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On games arising from multi-depot Chinese postman problems

Trine Tornøe Platz[1] Herbert Hamers[2]

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Abstract

This paper introduces cooperative games arising from multi-depot Chinese postman problems and explores the properties of these games. A multi-depot Chinese postman problem (MDCP) is represented by a connected (di)graph G, a set of k depots that is a subset of the vertices of G, and a non-negative weight function on the edges of G. A solution to the MDCP is a minimum weight tour of the (di)graph that visits all edges (arcs) of the graph and that consists of a collection of subtours such that the subtours originate from different depots, and each subtour starts and ends at the same depot. A cooperative Chinese postman (CP) game is induced by a MDCP by associating every edge of the graph with a different player. This paper characterizes globally and locally k-CP balanced and submodular (di)graphs. A (di)graph G is called globally (locally) k-CP balanced (respectively submodular), if the induced CP game of the corresponding MDCP problem on G is balanced (respectively submodular) for any (some) choice of the locations of the k depots and every non-negative weight function.

Keywords: Chinese postman problem, cooperative game, submodularity, balancedness.

JEL Classification Number: C71.

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1 Introduction

A Chinese postman problem (CPP) models the situation in which a postman must deliver mail to a given set of streets using the shortest possible route, under the constraint that he must start and end at the post office, see e.g. Edmonds and Johnson (1973).

A multi-depot Chinese postman problem (MDCP) is represented by a graph in which the edges of the graph correspond to the streets to be visited, a fixed set of k vertices serve as depots, and a non-negative weight function is defined on the edges. A solution to the problem is a minimum weight tour consisting of a collection of subtours such that every edge of the graph is visited, the subtours originate at different depots, and each subtour starts and ends at the same depot. The MDCP can be seen as a special case of the multi-depot capacitated arc routing problem described in, e.g., Wøhlk (2008), and Kansou and Yassine (2010). Applications of the MDCP include for example snow plowing and winter gritting, Wøhlk (2008). The CPP arises as a special case if only one depot is available, i.e. $k = 1$.

Chinese postman games were introduced in Hamers et al. (1999). We introduce a multi-depot version of these games. A multi-depot Chinese postman game is defined on a weighted connected (di)graph in which a set of vertices is fixed and referred to as the depots, and the players reside at the edges. More precisely, the choice of the location of the depots and the non-negative weight (or cost) function determines a specific CP game on this graph, since the value of a coalition in a multi-depot CP game is obtained by a cheapest collection of sub-tours starting and ending at the depots such that each subtour starts and ends at the same depot, and the subtours together visits all members of the coalition. Hence, the value of a coalition reflects the cheapest costs at which the coalition can be visited.

The aim of the paper is to explore the balancedness and submodularity of multi-depot CP games. In a balanced game, the core is non-empty, and submodular games have several desirable properties. For example, the Shapley value of a submodular game is the barycenter of the core, Shapley (1971). Furthermore, some solution concepts coincide for this class of games. The nucleolus is equal to the kernel, and the bargaining set coincides with the core, Maschler et al. (1972).

A connected graph or a strongly connected digraph G is said to be globally k-CP submodular (balanced) if for all $Q \in V(G)$ with $|Q| = k$, the induced CP game is submodular (balanced) for every non-negative weight function. G is locally k-CP submodular (balanced) if for some $Q \in V(G)$ with $|Q| = k$, the induced game is submodular (balanced) for all non-negative weight functions.

We characterize classes of globally and locally k-CP balanced and submodular graphs and digraphs. First we show that an undirected graph G is globally k-CP balanced if and only if it is locally k-CP balanced which in turn holds if and only if G is weakly Eulerian. For an undirected graph G, we show that for $k \in \{1, m-1, m\}$ where m is the number of vertices of G the graph is globally k-submodular if and only if it is weakly cyclic. Furthermore, we find that no connected graph is globally k-CP submodular, for $1 < k < |V(G)| - 1$. On the other hand, G is locally k-CP submodular for $k \in \{1, \ldots, |V(G)|\}$ if and only if G is weakly cyclic, and the depots can be located in a specific pattern. A strongly connected directed graph is globally as well as locally k-CP balanced. The characterization of globally k-CP submodular graphs depends again on the number of depots. A sufficient condition is provided for local k-CP submodularity.

The above characterization of k-CP balanced and submodular graphs follows an existing line of research in which game theoretical properties of the OR game are characterized by properties of the underlying network (graph). Hamers et al. (1999) introduced CP games and showed that weakly Eulerian graphs are CP-balanced, while Hamers (1997) showed that weakly cyclic graphs are CP-submodular. Full characterizations of the classes of CP-balanced and CP-submodular graphs were given in Granot et al. (1999). In Granot et al. (2004), the distinction between global and local requirements was made, and the authors characterized the classes of locally CP-submodular graphs and digraphs. Recently, Granot and Granot (2012) relaxed the local requirement on graphs and characterized the related CP-balanced graphs. Similar characterizations of classes of graphs exists for other types of OR-games. For example, Herer and Penn (1995) showed that graphs obtained as 1-sums of K_4 and outerplanar graphs characterize submodular Steiner-traveling salesman games. Okamoto (2003) showed that minimum vertex cover games are submodular if and only if the underlying graph is (K_3, P_3)-free, i.e., no induced subgraph is isomorphic to K_3 or P_3, and that minimum coloring games are submodular if and only if the underlying graph is complete multipartite. Deng (2000) showed that minimum coloring games are totally balanced if and only if the underlying graph is perfect. Hamers et al. (2011) showed that minimum coloring games have a Population Monotonic Allocation scheme if and only if the graph is $(P_4, 2K_2)$-free.

