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**Jens Leth Hougaard,**

**Juan D. Moreno-Ternero**

**and**

**Lars Peter Østerdal**

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FURTHER INFORMATION
Department of Business and Economics
Faculty of Social Sciences
University of Southern Denmark
Campusvej 55
DK-5230 Odense M
Denmark

Tel.: +45 6550 3271
Fax: +45 6550 3237
E-mail: lho@sam.sdu.dk
http://www.sdu.dk/ivoe

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# Rationing in the presence of baselines*

**Jens Leth Hougaard**[†] **Juan D. Moreno-Ternero**[‡]

**Lars Peter Østerdal**[§]

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**Abstract**

We analyze a general model of rationing in which agents have baselines, in addition to claims against the (insufficient) endowment of the good to be allocated. Many real-life problems fit this general model (e.g., bankruptcy with prioritized claims, resource allocation in the public health care sector, water distribution in drought periods). We introduce (and characterize) a natural class of allocation methods for this model. Any method within the class is associated with a rule in the standard rationing model, and we show that if the latter obeys some focal properties, the former obeys them too.

***JEL numbers***: *D63.*

***Keywords***: *rationing, baselines, claims, operators, solidarity.*

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†Corresponding author. Institute of Food and Resource Economics, University of Copenhagen, Rolighedsvej 25, DK-1958 Frederiksberg C., Denmark, phone +45 35 33 68 14, email: jlh@foi.dk

‡Department of Economics, Universidad Pablo de Olavide, and CORE, Université catholique de Louvain.

§Department of Business and Economics, University of Southern Denmark.

# 1 Introduction

The problem of dividing when there is not enough is one of the oldest problems in the history of economic thought. Problems of this sort (and possible solutions for them) are already documented in ancient sources, but their formalization is much more recent. O'Neill (1982) was indeed the first to introduce a simple model to analyze the problem in which a group of individuals have conflicting claims over an insufficient amount of a (perfectly divisible) good.[1] Such a model, which will be referred here as the *(standard) rationing model*, accommodates many real-life situations, such as the division of an insufficient estate to cover all its associated debts, the collection of a given tax from taxpayers, the distribution of commodities in a fixed-price setting, sharing the cost of a public facility, etc. It fails, however, to accommodate more complex rationing situations, such as those described next, in which not only claims, but also individual rights, references, or other objective entitlements, play an important role in the rationing process.

One example concerns bankruptcy laws, where some claims are typically prioritized. More precisely, bankruptcy codes normally list all claims that should be treated identically as various categories and assigns to them lexicographic priorities (e.g., Kaminski, 2006). Typically, there exists a category of *secured claims* (involving, for instance, unpaid salaries) receiving the highest priority, which implies that those claims are fully honored (if possible) before allocating the remaining part of the liquidation value among other categories. Such secured claims can be interpreted as a *right* to be considered in the allocation process.

Another example concerns university budgeting procedures, as considered by Pulido et al., (2002, 2008) under the name of *bankruptcy situations with references.* They analyze the real-life case of allocating a given amount of money among the various degree courses that are offered at a (public) Spanish university. The (verifiable) monetary needs of each course constitute their claims. In addition to these claims, there exist *reference* values for each course, which are set by the government independently, below their claims. It is typically the case that the available amount is sufficient to cover all those reference values, but falls short of the aggregate claim. As a result, the reference values are guaranteed for each course, and the remainder is divided according to the claims.

Other relevant practical cases also involving more complex rationing situations could be

[1]The reader is referred to Moulin (2002) or Thomson (2003, 2006) for recent surveys of the sizable related literature.

protocols for the reduction of pollution, water distribution in drought periods, or even some resource allocation procedures in the public health care sector, in which past consumption could be considered as an *entitlement*, and current needs as a claim.

The aim of this paper is to explore a more general model of rationing able to accommodate all the above situations. This model enriches the standard with a *baselines* profile (referring to either individual rights, references, or other objective entitlements), complementing the claims profile of a rationing problem.

We take first a *direct approach* to analyze this new model.[2] That is, we single out a natural class of rules which aims to encompass the real-life rationing situations mentioned above. In short, rules within this class first allocate each agent his baseline and then adjust this tentative allocation by using a (standard rationing) rule to distribute the remaining surplus, or deficit, relative to the initial endowment. In other words, the class arises after submitting the domain of standard rules to an *extension operator* reflecting the two-stage procedure described above.[3] We study the preservation of focal properties under such an operator, which can also be interpreted as the study of the robustness of the class.

We then take an *axiomatic approach* and study the implications of new axioms reflecting ethical or operational principles in this general context. We provide an axiomatic characterization for the class of rules just described.[4]

The rest of the paper is organized as follows. In Section 2, we describe the basic framework of the standard rationing model, as well as the new one to address more general rationing problems. In Section 3, we present our family of rules and study its robustness. In Section 4, we derive the family axiomatically. We conclude in Section 5 with some further insights. For a smooth passage, we defer all the proofs and provide them in the appendix.

[2] The terminology is borrowed from Thomson (2006).

[3] The notion of operators for the space of rules was first introduced by Thomson and Yeh (2008).

[4] There is yet a third approach to rationing problems that we do not consider here: the so-called *game theoretic approach*, which consists in modeling rationing problems as a transferable utility game and aims at identifying the likely outcome of such a game as the solution of the rationing problem. This approach has been taken, for instance, by Pulido et al., (2002, 2008) to analyze the *bankruptcy situations with references* described above, which, as we shall see later, constitute a specific case of our model.

# 2 Model and basic concepts

## 2.1 The benchmark framework

We study rationing problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$, with generic element $N$. Let $n$ denote the cardinality of $N$. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be $i$'s *claim* and $c \equiv (c_i)_{i \in N}$ the claims profile.[5] A *(standard rationing) problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the set of rationing problems with population $N$ and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *(standard) rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem. Each rule $R$ has a *dual* rule $R^*$ defined as $R^*(N, c, E) = c - R(N, c, C - E)$, for each $(N, c, E) \in \mathcal{D}$.

Some classical rules are the *constrained equal awards* rule, which distributes the endowment equally among all agents, subject to no agent receiving more than her claim; the *constrained equal losses* rule, which makes losses as equal as possible, subject to no one receiving a negative amount; and the *proportional* rule, which yields allocations proportionally to claims.[6]

Rules are typically evaluated in terms of the properties (axioms) they satisfy. The literature has provided a wide variety of axioms reflecting ethical or operational principles (e.g., Thomson, 2003; 2006). Here we concentrate on principles with a long tradition in the theory of justice (e.g., Moreno-Ternero and Roemer, 2006).[7]

We start with the requirement of allotting equal amounts to agents with equal claims. Formally, a rule satisfies *equal treatment of equals* if for each $(N, c, E) \in \mathcal{D}$, and each $i, j \in N$, we have $R_i(N, c, E) = R_j(N, c, E)$, whenever $c_i = c_j$.

A strengthening of the previous one says that agents with larger claims receive and lose at least as much as agents with smaller claims. That is, a rule is *order preserving*, if for

[5] For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^n$, let $z_M \equiv (z_i)_{i \in M}$. For each $i \in N$, let $z_{-i} \equiv z_{N \setminus \{i\}}$.

