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Discussion Papers on Business and Economics
No. 8/2009

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ISBN 978-87-91657-36-8

Sequential Legislative Lobbying

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Abstract

In this paper, we analyze the equilibrium of a sequential game-theoretical model of lobbying, due to Groseclose and Snyder (1996), describing a legislature that vote over two alternatives, where two opposing lobbies, Lobby 0 and Lobby 1, compete by bidding for legislators' votes. In this model, the lobbyist moving first suffers from a second mover advantage and will make an offer to a panel of legislators only if it deters any credible counter-reaction from his opponent, i.e., if he anticipates to win the battle. This paper departs from the existing literature in assuming that legislators care about the consequence of their votes rather than their votes per se. Our main focus is on the calculation of the smallest budget that he needs to win the game and on the distribution of this budget across the legislators. We study the impact of the key parameters of the game on these two variables and show the connection of this problem with the combinatorics of sets and notions from cooperative game theory.

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‡Financial support from the Spanish Ministry of Education and Science, Project No. SEJ2006-05455, is acknowledged.

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1 Introduction

In this paper, we consider a theoretical model of lobbying describing a legislature[1] that vote over two alternatives[2], and two opposing lobbies, Lobby 0 and Lobby 1, compete by bidding for legislators' votes[3]. We examine how the voting outcome and the bribes offered to the legislators depends on the lobbies' willingness to pay, legislators' preferences and the decision making process within the legislature.

There are many different ways to model the lobbying process. In this paper, we adopt the sequential model pioneered by Groseclose and Snyder (1996) and followed up by Banks (2000) and Diermeier and Myerson (1999). In their model, the competition between the two lobbies is described by a targeted offers game where each lobby gets to move only once, and in sequence. For most of the paper, Lobby 1 is pro-reform and moves first while Lobby 0 is pro-status quo and moves second. Votes are assumed to be observable. A strategy for each lobby is a profile of offers where the offer made to each legislator is assumed to be based on his/her vote and to be honored irrespective to the voting outcome. The net payoff of a lobby is its gross willingness to pay less the total amount of payments made to the legislators who ultimately vote for the policy advocated by this lobby. The legislators are assumed to care about the outcome of the vote process and about monetary offers. Therefore, voters do not truly act strategically as their voting behavior is simply a best response to the pair of offers made by the lobbies and is independent of the decisions of other legislators. We focus on the complete-information environment where the lobbies' and the legislators' preferences are known to the lobbies when they bid. We characterize the main features of the subgame perfect equilibrium of this game as a function of the following key parameters of the environment.

· The maximal willingness to pay of each lobby for winning[4] (i.e., to have their favorite policy selected). These two numbers represent the economic stakes under dispute and deter-

[1] We depart from voluminous literature based on the common agency setting in abandoning the assumption that policies are set by a single individual or by a cohesive, well-disciplined political party. In reality, most policy decisions, are made not by one person but by a group of elected representatives acting as a legislative body. Even when the legislature is controlled by a single party (as it is necessarily the case in a two-party system if the legislature consists of a unique chamber), the delegation members do not always follow the instructions of their party leaders.

[2] Hereafter, we will often refer to the two alternatives as being the status quo (alternative 0) versus the change or reform (alternative 1). While simplistic, many policy issues fit that formulation like for instance: to ratify or not a free-trade agreement, to forbid or not a free market for guns, to allow or not abortion.

[3] By legislators we mean here all individuals who have a constitutional role in the process of passing legislation. This may include individuals from what is usually referred to as being the executive branch like for instance the president or the vice-president.

[4] Or, under an alternative interpretation, their respective budgets.

mine the intensity and asymmetry of the competition.

· The voting rule describing the legislative process.

· The heterogeneity across legislator's preferences.

The binary setting considered in this paper is the simplest setting where we can tackle the joint influence of these three inputs on the final outputs. The first item consists of a single number per lobby: how much money this lobby is willing (able) to invest in this competition. The second item is also very simple. In this simplistic institutional setting, with no room for agenda setting or other sophisticated legislative action which would arise in the case of large multiplicity of issues[5], we only need to know what are the winning coalitions, i.e., the coalitions of legislators in position to impose the reform if the coalition unanimously supports this choice. Despite its apparent simplicity, this combinatorial object is extremely rich to accommodate a wide diversity of legislatures. Banks and Groseclose and Snyder focus on the standard majority game while Diermeier and Myerson consider the general case as we do. The third item describes the differences between the legislators others than those already attached to the preceding item if these legislators are not equally powerful or influent in the voting process. This "second" heterogeneity dimension refers to the differences between their intrinsic preferences for the reform versus the status quo. This difference measured in monetary units can be large or small and negative or positive. Diermeier and Myerson disregard this dimension by assuming that legislators are indifferent between the two policies while Banks and Groseclose and Snyder consider the general situation but derive their results under some specific assumptions. We assume that legislators prefer unanimously the reform to the status quo but differ with respect to the intensity of their preference.

All the papers in this literature assume that a legislator cares about his vote and not about the outcome of the vote as assumed in this paper. Legislators who care about outcomes are called *consequential* in contrast to *procedural* legislators who are those caring about their vote (Le Breton and Zaporozhets (2009)). Both assumptions are perfectly legitimate depending upon the type of policy issue under examination. The second assumption offers a technical advantage as legislators are not playing a game anymore since the votes of the other legislators do not influence their vote. In contrast, the first assumption preserves the game theoretical nature of the voting stage as many legislators want to know if they are pivotal and must therefore predict the voting behavior of the others.