The paper is organized as follows. In section 2, some terms and notions from game theory and graph theory are introduced. In section 3, the model is presented, and in section 4, we analyze k-CP games and characterize the classes of k-CP balanced and submodular graphs. Section 4.1 considers undirected graphs, while the case of directed graphs is analyzed in 4.2.

2 Preliminaries

Before we present the model, we first recall some definitions and terms from cooperative games and graph theory, respectively.

A cooperative (cost) game is a pair (N, c) (often referred to simply as c when no confusion arises) in which $N = \{1, \ldots, n\}$ is a finite set of players, and $c : 2^N \to \mathbb{R}$ is a function that assigns to every *coalition* $S \subseteq N$ a *cost* $c(S)$, with $c(\emptyset) = 0$. An allocation is $x \in \mathbb{R}^N$. The *core* of a game c is defined by

$$Core(c) = \{x \in \mathbb{R}^N | \sum_{i=1}^{n} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

Thus, the core is the set of efficient allocations in which no coalition has an incentive to split from the grand coalition. The core of a game may be empty. A game in which the core is non-empty is said to be *balanced*. Let c^S denote the restriction of c to the subsets of players in S. Then, if the subgame (S, c^S) is balanced for every $S \subseteq N$, the game, (N, c), is said to be *totally balanced*. A game is submodular if it holds for all $j \in N$ and all $S \subset T \subseteq N \setminus \{j\}$ that:

$$c(T \cup \{j\}) - c(T) \leq c(S \cup \{j\}) - c(S). \tag{2.1}$$

A submodular game is totally balanced, Shapley (1971).

An undirected (directed) *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty, finite set of *vertices*, and $E(G)$ is a set of (ordered) pairs of vertices called *edges (arcs)*. Throughout the paper, we let $m = |V(G)|$ denote the cardinality of $V(G)$. An edge $\{a, b\}$ joins the vertices a, b in an undirected graph. An arc (a, b) that joins the vertices a and b in a directed graph is directed from a to b and can only be traversed in this direction. An edge (arc) joining to vertices a to b is said to be *incident* to each of these vertices, and a and b are said to be *adjacent*.

A *(directed) walk* is a sequence of vertices and edges (arcs) $v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m$, in which $m \geq 0$, $v_0, \ldots, v_m \in V(G)$, and $e_1, \ldots, e_m \in E(G)$ such that $e_j = \{v_{j-1}, v_j\}$ for all $j \in \{1, \ldots m\}$. If $v_0 = v_m$, the walk is said to be closed. A *(directed) path* is a (directed) walk in which no edge (arc) or vertex is visited more than once. A (directed) circuit is a closed (directed) path. In a path $v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m$, the vertices v_0 and v_m are called the *endpoints* of the path. If there exists an undirected (directed) path between any two vertices in a graph, then the graph is said to be *connected* (*strongly connected*). In a connected graph G, an edge $b \in E(G)$ is called a *bridge* if it is a minimum cut set of cardinality 1. Removing a bridge results in a graph that is not connected. The set of all bridges in G is denoted $B(G)$. A graph G is said to be *Eulerian*, if the degree of every edge in $E(G)$ is even. We say that a connected graph G is *weakly Eulerian*, if removing all bridges in G results in a disconnected graph in which every connected component is Eulerian. Furthermore, we say that a graph G is *weakly cyclic*, if every edge in G belongs to a most one cycle. Note that a weakly cyclic graph is weakly Eulerian.

3 Chinese postman games

Let $G = (V(G), E(G))$ be a connected undirected (or strongly connected directed) graph in which $V(G)$ denotes the set of vertices, and $E(G)$ denotes the set of edges (arcs). Furthermore, let $Q \subseteq V(G)$ be a fixed subset of the vertices, which will be referred to as depots. Let $Q = \{v_0^1, \ldots, v_0^k\}$ with $k \in \{1, \ldots, m\}$, and let $S \subseteq E(G)$. Let $i \in \{1, ..., k\}$ and let $w_{v_0^i} = (v_0^i, e_1^i, v_1^i, ..., e_{m^i}^i, v_0^i)$ denote a closed walk that starts and ends at $v_0^i \in Q$. An empty walk may exist, that is $w_{v_0^i} = (v_0^i)$. Then an S-tour $d(S)$ in G with respect to Q is a collection of closed walks $d(S) = \{(w_{v_0^1}, ..., w_{v_0^k}\}$ such that every player in S is visited. We denote the set of S-tours associated with $S \subseteq E(G)$ by $D(S)$. Furthermore, let $t : E(G) \to [0 : \infty)$ be a non-negative weight function defined on the edges (arcs) of G. The cost of a walk $w_{v_0^i}$ is then equal to the sum of the weights on the edges visited, i.e., $cost(w_{v_0^i}) = \sum_{j=1}^{m^i} t(e_j^i)$. Note that the costs of an empty closed walk equals zero. Then for every $S \subseteq N$ we can derive the cost of an S-tour $d(S)$ as:

$$C_Q(d(S)) = \sum_{i=1}^{k} cost(w_{v_0^i}).$$

Let $\Gamma = (E(G), (G, Q), t)$ be a multi-depot CP problem in which $E(G)$ is the set of players, Q is a set of depots, and $t : E(G) \to [0 : \infty)$ is the non-negative weight function. The induced CP game (N, c_Q) is then defined by

$$c_Q(S) = \min_{d(S) \in D(S)} C_Q(d(S)).$$

That is, for any $S \subseteq N$, $c(S)$ equals the cost of a minimum weight S-tour in G. Note that for $|Q| = 1$, the multi depot CP situation and its induced game will coincide with the class introduced in Hamers (1999).[1]

An example of a CP game is given below. It shows that not all graphs are globally k-CP balanced.

