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Rationing with Baselines: The Composition Extension Operator*

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Abstract

We introduce a new operator for general rationing problems in which, besides conflicting claims, individual baselines play an important role in the rationing process. The operator builds onto ideas of composition, which are not only frequent in rationing, but also in related problems such as bargaining, choice, and queuing. We characterize the operator and show how it preserves some standard axioms in the literature on rationing. We also relate it to recent contributions in such literature.

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Keywords: *rationing, baselines, claims, operator, composition.*

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1 Introduction

In a seminal contribution, O'Neill (1982) introduced a simple model to analyze the problem in which a group of individuals have conflicting claims over an insufficient amount of a (perfectly divisible) good. Ever since, a sizable literature has emerged dealing with standard *rationing* problems fitting such model.[1] Recently, there has been a growing interest in analyzing more complex rationing situations in which not only claims, but also rights, references, or other objective entitlements, play an important role in the rationing process (e.g., Pulido et al., 2002, 2008; Kaminski, 2006; Ju et al., 2007; Hougaard et al., 2012, 2013).

The aim of this note is to explore an operator that allows us to move from the domain of standard rationing rules to the domain of rules in which a *baselines* profile complements the claims profile of a standard rationing problem.[2] Our operator is inspired by the properties of *composition*, which pertain to the way rules react to tentative allocations of the available amount to allocate (e.g., Moulin, 2000). More precisely, think of the following situation: after having divided the allocation of the available amount among its creditors, it turns out that the actual value of the amount is larger than was initially assumed. Then, two options are open: either the tentative division is canceled altogether and the actual problem is solved, or we add to the initial distribution the result of applying the rule to the remaining amount. The requirement of *composition up* is that both ways of proceeding should result in the same awards vectors. Think now of the dual case. Namely, after having divided the available amount among its creditors one finds that the actual value of the amount to divide falls short of what was assumed. Here again we can ignore the initial division and apply the rule to the revised problem, or we can apply the rule to the problem in which the initial claims are substituted by the (unfeasible) allocation initially proposed. The requirement of *composition down* is that both ways of proceeding should result in the same awards vectors.[3]

The operator we present here behaves in a similar way. More precisely, it associates with each rule defined for the standard rationing model a rule for the general model with baselines,

[1] The reader is referred to Thomson (2003) for a survey.

[2] The notion of operators for the domain of rationing rules was first introduced by Thomson and Yeh (2008).

[3] These properties are reminiscent of the so-called "path independence" axiom for choice functions (e.g., Plott, 1973). They also have a relative in the theory of axiomatic bargaining: the so-called "step-by-step negotiations" axiom introduced by Kalai (1977), which is the basis for the characterization of the egalitarian solution in such context. The same principle has also been frequently used in other related contexts like taxation, queuing, or resource allocation (e.g., Young, 1988; Moulin and Stong, 2002; Moreno-Ternero and Roemer, 2012).

which solves problems following a two-stage process. In the first stage, baselines (truncated by claims) either replace claims (if they are not collectively feasible) or are awarded to agents (if they are collectively feasible). In the second stage, the primitive rule is used to solve the standard problem that results after adjusting claims and endowment according to the process of the first stage. We characterize such a *composition extension operator* and show how it preserves some standard axioms in the literature on rationing.

A main focus of this note is to compare the composition operator directly to similar types of operators that have appeared in the literature. In particular, the so-called *baselines first operator* introduced in Hougaard et al., (2013). Both operators agree on how to extend a standard rationing rule to the setting of rationing in the presence of baselines, when truncated baselines are collectively feasible. They, however, disagree in the treatment of the opposite case; namely, when truncated baselines are not collectively feasible. The baselines-first extension operator proposes to adjust down the tentative allocation of those truncated baselines by means of the primitive rationing rule, applied to the resulting problem after replacing initial claims by (truncated) baselines. That is indeed equivalent to propose the allocation that the *dual* of the primitive rule yields for the problem in which initial claims are replaced by (the unfeasible) truncated baselines. The composition extension operator, however, advocates to use precisely the allocation proposed by the primitive rule to that latter problem. In doing so, the operator reflects the principle underlying the lower composition property, as done with the upper composition property for the other case (in which truncated baselines are collectively feasible).