[6] The reader is referred to Moulin (2002) or Thomson (2003, 2006) for their formal definitions as well as for further details about them.

[7] In what follows, we only consider properties that are either *punctual* (i.e., applying to a rule for each problem separately, point by point) or *relational* (i.e., linking the recommendations made by the rule for a finite set of different problems that are related in a certain way).

each $(N,c,E) \in \mathcal{D}$ and $i,j \in N$, $c_i \geq c_j$ implies that $R_i(N,c,E) \geq R_j(N,c,E)$ and $c_i - R_i(N,c,E) \geq c_j - R_j(N,c,E)$. The first inequality is referred to as *order preservation in gains*. The second inequality is referred to as *order preservation in losses*.

We now turn to several formulations of the principle of *solidarity*.[8] The first one says that when there is more to be divided, nobody should lose. Formally, a rule $R$ is *resource monotonic* (e.g., Roemer, 1986) if, for each $(N,c,E) \in \mathcal{D}$ and each $E' > E$, with $E' \leq \sum c_i$, we have $R(N,c,E) \leq R(N,c,E')$. The second one says that if an agent's claim increases, she should receive at least as much as she did initially. Formally, a rule $R$ is *claims monotonic* if, for each $(N,c,E) \in \mathcal{D}$, each $i \in N$, and each $c_i' > c_i$, we have $R_i(N,(c_i', c_{N\setminus\{i\}}),E) \geq R_i(N,(c_i, c_{N\setminus\{i\}}),E)$. A related property says that if an agent's claim and the endowment increase by the same amount, the agent's allocation should increase by at most that amount.[9] Formally, a rule $R$ satisfies *linked claims-resource monotonicity* (e.g., Moreno-Ternero and Villar, 2006b; Thomson and Yeh, 2008) if, for each $(N,c,E) \in \mathcal{D}$ and $i \in N$, $R_i(N,(c_i + \varepsilon, c_{N\setminus\{i\}}), E+\varepsilon) \leq R_i(N,c,E) + \varepsilon$.

We next introduce a solidarity property that pertains to possible variations in populations. It says that if new claimants arrive, each claimant initially present should receive at most as much as she did initially. Equivalently, if some claimants leave but there still is not enough to honor the claims all of the remaining claims, each remaining claimant should receive at least as much as she did initially. Formally, a rule $R$ is *population monotonic* (e.g., Thomson, 1983a,b) if, for each $(N,c,E) \in \mathcal{D}$ and $(N',c',E) \in \mathcal{D}$ such that $N \subseteq N'$ and $c_N' = c$, then $R_i(N',c',E) \leq R_i(N,c,E)$, for each $i \in N$. A related property says that if new claimants arrive and the endowment increases by the sum of their claims, then each claimant initially present should receive at least as much as she did initially. Formally, a rule $R$ satisfies *linked resource-population monotonicity* (e.g., Thomson and Yeh, 2008) if for each $(N,c,E) \in \mathcal{D}$ and each $(N',c',E') \in \mathcal{D}$ such that $N \subseteq N'$, $c_N' = c$, and $E = E'$, then $R_i(N,c,E) \leq R_i\left(N',c',E+\sum_{N'\setminus N} c_j'\right)$, for each $i \in N$.

The next axiom also amounts to simultaneous changes in the endowment and the population. It says that the arrival of new agents should affect all the incumbent agents in the same direction.

[8]Solidarity properties with respect to population changes, and axiomatizations based on it, were actually introduced by Thomson (1983a,b) in related models. Roemer (1986) introduced the solidarity notion referring to the available resource.

[9]In the dual context of taxation, the property reflects a "no donation paradox" condition (e.g., Ju and Moreno-Ternero, 2011).

In other words, agents cannot benefit from a change (either in the available wealth or in the number of agents) if someone else suffers from it. Formally, a rule $R$ satisfies *resource-population uniformity* (e.g., Chun, 1999; Moreno-Ternero and Roemer, 2006), if for each $(N, c, E) \in \mathcal{D}$ and $(N', c', E') \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then, either $R_i(N', c', E') \leq R_i(N, c, E)$, for each $i \in N$, or $R_i(N', c', E') \geq R_i(N, c, E)$, for each $i \in N$. This axiom implies resource monotonicity. In fact, it also implies the following axiom, which relates the allocation of a given problem to the allocations of the subproblems that appear when we consider a subgroup of agents as a new population and the amounts gathered in the original problem as the available endowment. The axiom requires that the application of the rule to each subproblem produces the allocation that the subgroup obtained in the original problem.[10] Formally, a rule $R$ is *consistent* if for each $(N, c, E) \in \mathcal{D}$, each $M \subset N$, and each $i \in M$, we have $R_i(N, c, E) = R_i(M, c_M, E_M)$, where $E_M = \sum_{i \in M} R_i(N, c, E)$. It turns out that consistency and resource monotonicity together imply resource-population uniformity.

To conclude this section, for any given property $\mathcal{P}$, $\mathcal{P}^*$ is the *dual property of* $\mathcal{P}$ if for each rule $R$, $R$ satisfies $\mathcal{P}$ if and only if its dual rule $\mathcal{P}^*$ satisfies $\mathcal{P}^*$. A property is *self-dual* if it coincides with its dual. Equal treatment of equals, order preservation, resource monotonicity, and consistency are self-dual properties. Order preservation in gains and order preservation in losses, claims monotonicity and linked claims-resource monotonicity, and population monotonicity and linked resource-population monotonicity are pairs of dual properties (e.g., Thomson, 2006).

## 2.2 The general framework

We now enrich the model to account for individual baselines that will be part of the rationing process. A *general (rationing) problem* (or problem with baselines) is a tuple consisting of a population $N \in \mathcal{N}$, a *baselines* profile $b \in \mathbb{R}^n_+$, a claims profile $c \in \mathbb{R}^n_+$, and an endowment $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. We denote by $\mathcal{E}^N$ the class of general problems with population $N$ and $\mathcal{E} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$. For each general problem $(N, b, c, E) \in \mathcal{E}$, let $t_i(b, c) = \min\{b_i, c_i\}$, for each $i \in N$, and $t(b, c) = \{t_i(b, c)\}_{i \in N}$ denote the corresponding (baseline-claim) truncated vector. Let $T = \sum_{i \in N} t_i(b, c)$. Given a general problem $(N, b, c, E) \in \mathcal{E}$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. A *(general) rule* on $\mathcal{E}$, $S \colon \mathcal{E} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each (general) problem $(N, b, c, E) \in \mathcal{E}$ an allocation $x = S(N, b, c, E)$ for the problem.

[10]See Thomson (2007) for an excellent survey of the many applications that have been made on this idea.