Example 3.1. Consider Figure 1 below. Let $Q = \{v_0, v_2\}$, and let $t(e) = 1$ for all edges

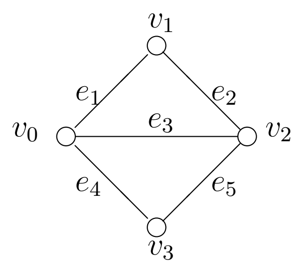

Figure 1: A non globally 2-CP balanced graph

in the graph. Then we see that $c(e_1, e_2, e_3) = c(e_3, e_4, e_5) = 3$, and $c(e_1, e_2, e_4, e_5) = 4$. However, since $c(N) = 6$, the worth of the grand coalition can not be allocated between the players such that $x(N) = c(N)$ without violating $x(S) \leq c(S)$ for some $S \subset N$. The game is therefore not balanced, and the graph is not globally 2-CP balanced. △

The above example showed that even when $k > 1$, some graphs are not globally k-CP balanced. Granot et al. (1999) show that all weakly cyclic graphs are globally 1-CP submodular, but for the case of $k > 1$, a weakly cyclic graph is not necessarily globally k-CP submodular, as the following example shows. This example also illustrates how different choices of Q with the same cardinality may lead to different games.

Example 3.2. Consider the undirected graph in Figure 2, and let $t(e) = 1$ for all edges in the graph. Let $Q = \{v_0, v_1\}$ and $Q' = \{v_0, v_3\}$. The induced games are shown in Table 1. The second row corresponds to the induced game when depots are located at $\{v_0, v_1\}$, while the third row illustrates the costs of the coalitions in the induced game,

[1] In Hamers (1999), the games are called delivery games, while the term Chinese postman (CP) games is used in subsequent publications on this topic.

when depots are located at $\{v_0, v_3\}$. For example, in case the depots are at $\{v_0, v_3\}$, the min. S-tour of $S = \{e_1, e_3\}$ is equal to $\{(v_0, e_1, v_1, e_1, v_0), (v_3, e_3, v_3)\}$ with an associated cost of 4. For ease of exposition a coalition $\{e_i, e_j\}$ is in the table written as ij.

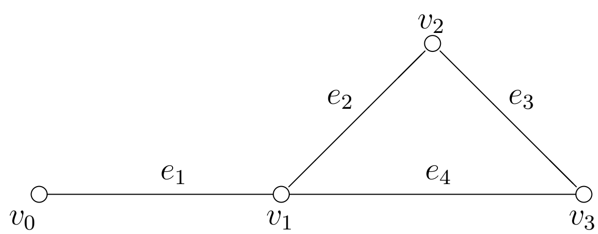

Figure 2

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$c_{\{v_0,v_1\}}(S)$	2	2	3	2	4	5	4	3	3	3	5	5	5	3	5
$c_{\{v_0,v_3\}}(S)$	2	3	2	2	4	4	4	3	3	3	5	5	5	3	5

Table 1: Two different CP games arising from the same graph

From Table 1, it is evident that different choices of depots lead to different games. Furthermore, while the two games are both balanced, only the first is submodular. The second CP game is not submodular since $c_{\{v_0,v_3\}}(e_1, e_2, e_3) - c_{\{v_0,v_3\}}(e_2, e_3) = 2 > 1 = c_{\{v_0,v_3\}}(e_1, e_2) - c_{\{v_0,v_3\}}(e_2)$. $\triangle$

While we have just shown that not all CP games are balanced or submodular, it is straightforward to verify that every CP game (N, c_Q) satisfies the following: $c_Q(S) \leq c_Q(T)$ for all $S \subset T \subseteq N$ (monotonicity), and $c_Q(S \cup T) \leq c_Q(S) + c_Q(T)$ for all S, T with $S \cap T = \emptyset$ (subadditivity). Furthermore, the worth of the grand coalition is independent of Q. That is, if we let $Q, Q' \subset V(G)$, $Q' \neq Q$ and consider the games (N, c_Q) and $(N, c_{Q'})$, then

$$c_Q(N) = c_{Q'}(N), \text{ for all } Q, Q' \subset V(G).$$

Indeed, since $c_Q(N)$ is the cost of a tour that visits every edge in G, also every vertex is visited by this tour, including every depot in Q'. But then it is also possible to visit all edges using tours originating from the depots in Q'. Finally, it can readily be verified that any graph with $|V(G)| \leq 3$ is k-CP submodular, and in the following, we therefore restrict attention to graphs $G = (V(G), E(G))$ with $|V(G)| \geq 4$.

4 Balanced and submodular k-CP graphs

In this section, we characterize k-CP balanced and k-CP submodular (di)graphs, and in addition, we consider both global and local requirements for each of the different properties.

4.1 Undirected k-CP graphs

We analyze first the case of undirected graphs and characterize k-CP balanced graphs. We find that the class of k-CP balanced graphs coincide with the class of CP-balanced graphs characterized in Granot et al. (1999).

Theorem 4.1. *Let $G = (V(G), E(G))$ be a connected graph, and let $k \in \{1, ..., m\}$. Then the following statements are equivalent:*

(i) G is weakly Eulerian,

(ii) G is globally k-CP balanced,

(iii) G is locally k-CP balanced.

Proof. $(i) \to (ii)$: Let $Q \subseteq V(G)$ be of cardinality k. Let $\Gamma = (E(G), (G, Q), t)$ be a multi depot CP problem for which (N, c) is the induced CP game. We have to show that (N, c) is balanced.