2 Preliminaries

2.1 The benchmark model

We study rationing problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of N. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i\in N}$ the claims profile.[4]

A *(standard rationing) problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$. Let $C \equiv \sum_{i\in N} c_i$. To avoid

[4]For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^n$, let $z_M \equiv (z_i)_{i\in M}$. For each $i \in N$, let $z_{-i} \equiv z_{N\setminus\{i\}}$.

unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the set of rationing problems with population N and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *(standard) rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$. The *proportional* rule, $P(N, c, E) = \frac{E}{C} \cdot c$, yielding allocations in proportion to claims is an example of a classical rule in this context.

Each rule R has a *dual* rule R^* defined as $R^*(N, c, E) = c - R(N, c, C - E)$, for each $(N, c, E) \in \mathcal{D}$.

2.2 The general framework

As in Hougaard et al., (2013) we consider the extended framework of a *general (rationing) problem* (or problem with baselines) consisting of a population $N \in \mathcal{N}$, a *baselines* profile $b \in \mathbb{R}^n_+$, a claims profile $c \in \mathbb{R}^n_+$, and an endowment $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. We denote by $\mathcal{E}^N$ the class of general problems with population N and $\mathcal{E} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$. For each general problem $(N, b, c, E) \in \mathcal{E}$, let $t_i(b, c) = \min\{b_i, c_i\}$, for each $i \in N$, and $t(b, c) = \{t_i(b, c)\}_{i \in N}$ denote the corresponding (baseline-claim) truncated vector. Let $T(b, c) = \sum_{i \in N} t_i(b, c)$.

Given a general problem $(N, b, c, E) \in \mathcal{E}$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. A *(general) rule* on $\mathcal{E}$, $S \colon \mathcal{E} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each (general) problem $(N, b, c, E) \in \mathcal{E}$ an allocation $x = S(N, b, c, E)$.

An (extension) operator associates with each rule defined for the standard model a rule for the general model. We consider an operator that solves problems following a two-stage process. In the first stage, baselines (truncated by claims) either replace claims (if they are not collectively feasible) or are awarded to agents (if they are collectively feasible). In the second stage, the primitive rule is used to solve the standard problem that results after adjusting claims and endowment according to the process of the first stage. In the spirit of the properties of *composition up* and *composition down*, described in the introduction, we refer to it as the *composition extension operator*.

Formally,

$$R^c(N, b, c, E) = \begin{cases} R(N, t(b, c), E), & \text{if } E \leq T(b, c) \\ t(b, c) + R(N, c - t(b, c), E - T(b, c)), & \text{if } E \geq T(b, c). \end{cases} \quad (1)$$

The composition extension operator can be directly compared to the *baselines-first* operator defined in Hougaard et al., (2013). The difference lies where $E \leq T(b,c)$, in which case the baselines-first extension operator uses the dual rule of R.[5]

Formally,

$$R^{bf}(N,b,c,E) = \begin{cases} R^*(N,t(b,c),E) & \text{if } \ E \leq T \\ t(b,c) + R(N,c-t(b,c),E-T) & \text{if } \ E \geq T \end{cases} \tag{2}$$

To use the dual rule, as in the baselines-first extension operator, is in many ways as natural as using the principle of composition down: Note that $R^*(N,t(b,c),E) = t(b,c) - R(N,t(b,c),T - E)$. Thus, the baselines-first extension operator also implements a natural principle for the case in which truncated baselines are not collectively feasible; namely, subtracting the part of the loss found by rationing with claims as truncated baselines.

In what follows, we dwell on the similarities and differences between the two operators. We start out with a comparison based on axiomatic characterizations.

3 An axiomatic characterization

We provide in this section a set of axioms characterizing the composition extension operator. These axioms are closely related to the characterization of the baselines-first extension operator and hence the reader is referred to Hougaard et al., (2013) for further details about most of them.

Baseline truncation: for each $(N,b,c,E) \in \mathcal{E}$, $S(N,b,c,E) = S(N,t(b,c),c,E)$.

Truncation of excessive claims: for each $(N,b,c,E) \in \mathcal{E}$ such that $E \leq T(b,c)$, $S(N,b,c,E) = S(N,b,t(b,c),E)$.

Baseline invariance: for each $(N,b,c,E) \in \mathcal{E}$ such that $E \geq T(b,c)$, and $k \in \mathbb{R}^n_+$ such that $k_j \leq t_j(b,c)$, for each $j \in N$, then $S(N,b,c,E) = k + S(N,b-k,c-k,E-\sum_{i \in N} k_i)$.