# 3 The direct approach to rationing in the presence of baselines

## 3.1 A family of rules

We look for rules to solve rationing problems in the presence of baselines. We propose to first assign agents their truncated baselines, and to allocate the resulting deficit, or surplus, using a standard rule for the standard problem that results after embedding baselines into claims. Specifically, a deficit is allocated according to the amounts already received by the agents, whereas a surplus is allocated according to the gap between their claims and what has already been allocated to them. An *extension operator* associates with each rule defined for the standard model a rule for the general model. We refer to our specific extension operator as the *Baselines First extension operator*.

Formally,

$$\widetilde{R}\,(N,b,c,E) = \begin{cases} t(b,c) - R(N,t(b,c),T-E) & \text{if } \; E \leq T \\ t(b,c) + R(N,c-t(b,c),E-T) & \text{if } \; E \geq T \end{cases} \tag{1}$$

We shall refer to $\widetilde{R}$ as the extended rule induced by $R$, via the Baselines First extension operator, or, simply, the extended rule induced by $R$.

Note that, if $b = 0$, then $\widetilde{R} \equiv R$. More interestingly, for any rule $R$, and any problem $(N, b, c, E)$, the induced extended rule $\widetilde{R}$ results in an allocation $x$ satisfying

$$x_i \leq t_i(b,c) \text{ for all } i \in N \text{ if and only if } E \leq T,$$

$$x_i \geq t_i(b,c) \text{ for each } i \in N \text{ if and only if } E \geq T.$$

As is the case with the so-called *Talmud* rule (e.g., Aumann and Maschler, 1985), rules obtained via our operator impose a rationing of the same sort for each individual and the whole society, but according to the profile of baselines.

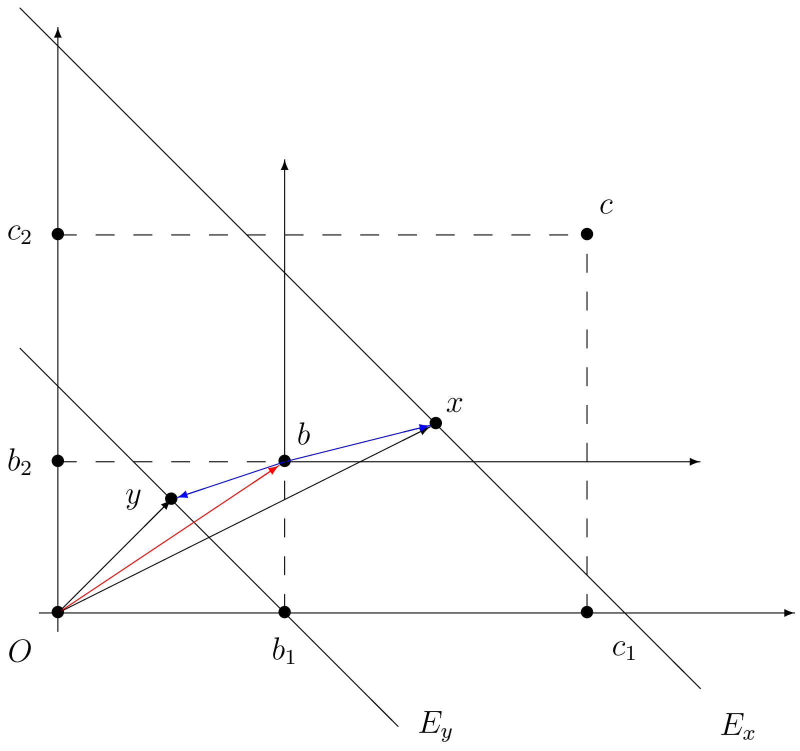


**Figure 1: Extended rules in the two-claimant case**. This figure illustrates how rules extended via de Baselines First operator behave for $N = \{1, 2\}$, and $b, c \in \mathbb{R}^N_+$, with $c_i > b_i$, for $i = 1, 2$. If the endowment is $E_x > b_1 + b_2$, then the proposed allocation $x$ can be decomposed as $b + (x - b)$ where $x - b$ is to be interpreted as the allocation for the problem arising after adjusting claims (and endowment) down by the baselines, i.e., $x = b + R(N, c - b, E_x - b_1 - b_2)$. If, however, $E_y < b_1 + b_2$ then the proposed allocation $y$ can be decomposed as $b - (b - y)$ where $b - y$ is to be interpreted as the allocation for the problem arising after replacing claims by baselines and the endowment by the difference between the aggregate baseline and the original endowment, i.e., $b - y = R(N, b, b_1 + b_2 - E_y)$.

It is not difficult to show that the following expression is equivalent to (1):

$$\widetilde{R}\,(N, b, c, E) = \begin{cases} R^*(N, t(b, c), E) & \text{if} \;\; E \leq T \\ t(b, c) + R(N, c - t(b, c), E - T) & \text{if} \;\; E \geq T \end{cases} \tag{2}$$

It follows from (2) that if each individual baseline is one half of the corresponding claim, then the rule induced by the constrained equal losses rule described above would solve the problem with baselines as the so-called Talmud rule would solve the original problem. Similarly, the rule induced by the constrained equal awards rule would solve the problem with baselines as the so-called *Reverse Talmud* rule (e.g., Chun et al., 2001) would solve the original problem. If instead of one half, baselines are equal to any other fixed proportion of claims, $\theta \in (0, 1)$, then the rule induced by the constrained equal losses rule would solve the problem with baselines as the corresponding member of the so-called TAL-family of rules (e.g., Moreno-Ternero and Villar, 2006a) would solve the original problem; also, the rule induced by the constrained equal awards rule would solve the problem with baselines as the corresponding member of the so-called Reverse TAL-family (e.g., van den Brink et al., 2008) would solve the original problem.

## 3.2 Preservation of axioms

As mentioned above, the Baselines First extension operator associates with each standard rule a new rule to solve (rationing) problems with baselines. A natural question is whether the rule so constructed inherits the properties of the original rule. We say that an axiom is *preserved* by our Baselines First extension operator if whenever a rule $R$ satisfies it, then the induced rule $\widetilde{R}$ satisfies the corresponding *general* version of the axiom.[11]

Our first result is that some of the solidarity axioms described above are preserved by the Baselines First extension operator.

**Theorem 1** *The properties of resource monotonicity, consistency, and resource-population uniformity are preserved by the Baselines First extension operator.*

However, many of the remaining well-known axioms in the benchmark framework are not preserved, as stated in the next result.