Define an allocation $x \in \mathbb{R}^N$ as:

$$x(e) = \begin{cases} 2t(e) & \text{if } e \in B(G), \\ t(e) & \text{otherwise.} \end{cases}$$

Since G is weakly Eulerian there exists a min N-tour that visits every bridge in the graph twice and all other edges exactly once. Then, $c(N) = \sum_{e \in E(G)} t(e) + \sum_{e \in B(G)} t(e)$, and consequently, x is efficient.

Furthermore, $c(S) \geq \sum_{e \in S} t(e) + \sum_{e \in B(G) \cap S} t(e) = x(S)$, where the inequality holds since every player in S must be visited at least once, and every player on a bridge must be visited twice, for any location of the depots. Hence, (N, c) is balanced, so we have that G is globally k-CP balanced.

$(ii) \to (iii)$. Follows by the definition of locally and globally k-CP balanced graphs.

$(iii) \to (i)$: Granot et al. (1999) show that any locally 1-CP balanced graph is weakly Eulerian. We therefore consider $|Q| > 1$. Let $\Gamma = (E(G), (G, Q), t)$ and assume that G is not weakly Eulerian. Consider a $v \in Q$, the corresponding $\Gamma_1 = (E(G), (G, v), t)$, and the induced CP-game $(N, c_{\{v\}})$. Then since $c_{\{v\}}(N) = c_Q(N)$ and $c_{\{v\}}(S) \geq c_Q(S)$, we may infer that $Core(c_Q) \subseteq Core(c_{\{v\}})$. From Granot et al. (1999), there exists a t such that $Core(c_{\{v\}}) = \emptyset$, and therefore $Core(c_Q) = \emptyset$. Thus, G is not locally k-CP balanced. □

We now turn to the submodularity of CP games and provide a lemma stating how the presence of certain structures in a k-CP problem precludes submodularity of the induced game. We start by introducing some notation.

Let P_l denote a path containing l, $l \geq 4$, vertices, and let S_4 denote a star graph with 4 vertices. Next, we introduce the forbidden structures. Let S_4^F denote a star graph with 4 vertices in which two of the vertices of degree 1 are associated with depots, while the other vertices are not. Furthermore, let P_l^F denote a path containing l vertices in which the two endpoints are associated with depots while no other vertices are. See the illustration in Figure 3.

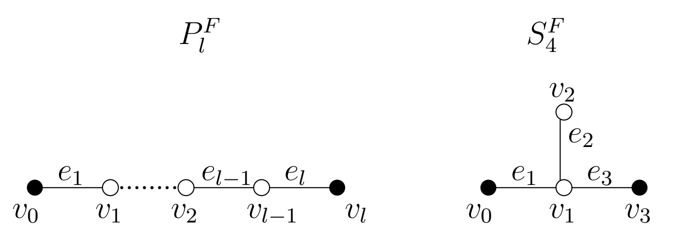

Figure 3: Forbidden structures

Let $G = (E(G), V(G))$ be an undirected connected graph, and let $Q \subseteq V(G)$. Then we say that G is (P_l^F, S_4^F)*−free with respect to* Q, if no structure P_l^F or S_4^F exists in G when depots are located at the vertices in Q. We state the following lemma.

Lemma 4.1. Let $G = (E(G), V(G))$ be a connected undirected graph. Let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced k-CP game. If G is submodular for all non-negative weight functions, then (N, c) is (P_l^F, S_4^F)− free wrt. Q.

Proof. Let $Q \subseteq V(G)$ be such that G is not $(P_l^F, S_4^F)-$ free wrt. Q. Then there exists a subgame (U, c^U) such that this subgame is defined on a structure P_l^F or S_4^F. We show that (N, c) is then not submodular. Consider P_l^F and let $t(e)$ be such that $t(e) = 1$ for edges e_1, e_{l-1} and e_l while $t(e) = 0$ for all other edges. Let $S = \{e_{l-1}\}$ and $T = \{e_{l-1}, e_l\}$. Then,

$$c(T \cup \{e_1\}) - c(T) = 2 > 0 = c(S \cup \{e_1\}) - c(S),$$

and the game is not submodular. A similar argument holds for S_4. □

We proceed to consider k-CP submodular graphs. We first characterize globally k-CP submodular graphs, for $k \in \{1, m-1, m\}$, and then we show that for $k \in \{2, \ldots, m-2\}$, no undirected connected graph is globally k-CP submodular.

Theorem 4.2. *Let $G = (V(G), E(G))$ be a connected graph and let $k \in \{1, m-1, m\}$. Then G is globally k-CP submodular if and only if G is weakly cyclic.*

Proof. Granot et al. (1999) showed that G is globally 1-CP submodular if and only if G is weakly cyclic, so it is sufficient for us to consider the case of $k \in \{m-1, m\}$. We first prove the 'if' part. We have to show (2.1). Let G be weakly cyclic. Furthermore, let $\Gamma = (E(G), (G, Q), t)$, let (N, c) be the induced game, and let $k \in \{m-1, m\}$. Then G consists of a set of edge-disjoint circuits and bridges. Let $C(G)$ and $B(G)$ denote the set of circuits and bridges, respectively, and observe that since $k \in \{m-1, m\}$, every edge in $E(G)$ is incident to at least one depot. Then, for any $S \subseteq N$, we have:

$$c(S) = \sum_{C \in C(G): C \cap S \neq \emptyset} \min\{\sum_{a \in C} t(a), \sum_{a \in S \cap C} 2t(a)\} + \sum_{b \in B(G): b \in S} 2t(b), \qquad (4.1)$$

Let $e \in N$, and let $S \subset T \subseteq N \setminus \{e\}$. We distinguish between two cases:

Case 1. $e \in B(G)$. From (4.1) it follows that $c(S \cup \{e\}) - c(S) = c(T \cup \{e\}) - c(T) = 2t(e)$ for all $e \in N$ and all $S \subset T \subseteq N \setminus \{e\}$, so (2.1) is satisfied.