Polar baseline equivalence: for each $(N,c,E) \in \mathcal{D}$, $S(N,0,c,E) = S(N,c,c,E)$.

The fourth axiom is new and thus deserves further explanation. It deals with the two polar cases of *non-informative* baselines and argues that both of them should be treated equally. More precisely, it states that a problem with null baselines and the corresponding problem in which baselines are equal to claims should be allocated identically.

[5]For the case $E \geq T(b,c)$, both operators agree with the proposal made by Pulido et al., (2002) for bankruptcy problems with objective entitlements, which are a specific instance of our bankruptcy problems with baselines.

These four axioms together characterize the range of the composition extension operator, introduced above.

Theorem 1 *A rule satisfies baseline truncation, truncation of excessive claims, baseline invariance and polar baseline equivalence if and only if it is the image of a standard rule via the composition operator.*

The axiom polar baseline equivalence is a plausible alternative to the following axiom, also introduced in Hougaard et al., (2013), which treats non-informative baselines in a different way, and that characterizes the baseline first operator when combined with the first three axioms described above (e.g., Hougaard et al., 2013; Theorem 5).

Baseline self-duality: for each $(N, c, E) \in \mathcal{D}$, $S(N, 0, c, E) = c - S(N, c, c, C - E)$.

One might argue that polar baseline equivalence has a higher appeal to deal with non-informative baselines, especially when there is not enough information to determine whether (non-informative) baselines are null or equal to claims. This speaks in favor of the composition extension operator when baselines are not clearly established.

4 Preservation of axioms

The composition extension operator associates with each standard rule a new rule to solve (rationing) problems with baselines. A natural question is whether the rule so constructed inherits the properties of the original rule. To explore this question, we consider several standard axioms in the literature on rationing problems, formulating in different ways the principle of solidarity, which has a long tradition in the theory of justice (e.g., Thomson, 1983; Roemer, 1986; Moreno-Ternero and Roemer, 2006).

The first group comprises fixed-population axioms.

Resource monotonicity: for each $(N, c, E) \in \mathcal{D}$ and each $E' > E$, with $E' \leq \sum c_i$, we have $R(N, c, E) \leq R(N, c, E')$.

Claims monotonicity: for each $(N, c, E) \in \mathcal{D}$, each $i \in N$, and each $c'_i > c_i$, we have $R_i(N, (c'_i, c_{N\setminus\{i\}}), E) \geq R_i(N, (c_i, c_{N\setminus\{i\}}), E)$.

Linked claims-resource monotonicity: for each $(N, c, E) \in \mathcal{D}$ and $i \in N$,
$R_i(N, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) \leq R_i(N, c, E) + \varepsilon$.

The second group comprises variable-population axioms.

Population monotonicity: for each $(N, c, E) \in \mathcal{D}$ and $(N', c', E) \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then $R_i(N', c', E) \leq R_i(N, c, E)$, for each $i \in N$.

Linked resource-population monotonicity: for each $(N, c, E) \in \mathcal{D}$ and each $(N', c', E') \in \mathcal{D}$ such that $N \subseteq N'$, $c'_N = c$, and $E = E'$, then $R_i(N, c, E) \leq R_i\left(N', c', E + \sum_{N' \setminus N} c'_j\right)$, for each $i \in N$.

Resource-population uniformity: for each $(N, c, E) \in \mathcal{D}$ and $(N', c', E') \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then, either $R_i(N', c', E') \leq R_i(N, c, E)$, for each $i \in N$, or $R_i(N', c', E') \geq R_i(N, c, E)$, for each $i \in N$.

Consistency: for each $(N, c, E) \in \mathcal{D}$, each $M \subset N$, and each $i \in M$, we have $R_i(N, c, E) = R_i(M, c_M, E_M)$, where $E_M = \sum_{i \in M} R_i(N, c, E)$.

We say that an axiom is *preserved* by our composition operator if whenever a rule R satisfies it, then the induced rule R^c satisfies the corresponding *general* version of the axiom.[6]

Our first preservation result refers to the properties that are individually preserved.

Theorem 2 *The properties of resource monotonicity, consistency, and resource-population uniformity are preserved by the composition extension operator.*

The second preservation result refers to the properties that are preserved in pairs, either by themselves or "with the assistance" of an additional property.

