $|N| \geqslant 2$. If $|N| = 2$, then by PO and REAS, $\sigma(N, \preceq, v) \subseteq PK_+(N, \preceq, v)$. By WUTPG, $\sigma(N, \preceq, v) = PK_+(N, \preceq, v)$. If $|N| > 2$ and $x \in \sigma(N, \preceq, v)$, then, by WRGP of σ, $x_S \in \sigma(S, \preceq, v_{S,x})$, and hence, $x_S \in PK_+(S, \preceq, v_{S,x})$, for all

$S \subseteq N$ with $|S| = 2$. By CRGP of PK_+, $x \in PK_+(N, \preceq, v)$. The opposite inclusion follows similarly by interchanging the roles of σ and PK_+. **q.e.d.**

The bounded positive prekernel inherits NE, AN, REAS, RGP, and CRGP from the positive prekernel. Hence, the bounded variant exclusively violates WUTPG of the axioms in Proposition 6.2, i.e., $PK_+^b(N, \preceq, v)$ is a singleton whenever $|N| = 2$ and $(N, \preceq)$ has not height 0. Hence, we note that the bounded positive prekernel is characterized by NE, AN, REAS, WRGP, CRGP, WUTPG$^{\text{free}}$ (requiring that the core is contained in the solution applied to any classical 2-person game), and BOUND.

Here, REAS cannot be deleted. Indeed, let $t > 0$ and, define the following solution σ by the requirement that $x \in \sigma(N, \preceq, v)$ if $s_{k\ell}(x, v) \leqslant (s_{\ell k}(x, v))_+$ for all $k, \ell \in N$, $k \neq \ell$, and $\max_{i \in S \not\ni j, S \in \mathcal{F}} e(S, x, v) = 0$ for all $i, j \in N, i \prec j$, and $v(N) - t \leqslant x(N) \leqslant v(N)$, then PK_+^b is a subsolution of σ that, hence, satisfies NE and WUTPG$^{\text{free}}$. Also, it satisfies BOUND and AN and the remaining properties (WRGP and CRGP) are easily deduced as well. However, σ does not coincide with PK_+^b already for 1-person games.

7 On the logical independence of the employed axioms

By generalizing or slightly modifying the solution concepts $\sigma^1, \ldots, \sigma^5$ of Orshan and Sudhölter (2010, Sect. 4.1), we will show that each of the axioms (1) NE, (2) REAS (respectively, BOUND), (3) COV, (4) RGP, (5) RCP (respectively, RCP$^{\text{free}}$) in Theorem 4.6 or Theorem 5.6, respectively, is logically independent of the remaining axioms. Indeed, for any $(N, \preceq, v) \in \Gamma$ with $\mathcal{F} = \mathcal{O}(N, \preceq)$, $\mathcal{R} = \mathcal{R}^{(N,\preceq)}$, and $\mu = \max_{\mathcal{T} \subseteq \mathcal{R}} e(\mathcal{T}, \nu(\mathcal{R}, v_{\mathcal{R}}), v_{\mathcal{R}})$, let

$$
\begin{aligned}
\sigma^1(N, \preceq, v) &= C(N, \preceq, v); \\
\sigma^{1b}(N, \preceq, v) &= C^b(N, \preceq, v); \\
\sigma^2(N, \preceq, v) &= X(N, \preceq, v); \\
\sigma^3(N, \preceq, v) &= \mathcal{NUC}((\max\{e(S, \cdot, v), t\})_{S \in \mathcal{F}}, X(N, v)) \text{ for some } t < 0; \\
\sigma^{3b}(N, \preceq, v) &= \{x \in \sigma^3(N, \preceq, v) \mid \max\{e(S, x, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} = t \text{ for all } k, \ell \in N, k \prec\cdot\, \ell\}; \\
\sigma^4(N, \preceq, v) &= \{x \in X(N, v) \mid e(\mathcal{T}, (x(Q))_{Q \in \mathcal{R}}, v_{\mathcal{R}}) \leqslant \mu \text{ for } \mathcal{T} \subseteq \mathcal{R} \text{ and } e(S, x, v) \leqslant 0 \text{ for } S \in \mathcal{F}_0\}; \\
s^{4b}(N, \preceq, v) &= \{x \in \sigma^4(N, \preceq, v) \mid \max\{e(S, x, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} = 0 \text{ for all } k, \ell \in N, k \prec\cdot\, \ell\}; \\
\sigma^5(N, \preceq, v) &= PK_+(N, \preceq, v); \text{ and} \\
\sigma^{5b}(N, \preceq, v) &= PK_+^b(N, \preceq, v).
\end{aligned}
$$