Case 2. $e \notin B(G)$. Then there exists a circuit $C \in C(G)$ such that $e \in C$. Let $A = \sum_{a \in C} t(a)$, $B = \sum_{a \in S \cap C} 2t(a)$, and $D = \sum_{a \in T \cap C} 2t(a)$. Then (4.1) implies that

$$c(S \cup \{e\}) - c(S) = \min\{A, B + 2t(e)\} - \min\{A, B\}, \text{ and}$$
$$c(T \cup \{e\}) - c(T) = \min\{A, D + 2t(e)\} - \min\{A, D\}.$$

Observe that $B \le D$. Now,

$$\begin{aligned}
&\text{if } A \le B \text{ then } c(S \cup \{e\}) - c(S) = c(T \cup \{e\}) - c(T) = 0,\\
&\text{if } A > D + 2t(e) \text{ then } A > B, \text{ and then } c(S \cup \{e\}) - c(S) = c(T \cup \{e\}) - c(T) = 2t(e),\\
&\text{if } A \le D + 2t(e) \text{ and } A > B, \text{ then } c(S \cup \{e\}) - c(S) = \min\{A - B, 2t(e)\}\\
&\qquad \ge \max\{0, A - D\} = A - \min\{A, D\} = c(T \cup \{e\}) - c(T).
\end{aligned}$$

where the last inequality follows from $B \le D$ and $A \le D + 2t(e)$. Thus, (2.1) is satisfied for all $e \in N$ and all $S \subset T \subseteq N \setminus \{e\}$.

Turning to the 'only if' part, assume that G is not weakly cyclic. We then have to show that there exists a weightfunction such that (N, c) is not submodular. If G is not weakly cyclic, there exists a subgraph G_1 in G as displayed in Figure 4. Consider Figure

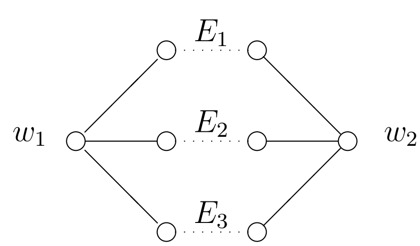

Figure 4

4 and let E_1, E_2 and E_3 denote the set of edges in each of the three paths between w_1 and w_2, respectively. Let the weight function on the edges of G be such that $t(e) = 1$ for every $e \in E_1 \cup E_2 \cup E_3$, while $t(e)$ is arbitrarily large for all $e \in E(G) \setminus E(G_1)$. Since $k \in \{m-1, m\}$, either w_1 or w_2 is in Q, and then since a (sub)tour in the graph must start and end at the same depot, an allocation $x \in \mathbb{R}^N$ in the core of the (sub)game $(E(G_1), c^{E(G_1)})$ must fulfill the following:

$$\begin{aligned}
x(E_1 \cup E_2) &\le c(E_1 \cup E_2) = |E_1| + |E_2|,\\
x(E_1 \cup E_3) &\le c(E_1 \cup E_3) = |E_1| + |E_3|,\\
x(E_2 \cup E_3) &\le c(E_2 \cup E_3) = |E_2| + |E_3|.
\end{aligned}$$

Adding the inequalities leads to

$$x(N) \le |E_1| + |E_2| + |E_3| < c(N),$$

and $(E(G_1), c^{E(G_1)})$ is not balanced. Consequently (N, c) is not totally balanced, and hence, not submodular. $\square$

Theorem 4.3. *Let $G = (V(G), E(G))$ be a connected graph. If $k \in \{2, \ldots, m-2\}$, then G is not globally k-CP submodular.*

Proof. Let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced k-CP game. Since $|V(G)| \geq 4$, there exists in G a connected subgraph with four vertices. The possible non-isomorphic structures of this subgraph are displayed in Figure 5. Then there exists

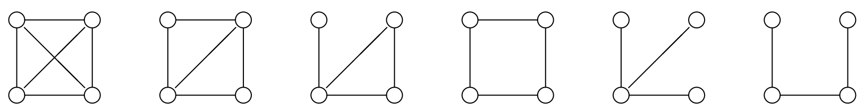

Figure 5: Non-isomorphic connected graphs with 4 vertices

a 3-player subgame (U, c^U) that is defined on P_4 or S_4. Since $2 \leq k \leq m-2$, we can in either case assign the k depots to the vertices of G, such that G is not (P_l^F, S_4^F)-free wrt. Q. It, therefore, follows from Lemma 4.1 that (N, c) is not submodular. Hence, G is not globally k-CP submodular. $\square$

The theorems above stated that only weakly cyclic graphs are globally k-CP submodular, and that this is only the case if there is just one depot or every edge is incident to a depot. In the following subsection, we relax the strict global requirement and consider undirected locally k-CP submodular graphs. The following theorem follows readily from Lemma 4.1

Theorem 4.4. *Let $G = (V(G), E(G))$ be a connected graph. If G is locally k-CP submodular, there exists a $Q \subseteq V(G)$ with cardinality k such that G is (P_l^F, S_4^F)-free wrt. Q.*

Being (P_l^F, S_4^F)-free is, however, not sufficient for connected graphs in general to be k-CP submodular, as the following example shows.