Theorem 3 *The following pairs are preserved by the composition extension operator: claims monotonicity and linked claims-resource monotonicity; population monotonicity and linked resource-population monotonicity, if assisted by resource monotonicity.*

In other words, if a rule R satisfies claims monotonicity and linked claims-resource monotonicity, then the induced rule R^c satisfies the corresponding two general properties. Also, if a rule R satisfies population monotonicity and linked resource-population monotonicity, and in addition is resource monotonic, then the induced rule R^c satisfies the corresponding three general properties.

[6]For ease of exposition, we skip the straightforward definitions of the general versions of each axiom introduced above.

5 Concluding remarks

We have introduced the composition extension operator within the framework of extended rationing problems with exogeneous baselines. The composition extension operator was compared to a natural counterpart - the baselines-first extension operator defined and analyzed in Hougaard et al., (2013). Both operators are in many ways equally natural. For example, concerning preservation of properties of the associated standard rationing rules the results are highly compatible. Yet, comparing the axiomatic foundation it appears that the way in which the composition extension operator treats the case of uninformative baselines is somewhat more desirable than that of the baselines-first operator.

Finally, we note that the composition extension operator we introduce here is reminiscent of the approach we take in Hougaard et al., (2012) to analyze rationing problems. Therein, we assume that, rather than being exogenously given, baselines associate with each rationing problem some (not necessarily feasible) allocation. For each baseline, we impose the two-stage *composition* process described above to associate a new rule with each rule. The resulting family of operators encompasses (and generalizes) two focal operators for the domain of rules in the standard rationing problem, known as the attribution of minimal rights and truncation operators, respectively (e.g., Thomson and Yeh, 2008).

6 Appendix: Proof of the results

Proof of Theorem 1

It is straightforward to see that, for any (standard) rule R, R^c satisfies the four axioms. Thus, we focus on the converse implication. Let S be a rule satisfying the four axioms and let (N, b, c, E) be a problem with baselines. If $E \geq T(b, c)$, then, by baseline invariance, and baseline truncation, $S(N, b, c, E) = S(N, 0, c - t(b, c), E - T(b, c))$. If $E \leq T(b, c)$, by baseline truncation and truncation of excessive claims, $S(N, b, c, E) = S(N, t(b, c), t(b, c), E)$. Moreover, by polar baseline equivalence, $S(N, t(b, c), t(b, c), E) = S(N, 0, t(b, c), E)$.

Let $R : \mathcal{D} \rightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$ be such that, for any $(N, c, E) \in \mathcal{D}$,

$$R\left(N, c, E\right) = S(N, 0, c, E).$$

In other words, R assigns to each problem the allocation that S yields for the corresponding problem with baselines in which baselines are null. Then,

$$S(N,b,c,E) = \begin{cases} R(N,t(b,c),E) & \text{if } \ E \leq T(b,c) \\ t(b,c) + R(N,c-t(b,c),E-T(b,c)) & \text{if } \ E > T(b,c) \end{cases},$$

which implies that $S \equiv R^c$. ■

Proof of Theorem 2

Note first that, by Theorem 1 in Hougaard et al., (2013), the three properties are preserved by the baselines-first extension operator. As this operator coincides with the composition extension operator for the case in which truncated baselines are collectively feasible, we only need to focus on the opposite case to prove the statements of the theorem.

Let R be a rule satisfying resource monotonicity. Let $(N,c,E) \in \mathcal{D}$ and each $E' > E$, with $E' \leq \sum c_i$. Let $b \in \mathbb{R}^n$ be a baseline profile and let $i \in N$ be a given agent. If $E < E' \leq T(b,c)$, then $R_i^c(N,b,c,E) = R_i(N,t(b,c),E)$ and $R_i^c(N,b,c,E') = R_i(N,t(b,c),E')$. Now, as R satisfies resource monotonicity the desired inequality follows.

Let R be a rule satisfying consistency. Let $(N,c,E) \in \mathcal{D}$ and $b \in \mathbb{R}^n_+$. Let $x = R^c(N,b,c,E)$. The aim is to show that, for any $M \subset N$,

$$R^c(M, b_M, c_M, \sum_{i \in M} x_i) = x_M.$$

Fix $M \subset N$ and let $E' = \sum_{j \in M} x_j$ and $T'(b,c) = \sum_{j \in M} t_j(b,c)$. Then, it is straightforward to show that $E \leq T(b,c)$ if and only if $E' \leq T(b,c)'$. If $E \leq T(b,c)$, then $x_i = R_i(N,t(b,c),E)$ for each $i \in N$, and thus $E' = \sum_{i \in M} R_i(N,t(b,c),E)$. Therefore, $R_i^c(M,b_M,c_M,E') = R_i(M,t_M(b,c),E')$ for each $i \in M$. Now, as R is consistent, it follows that $R_i(N,t(b,c),E) = R_i(M,t_M(b,c),E')$, for each $i \in M$, as desired.