Then it is easy to check that σ^i satisfies all axioms from (1) to (5) except (i), for all $i = 1, \ldots, 5$, and similarly for σ^{ib} and the bounded version of the axioms. Also, as mentioned, rint C_+ satisfies all axioms of Theorem 4.6 except CLOS. The solutions σ^1 and σ^{1b}, respectively, also show that NE is logically independent in the characterization of the positive prekernel (Proposition 6.1) and its bounded variant. Similarly, the (bounded) positive core has all properties except CRGP and the correspondence X only violates REAS in Proposition 6.2. The solution that assigns the positive prekernel to each $(N, \preceq, v)$ with

$|N| \leqslant 2$ and otherwise the set of all reasonable preimputations only violates WRGP, and its bounded variant may be used to show that RGP is independent of the remaining axioms in the characterization of the bounded positive prekernel. We now generalize the definition of the solution of Example 8.3 of the aforementioned article. Choose two players, say 1 and 2 and define, for all $(N, \preceq, v) \in \Gamma$, $\sigma^6(N, \preceq, v) = PK_+(N, \preceq, v)$ if $|N| \leq 2$ and $\{1,2\} \nsubseteq N$ or $C(N, \preceq, v) \neq \emptyset$. If $N = \{1,2\}$ and $C(N, \preceq, v) = \emptyset$ (hence the height of $(N, \preceq)$ is 0), put $\sigma^6(N, \preceq, v) = PK(N, v) \cup \{(v(\{1\}), v(N) - v(\{1\})), (v(N) - v(\{2\}), v(\{2\}))\}$. For $|N| \geqslant 3$, put $\sigma^6(N, \preceq, v) = \{x \in X(N, v) \mid x_S \in \sigma^6(S, \preceq, v_{S,x})$ for all $S \subseteq N, |S| = 2\}$ and let σ^{6b} be its bounded pendant. Hence,

$$\begin{aligned}
\sigma^6(N, \preceq, v) &= \{x \in X(N, v) \mid (x(Q))_{Q \in \mathcal{R}} \in \sigma^6(\mathcal{R}, v_{\mathcal{R}}), e(S, x, v) \leqslant 0 \text{ for all } S \in \mathcal{F}_0\} \text{ and} \\
\sigma^{6b}(N, \preceq, v) &= \{x \in \sigma^6(N, \preceq, v) \mid \max\{e(S, x, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} = 0 \text{ for all } k, \ell \in N, k \prec\cdot \ell\}.
\end{aligned}$$

As in the classical case, σ^6 (respectively, σ^{6b}) satisfies NE, REAS (respectively, BOUND), RGP, CRGP, and WUTPG (respectively, WUTPG$^{\text{free}}$); and it violates AN.

Let $t > 0$ and

$$\sigma^7(N, \preceq, v) = \{x \in X(N, v) \mid s_{k\ell}(x, v) \leqslant \max\{-t, s_{\ell k}(x, v)\} \text{ for all } k, \ell \in N, k \neq \ell\}.$$

Then σ^7 exclusively violates WUTPG and the variant defined by

$$\sigma^{7b}(N, \preceq, v) = \{x \in \sigma^6(N, \preceq, v) \mid \max\{e(S, x, v) \mid S \in \mathcal{F}, \ell \notin S \ni k\} = -t \text{ for all } k, \ell \in N, k \prec\cdot \ell\}$$

satisfies the remaining axioms of the bounded positive prekernel.

We don't know if ND is logically independent of the remaining axioms in the characterizing result of the (bounded) positive core. We remark that NE is logically independent of NE, REAS (BOUND), COV, and RGP on Γ^{free} (Orshan and Sudhölter 2010), i.e., there are solutions that satisfy NE, REAS, COV, and RGP, and do neither coincide with the prenucleolus nor with the positive core or its relative interior. But as soon as an axiom is added that guarantees that the positive core is the only solution that has also this property, then it is an open problem whether or not ND is still needed.

Moreover, we have to admit that we don't know whether RCRGP is logically independent of the remaining axioms in Theorem 5.6.

Finally, examples of the mentioned paper may be generalized to show that the infinity assumption on $|U|$ is necessary in the characterization of the (bounded) positive core and that for $|U| = 2$ the statement of Proposition 6.2 is no longer valid.

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