Example 4.1. Consider Figure 1 and assume that $Q = V(G)$ implying that every vertex in the graph is associated with a depot and that the graph is (P_l^F, S_4^F)-free wrt. Q. Let $S = \{e_4, e_5\}$ and $T = \{e_1, e_2, e_4, e_5\}$. Then $c(T \cup \{e_3\}) - c(T) = 2 > 1 = c(S \cup \{e_3\}) - c(S)$, and the game is not submodular. $\triangle$

This leads us to the following result:

Theorem 4.5. *Let G be a connected graph. If there exists a $Q \subseteq V(G)$ with cardinality k such that G is $(P_l^F, S_4^F)-$free wrt. Q, then G is locally k-CP submodular if and only if G is weakly cyclic.*

Proof. For the 'if ' part, let G be a weakly cyclic graph, let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced k-CP game. We have to show that (2.1) holds.

First, let G_1 be the weakly cyclic subgraph induced by the vertices of Q and every path between any two vertices of Q. Then since G is (P_l^F)-free wrt. Q, so is G_1, and since any edge $e \in E(G_1)$ lies on a path between depots, it must be that e is incident to at least one vertex of Q. Let $(E(G_1), c^{E(G_1)})$ be the CP game induced by the CP problem $\Gamma_1 = \{E(G_1), (G_1, Q), t_{G_1}\}$ on G_1, where t_{G_1} denotes the restriction of t to G_1. Let $C(G_1)$ denote the set of circuits in G_1. Then we can express $c^{E(G_1)}$ as

$$c^{E(G_1)}(S) = \sum_{C \in C(G_1): C \cap S \neq \emptyset} \min\{t(E(C)), 2t(S \cap E(C))\} + 2t(S \cap B(G)),$$

and it follows from the proof of Theorem 4.2 that $(E(G_1), c^{E(G_1)})$ is submodular.

Furthermore, since G is $(S_4^F)-$free, the edges in $E(G) \backslash E(G_1)$ can be partitioned into r sets $\{A_1, \ldots, A_r\}$ such that for each A_i with $i \in \{1, \ldots, r\}$, there exists an $a_i \in Q$ such that $G \setminus \{a_i\}$ is disconnected, and a path from a_i to any edge $e \in A_i$ contains no other depot than a_i. Let $G(A_i)$ denote the subgraph induced by the edges in A_i. Now, let $\Gamma_{a_i} = \{A_i, (G(A_i), a_i), t_{A_i}\}$ denote the CP problem on $G(A_i)$, and let $(A_i, c_{\{a_i\}})$ be the induced CP game. From Granot et al. (1999) it follows that $(A_i, c_{\{a_i\}})$ is submodular. Next, observe that the cost $c(S)$ of coalition S in the game (N, c) can be written as a sum

$$c(S) = c^{E(G_1)}(S \cap E(G_1)) + \sum_{i=1}^{r} c_{\{a_i\}}(S \cap A_i).$$

Now, since $(E(G_1), c^{E(G_1)})$ is submodular, and $(A_i, c_{\{a_i\}})$ is submodular for all $i \in \{1, ..., r\}$, it follows that (N, c) is submodular.

For the 'only if' part, note that if G is not weakly cyclic, there exists a subgraph G^* with a structure on the form of Figure 1. We will show that for every k, there exists a t and a location of the k depots in G for which (N, c) is not totally balanced and hence not submodular. For $k = 1$, the result follows from Granot and Hamers (2004). We consider $k \geq 2$.

Since G is connected, there exists a vertex $v_0 \in Q$ and a vertex $v \in V(G^*)$ such that the path from v_0 to v contains no other vertices in $V(G^*)$. Note that if $v_0 \in V(G^*)$, then $v = v_0$, and the path is empty. Assume wolog. that v lies on the path between w_1 and w_2 that consists of the edges in E_1. Let P_2 denote the (possibly empty) path from v to w_1 that visits only edges of E_1. Let $E(P_2)$ denote the set of edges in the path P_2, and let $t(E_1 \setminus E(P_2))$ denote the sum of the weights of the edges in $E_1 \setminus E(P_2)$. We now choose a weight function t such that $t(e) = 0$ for all $e \in E(P_1) \cup E(P_2)$, $t(E_1 \setminus E(P_2)) = 1$, $t(E_2) = t(E_3) = 1$, and $t(e) = 100$ for all other edges in G. Then, the cost of a min weight tour that visits every edge in G^* is equal to 4, while the cost of any min weight tour visiting all edges of $E_i \cup E_j$, $i \neq j$, $i, j \in \{1, 2, 3\}$ is equal to 2. Now, this implies that the subgame $(E(G^*), c^{E(G^*)})$ is not balanced. To see this note that an allocation $x \in \mathbb{R}^N$ in the core must fulfill the following:

$$\begin{aligned} x(E_1 \cup E_2) &\leq c(E_1 \cup E_2) = 2, \\ x(E_1 \cup E_3) &\leq c(E_1 \cup E_3) = 2, \\ x(E_2 \cup E_3) &\leq c(E_2 \cup E_3) = 2, \\ x(E(G^*)) &= c(E(G^*) = 4. \end{aligned} \tag{4.2}$$

If we add the inequalities, we get $2x(E(G^*)) = 6$ which implies that $x(E(G^*)) = 3 < 4 = c(E(G^*)$, and this contradicts the assumption that $x(E(G^*)) = c(E(G^*)$. Since the subgame is not balanced, (N, c) is not totally balanced and therefore not submodular. □

Theorem 4.5 shows that the class of locally k-CP submodular graphs does not coincide with the class of globally k-CP submodular graphs. This differs from the result in Granot et al. (1999) that the classes of globally and locally CP submodular graphs coincide. We now turn to the case of directed graphs.