**Theorem 2** *If a property is not self-dual then it is not preserved by the Baselines First extension operator.*

Not all self-dual properties are preserved. An obvious counterexample is equal treatment of equals, which is not satisfied by an induced rule if two agents with equal claims may have different baselines. A natural question is whether pairs of dual properties are preserved. An obvious counterexample is order preservation, which is the pair formed by order preservation in gains and order preservation in losses. This property is not necessarily satisfied by the rule induced by an order-preserving rule, if baselines are not ordered as claims. Nevertheless, some pairs are preserved, either by themselves or "with the assistance" of an additional property, as shown in the next result.[12]

**Theorem 3** *The following pairs are preserved by the Baselines First extension operator: claims monotonicity and its dual; population monotonicity and its dual, if assisted by resource monotonicity.*

In other words, if a rule $R$ satisfies claims monotonicity and linked claims-resource monotonicity, then the induced rule $\widetilde{R}$ satisfies the corresponding two general properties. Also, if a rule $R$ satisfies

[11]For ease of exposition, we skip the straightforward definitions of the general versions of each axiom introduced above. Just as an illustration, we say, for instance, that a rule $S$ satisfies claims monotonicity if for each $(N, b, c, E), (N, b, c', E) \in \mathcal{E}$ such that $c_i \leq c'_i$ for some $i \in N$, and $c'_{N\setminus\{i\}} \equiv c_{N\setminus\{i\}}$, we have that $S_i(N, b, c, E) \leq S_i(N, b, c', E)$. As for variable-population axioms, we say, for instance, that a rule $S$ satisfies consistency if for each $(N, b, c, E) \in \mathcal{E}$ and $M \subset N$, $\widetilde{R}(M, b_M, c_M, \sum_{i\in M} x_i) = x_M$, where $x = \widetilde{R}\,(N, b, c, E)$.

[12]The terminology is borrowed from Hokari and Thomson (2008).

population monotonicity and linked resource-population monotonicity, and in addition is resource monotonic, then the induced rule $\widetilde{R}$ satisfies the corresponding three general properties.

To conclude with this section, we focus on equal treatment of equals and order preservation, which, as mentioned above, are not preserved by the Baselines First extension operator. Some caveats are in order.

First, equal treatment of equals should not be expected to be preserved because the direct extension of this property to the enriched model does not include hypotheses about baselines. One might argue that this property does not have a direct transcription to the general model, that two agents should be considered equals only if their claims and their baselines are equal. Thus, the proper (and straightforward) statement should be the following: On the subdomain of problems in which baselines are equals, equal treatment of agents with equal claims (on that domain, such agents can be said to be *truly* equals) is preserved by the Baselines First extension operator.

As for order preservation, we impose additional conditions on baselines, to guarantee its preservation. We say that *baselines are ordered like claims* if whenever $c_i \leq c_j$ then $b_i \leq b_j$. Similarly, we say that *claim-baseline differences are ordered like claims* if whenever $c_i \leq c_j$ then $c_i - b_i \leq c_j - b_j$.

**Theorem 4** *If baselines are ordered like claims, and $R$ is order preserving, then $\widetilde{R}$ is order preserving in gains. If baselines and claim-baseline differences are ordered like claims then order preservation is preserved by the Baselines First extension operator.*

| **Properties** | **Preservation** |
|---|---|
| Resource monotonicity | + |
| Consistency | + |
| Resource-population uniformity | + |
| Claims monotonicity + its dual | + |
| Population monotonicity + its dual | + (RM) |
| Equal treatment of equals | + (*) |
| Order preservation | + (**) |

**Table 1: Preservation of axioms**. This table summarizes the behavior of the Baselines First extension operator with respect to the properties we have considered. A plus sign indicates that the property (or the pair of properties) is preserved. A parenthesis after the plus sign indicates that the preservation occurs with a proviso. For instance, under the presence of resource monotonicity (RM), the pair of properties formed by population monotonicity and its dual is preserved. Similarly,

if baselines are equals for agents with equal claims (*), equal treatment of equals is preserved. Finally, if baselines and claim-baseline differences are ordered like claims (**), then order preservation is preserved.

# 4 The axiomatic approach

In this section, we take a different approach to the problem of rationing in the presence of baselines. We now provide some (new) axioms conveying natural ways of taking baselines into account, while designing the rationing scheme, and study their implications. As we shall show, a combination of these axioms will lead to a characterization of the family of rules presented above.

Our first axiom requires a baseline to be disregarded if it is above the corresponding claim.[13] Formally, a rule $S$ satisfies *baseline truncation* if, for each $(N, b, c, E) \in \mathcal{E}$, $S(N, b, c, E) = S(N, t(b, c), c, E)$. The rationale is that as no agent can get more than her claim, as stated in the definition of (general) rules, baselines above that level should be considered irrelevant.

The second axiom requires to disregard the amount of a claim exceeding its corresponding baseline, whenever truncated baselines cannot be jointly covered. Formally, a rule $S$ satisfies *truncation of excessive claims* if, for each $(N, b, c, E) \in \mathcal{E}$ such that $E \leq T$, $S(N, b, c, E) = S(N, b, t(b, c), E)$. The rationale for this idea is somewhat related to the rationale for the previous one. Namely, as not all baselines can be honored, no agent will achieve more than her baseline and, thus, the portion of their claims above their baselines should be considered irrelevant.

The third axiom complements the previous one as it refers to a situation where all truncated baselines can be covered. It states that, in such a case, if an individual's claim and baseline are reduced by an amount $k_i$, then such reduction in the endowment should be taken from that individual while the others remain unaffected. Formally, a rule $S$ satisfies *baseline invariance* if for each $(N, b, c, E) \in \mathcal{E}$ such that $E \geq T$, and $k \in \mathbb{R}^n_+$ such that $k_j \leq t_j(b, c)$, for each $j \in N$, then $S(N, b, c, E) = k + S(N, b - k, c - k, E - \sum_{i \in N} k_i)$. In particular, the axiom says that, when all truncated baselines can be covered, the rationing problem can be solved in two stages; first, a certain amount is granted to all agents, second, we solve the resulting problem after adjusting down baselines and claims by such amount, and the endowment by the corresponding aggregated amount.[14]

Finally, we consider a fourth axiom dealing with the two polar cases of *non-informative* baselines, and inspired by the notion of self-duality from the standard model of rationing. It states that a rule should allocate amounts for a problem with null baselines in the same way as it allocates losses for the corresponding problem in which baselines are equal to claims. Formally, a rule $S$ satisfies *polar*

[13]This property is reminiscent of the so-called independence of irrelevant claims axiom (e.g., Dagan, 1996).

[14]This is somehow reminiscent of the composition from minimal rights axiom (e.g., Dagan, 1996).

*baseline self-duality* if, for each $(N, c, E) \in \mathcal{D}$, $S(N, 0, c, E) = c - S(N, c, c, C - E)$.[15]

As the next theorem shows, these four axioms together characterize our family of rules introduced above.

**Theorem 5** *A rule satisfies baseline truncation, truncation of excessive claims, baseline invariance and polar baseline self-duality if and only if it is obtained via the Baselines First extension operator.*

As shown in the appendix, Theorem 5 is tight. It turns out that the first three axioms of its statement characterize the family of extended rules arising from using (possibly) different rules when $T \geq E$ or $T \leq E$ in the definition of the Baselines First extension operator. More precisely, let us define the *generalized Baselines First extension operator* (from the cartesian product of the space of standard rules to the set of general rules) by

$$\widetilde{RU}\,(N, b, c, E) = \begin{cases} t(b, c) - R(N, t(b, c), T - E) & \text{if } E \leq T \\ t(b, c) + U(N, c - t(b, c), E - T) & \text{if } E > T \end{cases},$$

where $R$ and $U$ are standard (rationing) rules. Then, we have the following:

**Theorem 6** *A rule satisfies baseline truncation, truncation of excessive claims and baseline invariance if and only if it is obtained via the generalized Baselines First extension operator.*

# 5 Final remarks

We have formulated a general framework to analyze general rationing problems in which agents have claims over the (insufficient) endowment, but also relevant baselines for the allocation. An individual baseline can be interpreted as an objective entitlement (as in budgeting situations), a right (as in the case of unpaid salaries in bankruptcy situations), or as a measure of needs (as in health care prioritization). We have introduced a natural family of rules and argued that this family encompasses a series of real-life rationing situations. It is somewhat surprising that with little or no structure on the arbitrarily chosen baseline profiles, this family still proves relatively robust in preserving a series of well known and desirable properties from the standard rationing model.