To conclude, the statement on resource-population uniformity follows from the fact that such axiom is equivalent to the combination of resource monotonicity and consistency (e.g., Hougaard et al., 2013). ■

Proof of Theorem 3

Again, by Theorem 3 in Hougaard et al., (2013), we only need to focus on the case in which truncated baselines are collectively infeasible to prove the statements of the theorem.

Let R be a rule satisfying claims monotonicity and linked claims-resource monotonicity. Our aim is to show that R^c satisfies the general versions of the two properties.

Claims monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $i \in N$, such that $c_i \le c'_i$. Let $b \in \mathbb{R}^n_+$ be a baseline profile, $T(b, c) = \sum_{j\in N} t_j(b, c)$, and $T'(b, c) = \sum_{j\in N} t_j(b, c')$.[7]

If $E \le T(b, c)$, then, $R^c_i(N, b, c, E) = R_i(N, t(b, c), E)$ and

$$R^c_i(N, b, c', E) = R_i(N, t(b, c'), E) = R_i(N, (t_{N\setminus\{i\}}(b, c), t_i(b, c')), E).$$

As R satisfies claims monotonicity, the desired inequality follows.

If $T(b, c) < E < T'(b, c)$, then, $R^c_i(N, b, c, E) = t_i(b, c) + R_i(N, c - t(b, c), E - T(b, c))$, and $R^c_i(N, b, c', E) = R_i(N, t(b, c'), E)$. Now, this case implies that $t_i(b, c) = c_i$ (otherwise, $t_i(b, c) = b_i$ and hence $T(b, c) = T'(b, c)$). Thus, by boundedness, $R^c_i(N, b, c, E) = c_i$. Now, by resource monotonicity and claims monotonicity of R, $R_i(N, t(b, c'), E) \ge R_i(N, t(b, c), T(b, c)) = t_i(b, c) = c_i$, from where the desired inequality follows.

Linked claims-resource monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $i \in N$. Let $b \in \mathbb{R}^n_+$ be a baseline profile, $\varepsilon > 0$ and $c' = (c_i + \varepsilon, c_{N\setminus\{i\}})$. Let $T'(b, c) = T + t_i(b, c') - t_i(b, c)$. Then, $t_i(b, c') \le t_i(b, c) + \varepsilon$ and $T \le T'(b, c) \le T + \varepsilon$.

If $E \le T'(b, c) - \varepsilon$, then $R^c_i(N, b, c, E) = R_i(N, t(b, c), E)$ and $R^c_i(N, b, c', E + \varepsilon) = R_i(N, t(b, c'), E + \varepsilon) = R_i(N, (t_i(b, c'), t_{-i}(b, c)), E + \varepsilon)$. By claims monotonicity of R, $R_i(N, t(b, c'), E + \varepsilon) \le R_i(N, (t_i(b, c) + \varepsilon, t_{-i}(b, c)), E + \varepsilon)$. By linked claims-resource monotonicity of R, $R_i(N, (t_i(b, c) + \varepsilon, t_{-i}(b, c)), E + \varepsilon) \le R_i(N, t(b, c), E) + \varepsilon$, from where the desired inequality follows.

If $T'(b, c) - \varepsilon < E < T(b, c)$, then $R^c_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = t_i(b, c') + R_i(N, (c_i + \varepsilon - t_i(b, c'), (c - t(b, c))_{-i}), E + \varepsilon - T'(b, c))$, and $R^c_i(N, b, c, E) = t_i(b, c') - R_i(N, t(b, c), T - E)$. Thus, the desired inequality becomes

$$\varepsilon - t_i(b, c') + t_i(b, c) \ge R_i(N, t(b, c), T - E) + R_i(N, (c_i + \varepsilon - t_i(b, c'), (c - t(b, c))_{-i}), E + \varepsilon - T'(b, c)) \quad (3)$$

Now, by balance and boundedness, the right hand side of (3) is bounded above by $T - E + E + \varepsilon - T'(b, c)$, which is precisely the left hand side of (3).