4.2 Directed k-CP graphs

In this section, we consider strongly connected directed graphs, and as in the previous section, we characterize k-CP balanced and k-CP submodular graphs.

First, we extend to the case of multiple depots in the underlying graph the result from Granot et al. (1999) that every strongly connected digraph is CP-balanced. Furthermore, we show that the classes of globally k-CP balanced and locally k-CP balanced graphs coincide.

Theorem 4.6. *Let $G = (V(G), E(G))$ be a strongly connected, directed graph, and let $k \in \{1, ..., m\}$. Then G is globally k-CP balanced.*

Proof. Let $\Gamma = (E(G), (G, Q), t)$ where $Q \subset V(G)$ is of cardinality k, and let $(E(G), c)$ be the induced game. Consider the game $(E(G), c^*)$ for which the linear programming problem in (4.3) represents a linear production game formulation of $(E(G), c^*)$. Let x_{ij} denote the flow in arc (v_i, v_j) and t_{ij} the cost of the arc and consider the LP-problem below.

$$\begin{aligned} c^*(S) = \min \sum_{i,j \in E(G)} t_{ij} x_{ij} \\ \text{subject to} \\ \sum_{j \in E(G)} x_{ji} - \sum_{j \in E(G)} x_{ij} = 0 \text{ for all } i \in E(G) \\ x_{ij} \geq 1 \text{ for all arcs } (v_i, v_j) \in S \\ x_{ij} \geq 0 \text{ for all arcs } (v_i, v_j) \notin S \end{aligned} \tag{4.3}$$

If $S = E(G)$, the solution to the problem is a minimum cost circulation in which the flow in every arc is at least 1. According to Orloff (1974), this is equivalent to an optimal Chinese postman tour of $E(G)$ with cost function t. Thus, $c(E(G)) = c^*(E(G))$. For $S \subset E(G)$, the solution to the linear programming problem will be a minimum cost circulation that may consist of several disconnected min cost subtours. In the k-CP problem, each subtour must visit a depot, and consequently, $c(S) \geq c^*(S)$. Owen (1975) has shown that $(E(G), c^*)$ is a totally balanced game, and since $c(E(G)) = c^*(E(G))$, and $c(S) \geq c^*(S)$ for each $S \subset E(G)$, this implies that $(E(G), c)$ is balanced. Hence, G is globally k-CP balanced. □

From Theorem 4.6 and the definition of globally and locally k-CP balanced graphs, we get the following corollary.

Corollary 4.1. Let $G = (V(G), E(G))$ be a strongly connected, directed graph, and let $k \in \{1, ..., m\}$. Then G is locally k-CP balanced.

Next, we turn to the submodularity of games arising from directed graphs, and give a characterization of globally k-CP submodular digraphs. First, however, we state a few definitions.

A directed weakly cyclic graph is a 1-sum of directed circuits, where a 1-sum of two graphs H and G is the graph that arises from coalescing one vertex in H with a vertex in G. Furthermore, we say that a directed circuit C is *internal* if it shares at least two vertices with other circuits, and we let $|C|$ denote the number of edges in C. Furthermore, let $\mathcal{C}(G)$ denote the set of internal circuits in G. We are now ready to characterize globally k-CP submodular digraphs.

Theorem 4.7. *Let $G = (V(G), E(G))$ be a strongly connected digraph, and let $k > m - \min\{|C||C \in \mathcal{C}(G)\}$ if $\mathcal{C}(G) \neq \emptyset$, and let $k \geq 1$ if $\mathcal{C}(G) = \emptyset$. Then G is globally k-CP submodular if and only if G is weakly cyclic.*

Proof. For the 'if' part, let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced game. We have to prove that (2.1) holds. Recall that G is weakly cyclic. We consider first the case of $\mathcal{C}(G) \neq \emptyset$. By assumption on k, every internal circuit contains at least one depot, while the non-internal circuits may contain no depots. Consider an $e \in E(G)$, and let C^* denote the circuit containing e. Let $t(C^*)$ denote the sum of the weights on the edges in C^*. We distinguish between three cases:

Case 1. $T \cap E(C^*) \neq \emptyset$: Then $c(T \cup \{e\}) - c(T) = 0$, and (2.1) clearly holds.

Case 2. $T \cap E(C^*) = \emptyset$ and $Q \cap V(C^*) \neq \emptyset$: Then $c(T \cup \{e\}) - c(T) = c(S \cup \{e\}) - c(S) = t(C^*)$, and (2.1) holds.

Case 3. $T \cap E(C^*) = \emptyset$ and $Q \cap V(C^*) = \emptyset$: Then C^* is a non-internal circuit. Note that since C^* is non-internal it can at most share one vertex with other circuits (but may share this vertex with more than one circuit). Let $\mathcal{C}^*$ denote the set of circuits connected to C^*. Then either: $T \cap E(\mathcal{C}^*) \neq \emptyset$, in which case $c(T \cup \{e\}) - c(T) = t(C^*) \leq (S \cup \{e\}) - c(S)$, or $T \cap E(\mathcal{C}^*) = \emptyset$ in which case $c(T \cup \{e\}) - c(T) = (S \cup \{e\}) - c(S) = t(C^*) + t(C_{min})$, where $C_{min} \in \mathcal{C}^*$ denotes a circuit connected to C such that $t(C_{min}) \leq t(C')$ for all $C' \in \mathcal{C}^*$, and C_{min} contains a depot. In both cases (2.1) holds.