Our contribution is somehow reminiscent of a route previously explored for cooperative models of bargaining. A variety of studies has extended Nash's original bargaining model by specifying an additional reference point to the disagreement point, which plays a role in the bargaining solution.

[15] Note that the standard definition of self-duality does not make sense in the present context as there is no clear definition of a loss when both baselines and claims are in play. That is why we restrict the notion only to the two polar cases of *non-informative* baselines.

Such a reference point can be interpreted as a status quo, as a first step towards the final compromise, or simply as a vector of claims (e.g., Gupta and Livne, 1988; Thomson, 1994). It is the obvious counterpart to the baselines profile in our setting.

In a companion paper (Hougaard et al., 2011) we have dealt with an alternative model of rationing in the presence of baselines, in which baselines are endogenously generated. Instead of assuming that baselines are exogenously given, we assume that they are determined as a function of the components of a standard rationing problem. Instances of such endogenous baselines are the so-called *lower bounds*, which guarantee each claimant a minimal amount as a function of their claims and the available amount (e.g., Moreno-Ternero and Villar, 2004). In that model, rather than defining an extension operator (to move from the set of rules for standard problems to the set of rules for problems with baselines), we define a family of operators (one for each baseline) on the space of standard rules. Each operator associates each rule with another defined by a two-step procedure, somewhat similar to the one we describe in this paper, and inspired by the so-called composition properties, which pertain to the way in which (standard) rules allocate tentative estimations of the endowment (e.g., Moulin, 2000). The resulting family of composition operators encompasses (and generalizes) two focal operators, known as the attribution of minimal rights and truncation operators respectively (e.g., Thomson and Yeh, 2008).

Finally, we believe that our work can also shed some light in the search of a dynamic rationale for some classical rules. More precisely, imagine we consider a sequence of rationing problems (involving the same group of agents), at different periods of time, whose period-wise allocations might not only be determined by the data of the rationing problem at such period, but also by the allocations in previous periods. A plausible way to start approaching this issue would be by assuming that, at each period, the corresponding rationing problem is enriched by an index summarizing the amounts each agent obtained in the previous period. If so, we would just be providing an alternative interpretation for the baselines profile that we consider in our model. Needless to say, it would be interesting to go beyond this point and, ultimately, to provide a dynamic rationale for some classical rules, as recently done by Fleurbaey and Roemer (2011) for the three canonical axiomatic bargaining solutions.

# 6 Appendix: Proofs of the results

## Proof of Theorem 1

**Resource monotonicity**. Let $R$ be a rule satisfying resource monotonicity. Let $(N, c, E) \in \mathcal{D}$ and each $E' > E$, with $E' \leq \sum c_i$. Let $b \in \mathbb{R}^n$ be a baseline profile and let $i \in N$ be a given agent. The aim is to show that $\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_i(N, b, c, E')$. To do so, we distinguish three cases.

**Case 1**: $E < E' \leq T$.

Then, $\widetilde{R}_i(N, b, c, E) = R_i^*(N, t(b, c), E)$ and $\widetilde{R}_i(N, b, c, E') = R_i^*(N, t(b, c), E')$. Now, as resource monotonicity is a self-dual property, it follows that $R^*$ satisfies resource monotonicity too and hence $R_i^*(N, t(b, c), E) \leq R_i^*(N, t(b, c), E')$, as claimed.

**Case 2**: $T \leq E < E'$.

Then,

$$\widetilde{R}_i(N, b, c, E) = t_i(b, c) + R_i(N, c - t(b, c), E - T) \leq t_i(b, c) + R_i(N, c - t(b, c), E' - T) = \widetilde{R}_i(N, b, c, E'),$$

where the inequality follows from the fact that $R$ satisfies resource monotonicity.

**Case 3**: $E < T < E'$.

Then, the definition guarantees that $\widetilde{R}_i(N, b, c, E) \leq t_i(b, c) \leq \widetilde{R}_i(N, b, c, E')$.

**Consistency**. Let $R$ be a rule satisfying consistency. Let $(N, c, E) \in \mathcal{D}$ and $b \in \mathbb{R}_+^n$. Let $x = \widetilde{R}(N, b, c, E)$. The aim is to show that, for any $M \subset N$,

$$\widetilde{R}(M, b_M, c_M, \sum_{i \in M} x_i) = x_M.$$

Fix $M \subset N$ and let $E' = \sum_{i \in M} x_i$ and $T' = \sum_{j \in M} t_j(b, c)$. Thus, $E \leq T$ if and only if $E' \leq T'$. We then distinguish two cases.

**Case 1**: $E \leq T$.

Then, $x_i = t_i(b, c) - R_i(N, t(b, c), T - E)$ for each $i \in N$, and thus $T' - E' = \sum_{i \in M} R_i(N, t(b, c), T - E)$. Therefore, $\widetilde{R}_i(M, b_M, c_M, E') = t_M(b, c) - R_i(M, t_M(b, c), T' - E')$ for all $i \in M$. Now, as $R$ is consistent, it follows that $R_i(N, t(b, c), T - E) = R_i(M, t_M(b, c), T' - E')$, for each $i \in M$, which concludes the proof for this case.

**Case 2**: $E \geq T$.

Then, $x_i = t_i(b, c) + R_i(N, c - t(b, c), E - T)$ for each $i \in N$, and thus $T' - E' = \sum_{i \in M} R_i(N, c - t(b, c), E - T)$. Therefore, $\widetilde{R}_i(M, b_M, c_M, E') = t_i(b, c) + R_i(M, c_M - t_M(b, c), E' - T')$ for all $i \in M$. Now, as $R$ is consistent, it follows that $R_i(N, c - t(b, c), E - T) = R_i(M, c_M - t_M(b, c), E' - T')$, for each $i \in M$, which concludes the proof for this case.

**Resource-population uniformity** follows from the first two statements and the relationship among the axioms described in Section 2. ■

## Proof of Theorem 2

Let $\mathcal{P}$ be a property and $\mathcal{P}^*$ be its dual. Let $R$ be a rule satisfying $\mathcal{P}$, but not $\mathcal{P}^*$. Then, $R^*$, the dual rule of $R$, satisfies $\mathcal{P}^*$ but not $\mathcal{P}$.

If $\mathcal{P}$ is "punctual" then there exists a problem $(N, c, E) \in \mathcal{D}$ for which $R^*$ violates $\mathcal{P}$. We then consider the corresponding problem with baselines $(N, b, c, E) \in \mathcal{E}$ in which $b = c$. It then follows that $\widetilde{R}(N, b, c, E) = R^*(N, c, E)$ and hence, we conclude that $\widetilde{R}$ violates $P$.