As for the second statement of the theorem, let R be a rule satisfying resource monotonicity, population monotonicity and linked resource-population monotonicity. By Theorem 2, R^c satisfies the general property of resource monotonicity. Our aim is to show that R^c also satisfies the general properties of population monotonicity and linked resource-population monotonicity.

Population monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E') \in \mathcal{D}$ be such that $N \subseteq N'$, $c'_N = c$ and $E = E'$. Let $b \in \mathbb{R}^n_+$ and $b' \in \mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N = b$. Note

[7]Note that $t_{N\setminus\{i\}}(b, c') \equiv t_{N\setminus\{i\}}(b, c)$, $t_i(b, c') \ge t_i(b, c)$ and thus, $T'(b, c) \ge T(b, c)$.

that $t_j(b', c') = t_j(b, c)$ for each $j \in N$. Finally, let $T(b, c) = \sum_{j \in N} t_j(b, c)$ and $T'(b, c) = \sum_{j \in N'} t_j(b', c')$. If $E \leq T(b, c)$, $R_i^c(N', b', c', E) = R_i(N', t(b', c'), E)$ and $R_i^c(N, b, c, E) = R_i(N, t(b, c), E)$. As R satisfies population monotonicity, the desired inequality follows.

Linked resource-population monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E) \in \mathcal{D}$ be such that $N \subseteq N'$ and $c'_N = c$. Let $b \in \mathbb{R}_+^n$ and $b' \in \mathbb{R}_+^{n'}$ be two baseline profiles such that $b'_N = b$. Note that $t_j(b', c') = t_j(b, c)$ for each $j \in N$. Finally, let $T(b, c) = \sum_{j \in N} t_j(b, c)$, and $T'(b, c) = \sum_{j \in N'} t_j(b', c')$. If $E \leq T(b, c) - \sum_{N' \setminus N} \left(c'_j - t_j(b', c')\right)$, $E \leq T(b, c)$ and $E' \leq T'(b, c)$ and, therefore, $R_i^c(N', b', c', E') = R_i(N', t(b''), E')$ and $R_i^c(N, b, c, E) = R_i(N, t(b, c), E)$. By resource monotonicity and population monotonicity of R, the desired inequality follows. ■

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Appendix that is not part of the submission for publication

We here show that Theorem 1 is tight. In what follows, let A denote the (standard) constrained equal awards rule and P denote the (standard) proportional rule.

- Let S be defined by

$$S(N,c,b,E)=\begin{cases} P(N,t(b,c),E), & \text{if } E\leq T(b,c) \\ t(b,c)+P(N,c-t(b,c),E-T(b,c)) & \text{if } E\geq T(b,c) \quad \text{and} \quad c_i\geq b_i \quad \text{for each } i\in \end{cases}$$
$$\qquad t(b,c)+A(N,c-t(b,c),E-T(b,c)) \quad \text{if } E\geq T(b,c) \quad \text{and} \quad c_i< b_i \quad \text{for some } i\in$$

  It is straightforward to show that S is a well-defined rule that satisfies truncation of excessive claims, baseline invariance, and polar baseline equivalence, but not baseline truncation.

- Let S be defined by

$$S\left(N,b,c,E\right)=\begin{cases} P(N,c,E) & \text{if } E\leq T(b,c) \\ t(b,c)+P(N,c-t(b,c),E-T(b,c)) & \text{if } E\geq T(b,c) \end{cases}$$

  It is straightforward to show that S is a well-defined rule that satisfies baseline truncation, baseline invariance, and polar baseline equivalence, but not truncation of excessive claims.

- Let S be defined by

$$S\left(N,b,c,E\right)=\begin{cases} t(b,c)-P(N,t(b,c),T-E) & \text{if } E\leq T(b,c) \\ P(N,c,E) & \text{if } E\geq T(b,c) \end{cases}$$

  It is straightforward to show that S is a well-defined rule that satisfies baseline truncation, truncation of excessive claims, and polar baseline equivalence, but not baseline invariance.

- Let S be defined by

$$S\left(N,b,c,E\right)=\begin{cases} t(b,c)-A(N,t(b,c),T-E) & \text{if } E\leq T(b,c) \\ t(b,c)+P(N,c-t(b,c),E-T(b,c)) & \text{if } E\geq T(b,c) \end{cases}$$

  It is straightforward to show that S is a well-defined rule that satisfies baseline truncation, truncation of excessive claims and baseline invariance, but not polar baseline equivalence.