If $\mathcal{C}(G) = \emptyset$, then for each directed circuit C in G, C either contains a depot, or C is connected to a circuit containing a depot (since $k \geq 1$). Thus, we can argue as in the situation above.

For the 'only if' part: If G is not weakly cyclic, there exists a subgraph G_1 in G as in Figure 6. Consider Figure 6, and let the set of arcs contained in the three directed paths between w_1 and w_2 be given by E_1, E_2 and E_3 respectively. Let $t(e) = 1$ for all $e \in E_1 \cup E_2 \cup E_3$, and let $t(e)$ be arbitrarily large for all $e \in E(G) \setminus E(G_1)$. If we

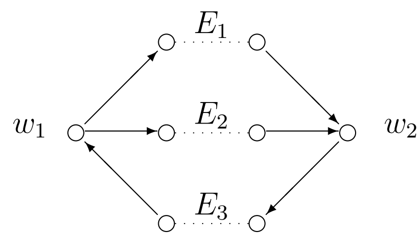

Figure 6: Non-weakly cyclic directed graph

assume, for example, that $w_1 \in Q$, then

$$\begin{aligned} c(E_1 \cup E_2 \cup E_3) + c(E_3) &= (|E_1| + |E_2| + 2|E_3|) + (|E_3| + \min\{|E_1|, |E_2|\}) \\ &> (|E_1| + |E_3|) + (|E_2| + |E_3|) \\ &= c(E_1 \cup E_3) + c(E_2 \cup E_3), \end{aligned}$$

and therefore, $c(E_1 \cup E_2 \cup E_3) - c(E_1 \cup E_3) > c(E_2 \cup E_3) - c(E_3)$,

which shows that the game is not submodular. Hence, G is not globally k-CP submodular. □

The theorem above showed that even if every vertex is associated with a depot, only weakly cyclic digraphs are globally k-CP submodular. If, on the other hand, there are too few depots in the multi-depot CP problem, a connected digraph is not globally k-CP submodular.

Theorem 4.8. *Let G be a strongly connected directed graph for which $\mathcal{C}(G) \neq \emptyset$, and let $1 < k \leq m - \min\{|C||C \in \mathcal{C}(G)\}$. Then G is not globally k-CP submodular.*

Proof. From the proof of Theorem 4.7 only weakly cyclic digraphs can be globally k-CP submodular. To prove the present theorem, we therefore only need to show that a weakly cyclic digraph is not globally k-CP submodular for $1 < k \leq m - \min\{|C||C \in \mathcal{C}(G)\}$. Let $\Gamma = (E(G), (G, Q), t)$, let (N, c) be the induced game, and let $1 < k \leq m - \min\{|C||C \in \mathcal{C}(G)\}$. Then we can choose a $Q \subset V(G)$ such that at least one internal circuit C_0 does not contain a depot but is connected to two different circuits C_1 and C_2 both containing depots, for example as in Figure 7. To see that the induced game is not submodular for every non-negative weight function, choose t such that $t(C_2) \leq t(C_1)$ and consider $S = \{E(C_0)\}$ and $T = \{E(C_0), E(C_1)\}$. Then since $t(C_2) \leq t(C_1)$, we see that $c(S \cup E(C_2)) = c(S)$ whereas $c(T \cup E(C_2)) - c(T) = t(C_2)$. Thus, the induced

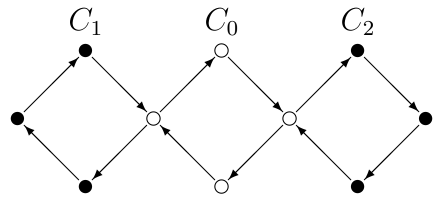

Figure 7: Internal circuit C_0 containing no depots in a weakly cyclic digraph

game is not submodular for all non-negative weight functions, and G is not globally k-CP submodular. □

The final result of this section shows that all directed weakly cyclic graphs are locally k-CP submodular.

Theorem 4.9. *Let G be a strongly connected digraph. If G is weakly cyclic, then G is locally k-CP submodular.*

Proof. Let G be weakly cyclic, let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced game. We need to show that (2.1) holds. For every $k \in \{1, \ldots, m\}$, let Q be chosen as follows: If possible, place a depot in (at least) one vertex in each circuit. Otherwise, assign depots such that the subgraph consisting only of circuits containing depots is a strongly connected graph. Now, let G_1 denote the subgraph consisting of all circuits in G that contain depots according to Q. Then G_1 is strongly connected. Let $e \in E(G)$, and let C^* denote the circuit containing e. We consider two cases:

Case 1. $e \in E(G_1)$: If $c(T \cup \{e\}) - c(T) = 0$, then (2.1) holds, and otherwise, $c(T \cup \{e\}) - c(T) = c(S \cup \{e\}) - c(S) = t(C^*)$.

Case 2. If $e \in E(G) \setminus E(G_1)$, there exists a vertex $v \in Q$ such that every $S \cup \{e\}$-tour must pass v in order to visit e. Thus, for any $S \subseteq N \setminus \{e\}$ there is a min. weight $S \cup \{e\}$-tour in which e is serviced by v, along with all other edges in C^*. Then for every $e \in E(G) \setminus E(G_1)$ there exists a submodular one-depot CP game $(N, c_{\{v\}})$ such that $c_Q(S \cup \{e\}) - c_Q(S) = c_{\{v\}}(S \cup \{e\}) - c_{\{v\}}(S)$ for all $S \subset T \subseteq N \setminus \{e\}$, and (2.1) is fulfilled. □

From Theorem 4.7 and Theorem 4.9, it follows that the set of locally k-CP submodular digraphs is a superset of the set of globally k-CP submodular digraphs.

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