If $\mathcal{P}$ is "relational" a similar argument applies. For ease of exposition, we assume that $\mathcal{P}$ only involves a finite collection of problems. Formally, if $R^*$ violates $\mathcal{P}$ then there exists a collection of problems $\{(N^j, c^j, E^j)\}_{j=1,\dots,k} \subset \mathcal{D}$ for which $\mathcal{P}$ is violated. Now, consider the corresponding problems with baselines $\{(N^j, b^j, c^j, E^j)\}_{j=1,\dots,k} \subset \mathcal{E}$ where, for each $j = 1, \dots, k$,

$$b_i^j = \begin{cases} \max_{l=1,\dots,k}\{c_i^l\}, & \text{if } i \in \bigcap_{l=1,\dots,k} N^l \\ c_i^j & \text{if } i \in N^j \setminus \bigcap_{l=1\dots k} N^l \end{cases}$$

It is straightforward to show that, for each $j = 1, \dots, k$, $\widetilde{R}(N^j, b^j, c^j, E^j) = R^*(N^j, c^j, E^j)$. Therefore, $\widetilde{R}$ violates $\mathcal{P}$. ■

## Proof of Theorem 3

Let $R$ be a rule satisfying claims monotonicity and linked claims-resource monotonicity. Our aim is to show that $\widetilde{R}$ satisfies the general versions of the two properties.

**Claims monotonicity**. Let $(N, c, E) \in \mathcal{D}$ and $i \in N$, such that $c_i \leq c_i'$. Let $b \in \mathbb{R}_+^n$ be a baseline profile, $T = \sum_{j \in N} t_j(b, c)$, and $T' = \sum_{j \in N} t_j(b, c')$.[16] The aim is to show that

$$\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_i(N, b, c', E). \tag{3}$$

We distinguish three cases:

**Case 1**: $E \leq T$.

Then, $\widetilde{R}_i(N, b, c, E) = R_i^*(N, t(b, c), E)$ and

$$\widetilde{R}_i(N, b, c', E) = R_i^*(N, t(b, c'), E) = R_i^*(N, (t_{N\setminus\{i\}}(b, c), t_i(b, c')), E).$$

As $R$ satisfies linked claims-resource monotonicity, it follows that $R^*$ satisfies claims monotonicity, from where we obtain (3).

**Case 2**: $E \geq T'$.

Then, $\widetilde{R}_i(N, b, c, E) = t_i(b, c) + R_i(N, c - t(b, c), E - T)$, and

$$\widetilde{R}_i(N, b, c', E) = t_i(b, c') + R_i(N, c' - t(b, c'), E - T') =$$

$$= t_i(b, c') + R_i(N, (c - t(b, c))_{N\setminus\{i\}}, c_i' - t_i(b, c')), E - T - t_i(b, c') + t_i(b, c)).$$

Let $\varepsilon = t_i(b, c') - t_i(b, c) \geq 0$. Then, (3) is equivalent to

$$R_i(N, c - t(b, c), E - T) \leq \varepsilon + R_i(N, ((c - t(b, c))_{N\setminus\{i\}}, c_i' - t_i(b, c')), E - T - \varepsilon). \tag{4}$$

We distinguish three subcases.

**Case 2.1**: $b_i > c_i'$.

[16] Note that $t_{N\setminus\{i\}}(b, c') \equiv t_{N\setminus\{i\}}(b, c)$, $t_i(b, c') \geq t_i(b, c)$ and thus, $T' \geq T$.

Here, $t_i(b,c) = c_i < c'_i = t_i(b,c')$ (and thus $\varepsilon = c'_i - c_i$). Then, (4) becomes

$$R_i(N,((c-t(b,c))_{N\setminus\{i\}},0),E-T) \leq \varepsilon + R_i(N,((c-t(b,c))_{N\setminus\{i\}},0),E-T-\varepsilon).$$

Now, by claims monotonicity of $R$,

$$R_i(N,((c-t(b,c))_{N\setminus\{i\}},0),E-T) \leq R_i(N,((c-t(b,c))_{N\setminus\{i\}},\varepsilon),E-T).$$

And, by linked claims-resource monotonicity of $R$,

$$R_i(N,((c-t(b,c))_{N\setminus\{i\}},\varepsilon),E-T) \leq \varepsilon + R_i(N,((c-t(b,c))_{N\setminus\{i\}},0),E-T-\varepsilon),$$

which concludes the proof in this case.

**Case 2.2**: $c'_i \geq b_i \geq c_i$.

Here, $t_i(b,c) = c_i \leq b_i = t_i(b,c')$ (and thus $\varepsilon = b_i - c_i$). Then, (4) becomes

$$R_i(N,((c-t(b,c))_{N\setminus\{i\}},0),E-T) \leq b_i - c_i + R_i(N,((c-t(b,c))_{N\setminus\{i\}},c'_i - b_i),E-T-b_i+c_i).$$

Now, by claims monotonicity of $R$,

$$R_i(N,((c-t(b,c))_{N\setminus\{i\}},0),E-T) \leq R_i(N,((c-t(b,c))_{N\setminus\{i\}},c'_i - c_i),E-T).$$

And, by linked claims-resource monotonicity of $R$,

$$R_i(N,((c-t(b,c))_{N\setminus\{i\}},c'_i-c_i),E-T) \leq \varepsilon + R_i(N,((c-t(b,c))_{N\setminus\{i\}},c'_i-c_i-\varepsilon),E-T-\varepsilon),$$

which concludes the proof in this case.

**Case 2.3**: $c_i > b_i$.

Here, $t_i(b,c) = b_i = t_i(b,c')$ (and thus $\varepsilon = 0$), from where (4) trivially follows as a consequence of the fact that $R$ satisfies claims monotonicity.

**Case 3**: $T < E < T'$. Then, $\widetilde{R}_i(N,b,c,E) = t_i(b,c) + R_i(N,c-t(b,c),E-T)$, and $\widetilde{R}_i(N,b,c',E) = R^*_i(N,t(b,c'),E) = t_i(b,c') - R_i(N,(t_{N\setminus\{i\}}(b,c),t_i(b,c')),T'-E)$. Thus, in order to prove (3), it suffices to show that

$$R_i(N,(t_{N\setminus\{i\}}(b,c),t_i(b,c')),T'-E) + R_i(N,c-t(b,c),E-T) \leq t_i(b,c') - t_i(b,c) \tag{5}$$

Note that Case 3 implies that $t_i(b,c) = c_i$ (otherwise, $t_i(b,c) = b_i$ and hence $T = T'$). Thus, by boundedness, $R_i(N,c-t(b,c),E-T) = 0$. Also, by balance and boundedness, $R_i(N,(t_{N\setminus\{i\}}(b,c)),t_i(b,c')),T'-E) \leq T'-E = T-E+t_i(b,c')-t_i(b,c) \leq t_i(b,c')-t_i(b,c)$, from where (5) follows.

**Linked claims-resource monotonicity**. Let $(N,c,E) \in \mathcal{D}$ and $i \in N$. Let $b \in \mathbb{R}^n_+$ be a baseline profile, $\varepsilon > 0$ and $c' = (c_i+\varepsilon, c_{N\setminus\{i\}})$. Let $T' = T + t_i(b,c') - t_i(b,c)$. Then, $t_i(b,c') \leq t_i(b,c)+\varepsilon$ and $T \leq T' \leq T+\varepsilon$. The aim is to show that

$$\widetilde{R}_i(N,b,(c_i+\varepsilon,c_{N\setminus\{i\}}),E+\varepsilon) \leq \widetilde{R}_i(N,b,c,E)+\varepsilon \tag{6}$$

We distinguish several cases:

**Case 1**: $E \leq T' - \varepsilon$.

Then, $\widetilde{R}_i(N, b, c, E) = R_i^*(N, t(b, c), E)$ and $\widetilde{R}_i(N, b, c', E+\varepsilon) = R_i^*(N, t(b, c'), E+\varepsilon) = R_i^*(N, (t_i(b, c'), t_{-i}(b, c))$ $\varepsilon)$. By claims monotonicity $R^*$, $R_i^*(N, t(b, c'), E+\varepsilon) \leq R_i^*(N, (t_i(b, c)+\varepsilon, t_{-i}(b, c)), E+\varepsilon)$. By linked claims-resource monotonicity of $R^*$, $R_i^*(N, (t_i(b, c)+\varepsilon, t_{-i}(b, c)), E+\varepsilon) \leq R_i^*(N, t(b, c), E) + \varepsilon$, from where (6) follows.

**Case 2**: $E \geq T$.

Then, $\widetilde{R}_i(N, b, c, E) = t_i(b, c) + R_i(N, c - t(b, c), E - T)$ and $\widetilde{R}_i(N, b, (c_i+\varepsilon, c_{N\setminus\{i\}}), E+\varepsilon) = t_i(b, c') + R_i(N, ((c_i+\varepsilon, c_{N\setminus\{i\}}) - (t_i(b, c'), t_{-i}(b, c))), E+\varepsilon-T')$. As $(E-\varepsilon-T') - (E-T) = \varepsilon - (t_i(b, c') - t_i(b, c))$ and $(c_i + \varepsilon - t_i(b, c')) - (c_i - t_i(b, c)) = \varepsilon - (t_i(b, c') - t_i(b, c))$, (6) follows from linked claims-resource monotonicity of $R$.

**Case 3**: $T' - \varepsilon < E < T$.

Then, $\widetilde{R}_i(N, b, (c_i+\varepsilon, c_{N\setminus\{i\}}), E+\varepsilon) = t_i(b, c') + R_i(N, (c_i+\varepsilon-t_i(b, c'), (c-t(b, c))_{-i}), E+\varepsilon-T')$, and $\widetilde{R}_i(N, b, c, E) = t_i(b, c') - R_i(N, t(b, c), T-E)$. Thus, (6) becomes

$$\varepsilon - t_i(b, c') + t_i(b, c) \geq R_i(N, t(b, c), T-E) + R_i(N, (c_i+\varepsilon-t_i(b, c'), (c-t(b, c))_{-i}), E+\varepsilon-T') \quad (7)$$

Now, by balance and boundedness, the right hand side of (7) is bounded above by $T - E + E + \varepsilon - T'$, which is precisely the left hand side of (7).

Let now $R$ be a rule satisfying resource monotonicity, population monotonicity and linked resource-population monotonicity. By Theorem 1, $\widetilde{R}$ satisfies the general property of resource monotonicity. Our aim is to show that $\widetilde{R}$ also satisfies the general properties of population monotonicity and linked resource-population monotonicity.

**Population monotonicity**. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E') \in \mathcal{D}$ be such that $N \subseteq N'$, $c'_N = c$ and $E = E'$. Let $b \in \mathbb{R}^n_+$ and $b' \in \mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N = b$. Note that $t_j(b', c') = t_j(b, c)$ for each $j \in N$. Finally, let $T = \sum_{j\in N} t_j(b, c)$ and $T' = \sum_{j\in N'} t_j(b', c')$. The aim is to show that

$$\widetilde{R}_i(N', b', c', E) \leq \widetilde{R}_i(N, b, c, E), \quad (8)$$

for each $i \in N$.

We distinguish several cases:

**Case 1**: $E \leq T$.

Then, $\widetilde{R}_i(N', b', c', E) = R_i^*(N', t(b', c'), E)$ and $\widetilde{R}_i(N, b, c, E) = R_i^*(N, t(b, c), E)$. As $R$ satisfies linked resource-population monotonicity, it follows that $R^*$ satisfies population monotonicity, from where we obtain (8).

**Case 2**: $E \geq T'$.

Then, $\widetilde{R}_i(N', b', c', E) = t_i(b', c') + R_i(N', c' - t(b', c'), E - T')$ and $\widetilde{R}_i(N, b, c, E) = t_i(b, c) + R_i(N, c - t(b, c), E - T)$. As $R$ satisfies resource monotonicity and population monotonicity, (8) follows.

**Case 3**: $T < E < T'$.

Then, the definition guarantees that $\widetilde{R}_i(N', b', c', E) \leq t_i(b', c') = t_i(b, c) \leq \widetilde{R}_i(N, b, c, E)$.

**Linked resource-population monotonicity**. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E) \in \mathcal{D}$ be such that $N \subseteq N'$ and $c'_N = c$. Let $b \in \mathbb{R}^n_+$ and $b' \in \mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N = b$. Note that $t_j(b', c') = t_j(b, c)$ for each $j \in N$. Finally, let $T = \sum_{j \in N} t_j(b, c)$, and $T' = \sum_{j \in N'} t_j(b', c')$. The aim is to show that, for each $i \in N$,

$$\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_i(N', b', c', E'), \tag{9}$$

where $E' = E + \sum_{N' \setminus N} c'_j$.

We distinguish several cases:

**Case 1**: $E \leq T - \sum_{N' \setminus N} \left( c'_j - t_j(b', c') \right)$.

Then, $E \leq T$ and $E' \leq T'$ and, therefore, $\widetilde{R}_i(N', b', c', E') = R^*_i(N', t(b', c'), E')$ and $\widetilde{R}_i(N, b, c, E) = R^*_i(N, t(b, c), E)$. By resource monotonicity and population monotonicity of $R^*$, (9) follows.

**Case 2**: $T - \sum_{N' \setminus N} \left( c'_j - t_j(b', c') \right) \leq E \leq T$.

Then, $E \leq T$ whereas $E' \geq T'$ and, hence, the definition guarantees that $\widetilde{R}_i(N', b', c', E') \geq t_i(b', c') = t_i(b, c) \geq \widetilde{R}_i(N, b, c, E)$, as desired.

**Case 3**: $E \geq T$.

Then, $E' \geq T'$ and, hence, $\widetilde{R}_i(N', b', c', E') = t_i(b', c') + R_i(N', c' - t(b', c'), E' - T')$ and $\widetilde{R}_i(N, b, c, E) = t_i(b, c) + R_i(N, c - t(b, c), E - T)$. It is straightforward to show that $E' - T' = E - T + \sum_{N' \setminus N} \left( c'_j - t_j(b', c') \right)$. Thus, (9) follows from the fact that $R$ satisfies resource -and-population monotonicity. ■

## Proof of Theorem 4

We only give the proof of the second statement, as the other two are straightforward. Let $R$ be a rule satisfying order preservation and let $(N, b, c, E)$ be a problem for which baselines are ordered like claims. Let $i, j \in N$ be such that $c_i \leq c_j$ (and hence $b_i \leq b_j$). We aim to show that

$$\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_j(N, b, c, E)$$

To do so, we distinguish two cases.

**Case 1**: $E \leq T$.

Then, $\widetilde{R}_i(N, b, c, E) = R^*_i(N, t(b, c), E)$ and $\widetilde{R}_j(N, b, c, E) = R^*_j(N, t(b, c), E)$. Now, as order preservation is a self-dual property, it follows that $R^*$, the dual rule of $R$, is order preserving too. As $b_i \leq b_j$ and $c_i \leq c_j$, it follows that $t_i(b, c) \leq t_j(b, c)$. Altogether, $\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_j(N, b, c, E)$, as desired.

**Case 2**: $E \geq T$.

Then, $\widetilde{R}_i(N,b,c,E) = t_i(b,c) + R_i(N,c-t(b,c),E-T)$ and $\widetilde{R}_j(N,b,c,E) = t_j(b,c) + R_j(N,c-t(b,c),E-T)$. Note that, as mentioned above, $t_i(b,c) \leq t_j(b,c)$. Thus, if $c_i - t_i(b,c) \leq c_j - t_j(b,c)$, the desired inequality would trivially follow from the fact that $R$ satisfies order preservation. If, on the contrary, $c_i - t_i(b,c) \geq c_j - t_j(b,c)$ the fact that $R^*$ satisfies order preservation guarantees that $R_i^*(N,c-t(b,c),C-E) \geq R_j^*(N,c-t(b,c),C-E)$. Thus,

$$t_i(b,c) + R_i(N,c-t(b,c),E-T) = c_i - R_i^*(N,c-t(b,c),C-E) \leq$$

$$\leq c_j - R_j^*(N,c-t(b,c),C-E) = t_j(b,c) + R_j(N,c-t(b,c),E-T),$$

as desired. ■

## Proof of Theorem 5

It is straightforward to see that, for any (standard) rule $R$, $\widetilde{R}$ satisfies the four axioms. Thus, we focus on the converse implication. Let $S$ be a rule satisfying the four axioms and let $(N,b,c,E)$ be a problem with baselines. We distinguish two cases.

**Case 1**: $E \geq T$.

By baseline invariance, $S(N,b,c,E) = t(b,c)+S(N,b-t(b,c),c-t(b,c),E-T)$. Note that if $b_i \leq c_i$, then $b_i - t_i(b,c) = 0 < c_i - t_i(b,c) = c_i - b_i$, whereas if $b_i \geq c_i$, then $b_i - t_i(b,c) = b_i - c_i > 0 = c_i - t_i(b,c)$. Thus, by baseline truncation, $S(N,b-t(b,c),c-t(b,c),E-T) = S(N,0,c-t(b,c),E-T)$.

**Case 2**: $E \leq T$.

By baseline truncation, $S(N,b,c,E) = S(N,t(b,c),c,E)$. And, by truncation of excessive claims, $S(N,t(b,c),c,E) = S(N,t(b,c),t(b,c),E)$. Finally, by polar baseline self-duality, $S(N,t(b,c),t(b,c),E) = t(b,c) - S(N,0,t(b,c),T-E)$.

To summarize,

$$S(N,b,c,E) = \begin{cases} t(b,c) - S(N,0,t(b,c),T-E) & \text{if } E \leq T \\ t(b,c) + S(N,0,c-t(b,c),E-T) & \text{if } E > T \end{cases}. \tag{10}$$

Let $R : \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$ be such that, for any $(N,c,E) \in \mathcal{D}$,

$$R(N,c,E) = S(N,0,c,E).$$

In other words, $R$ assigns to each problem the allocation that $S$ yields for the corresponding problem with baselines in which baselines are null. Hence, (10) becomes

$$S(N,b,c,E) = \begin{cases} t(b,c) - R(N,t(b,c),T-E) & \text{if } E \leq T \\ t(b,c) + R(N,c-t(b,c),E-T) & \text{if } E > T \end{cases},$$

which implies that $S \equiv \widetilde{R}$. ■

We conclude by showing the tightness of the result. In what follows, let $A$ denote the (standard) constrained equal awards rule and $P$ denote the (standard) proportional rule.[17]

- Let $S$ be defined by

$$S(N,c,b,E)=\begin{cases} P(N,t(b,c),E), & \text{if } E\leq T \\ t(b,c)+P(N,c-t(b,c),E-T) & \text{if } E\geq T \text{ and } c_i\geq b_i \text{ for each } i\in N \\ t(b,c)+A(N,c-t(b,c),E-T) & \text{if } E\geq T \text{ and } c_i<b_i \text{ for some } i\in N, \end{cases}$$

  It is straightforward to show that $S$ is a well-defined rule that satisfies truncation of excessive claims, baseline invariance, polar baseline self-duality and polar baseline equivalence, but not baseline truncation.

- Let $S$ be defined by

$$S\left(N,b,c,E\right)=\begin{cases} P(N,c,E) & \text{if } E\leq T \\ t(b,c)+P(N,c-t(b,c),E-T) & \text{if } E\geq T \end{cases}$$

  It is straightforward to show that $S$ is a well-defined rule that satisfies baseline truncation, baseline invariance, polar baseline self-duality and polar baseline equivalence, but not truncation of excessive claims.

- Let $S$ be defined by

$$S\left(N,b,c,E\right)=\begin{cases} t(b,c)-P(N,t(b,c),T-E) & \text{if } E\leq T \\ P(N,c,E) & \text{if } E\geq T \end{cases}$$

  It is straightforward to show that $S$ is a well-defined rule that satisfies baseline truncation, truncation of excessive claims, polar baseline self-duality and polar baseline equivalence, but not baseline invariance.

- Let $S$ be defined by

$$S\left(N,b,c,E\right)=\begin{cases} t(b,c)-A(N,t(b,c),T-E) & \text{if } E\leq T \\ t(b,c)+P(N,c-t(b,c),E-T) & \text{if } E\geq T \end{cases}$$

  It is straightforward to show that $S$ is a well-defined rule that satisfies baseline truncation, truncation of excessive claims and baseline invariance, but neither polar baseline self-duality nor polar baseline equivalence.

[17]Formally, for each $(N,c,E)\in\mathcal{D}$, $A\left(N,c,E\right)=(\min\{c_i,\lambda\})_{i\in N}$ where $\lambda>0$ is chosen so that $\sum_{i\in N}\min\{c_i,\lambda\}=E$, whereas $P\left(N,c,E\right)=\frac{E}{C}\cdot c$.

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