The first contribution consists in identifying the conditions under which the lobby moving

[5]Many formal models of the legislative process have been developed by social scientists to deal with more complicated choice environments. We refer the reader to Grossman and Helpman (2001) for lobbying models with more than two alternatives.

first will make positive offers to some legislator. In this sequential game, the lobby moving last has an advantage as it can react optimally to the offers of its opponent without any further possibility or reaction. If the asymmetry is too weak, Lobby 1 will abandon the prospect of influencing the legislature as it will be rationally anticipating its defeat; in fact, it will make offers only if it anticipates to win for sure. If it does not make any offer, it is enough for Lobby 0 to compensate a minimal winning coalition of legislators for their intrinsic preferences towards reform. Lobby 1 will participate if its willingness to pay or budget is larger than the willingness to pay or budget of Lobby 0. This minimal amount of asymmetry, that we call the *victory threshold*, defines by how much the stake of Lobby 1 must overweight the stake of Lobby 0 to make sure that Lobby 1 wins the game. Our first result states that the calculation of the victory threshold amounts to calculate the supremum of a linear form over a convex polytope which is closely related to the polytope of balanced families of coalitions introduced in cooperative game theory to study the core and other solutions. The practical value of this result relies on the fact that we can take advantage of the voluminous amount of work which has been done on the description of balanced collections. When heterogeneity across legislators' preferences is ignored, the victory threshold only depends upon the simple game describing the rules of the legislature. It corresponds to what has been called by Diermeier and Myerson, the hurdle factor of the legislature. Quite surprisingly, this single parameter acts a summary statistic as long as we want to predict the minimal budget that Lobby 1 needs to invest to win the game. We will illustrate the connections between the computation of the hurdle factor and the *covering problem*, which is one of the most famous, but also difficult, problem in the combinatorics of sets or hypergraphs[6].

The second contribution consists in showing that the victory threshold can be alternatively calculated, surprisingly, as the maximum of specific criteria of equity over the set of imputations of a cooperative game with transferable utility (TU game) attached to the simple game of the legislature. The specific equity criterion is the minimum over all coalitions of the ratio of the difference between what the members of the coalition get in the imputation and what they could get on their own and the size of the coalition, i.e., the first component in the lexicographic order supporting the per-capita nucleolus that was introduced by Grotte (1970). The connection with the theory of cooperative games turns out to be even more surprising as it allows to provide a complete characterization of the second dimension of the optimal offer strategy of Lobby 1. From what precedes, we know that the size of the lobbying budget is the victory threshold times the willingness to pay (or budget) of Lobby

[6]Of course, once it is noted that the hurdle factor is the fractional covering number of a specific hypergraph, we can take advantage of the enormous body of knowledge in that area of combinatorics.

0. It remains to understand how this budget is going to be allocated across the legislators. This is of course an important question as we would like to understand what are the characteristics of a legislator which determine the willingness of Lobby 1 to buy its support and the amount that he will receive for the selling his vote. As already discussed, legislators may differ along two lines: the intensity of their preference for Lobby 1 and their position/power in the legislature. Likely the price of the vote of a legislator will be a function of both parameters. We show that the set of equilibrium offers is the per-capita least core of the cooperative game used to calculate the victory threshold. We investigate their dependency upon the desirability of the legislators and we show that it is not always the case that more desirable legislators receive better offers. We also show how to calculate these prices in the case of some important real world simple games. One important conclusion is that these prices have little to do with the power of a legislator as calculated through either the Banzhaf index (Banzhaf (1965), (1968)) or the Shapley-Shubik index (Shapley and Shubik (1954)). This suggests that the axiomatic theory of power measurement may not be fully relevant to predict the payoffs of the players in a game like this one[7].

Related Literature

The literature on lobbying is very dispersed and voluminous[8]. The closest papers to ours are Banks (2000), Dekel, Jackson and Wolinsky (2006a,b), Diermeier and Myerson (1999), Groseclose and Snyder (1996), Le Breton and Zaporozhets (2009), Young (1978a,b,c) and Shubik and Young (1978d). All these papers consider a binary setting but in contrast to this paper, they assume that legislators care about their vote and money rather than the outcome. As already mentioned, the two-round sequential vote buying model that we consider is from the fundamental contribution of Groseclose and Snyder. Banks as well as Diermeier and Myerson also consider this game. Their specific assumptions and focus are however quite different from ours. Banks and Groseclose and Snyder are primarily interested in identifying the number and the identity of the legislators who will receive an offer in the case of the simple majority game. By considering this important but specific symmetric game, they eliminate the possibility of evaluating the impact of the legislative power on the outcome. However, they consider more general profiles of legislators' preferences: Instead of our unanimity assumption in favor of a reform, Banks assumes that a majority of legislators has an intrinsic preference for the status quo. This implies that Lobby 1 needs to bribe at least a majority to win; Banks provides conditions on the profile under which this majority will be minimal or maximal but does not determine the optimal size in the general case. Diermeier and

[7]This echoes Snyder, Ting and Ansolabehere (2005).

[8]We refer the reader to Grossman and Helpman (2001) for a description of the state of the art.

Myerson assume instead that legislators do not have any intrinsic preference but consider an arbitrary simple game. Their main focus is on the architecture of multicameral legislatures and on the optimal behavior of each chamber under the presumption that it can select its own hurdle factor to maximize the aggregate offer made to its members. Our paper is very much related to the contributions of Young who has analyzed a similar game and derived independently Proposition 2. He should receive credit for being the first one to point out the relevance of the least core and the nucleolus to predict some dimensions of the equilibrium strategies of the lobbyists.

Dekel, Jackson and Wolinsky examine an open-ended sequential game where lobbies alternate in increasing their offers to legislators. By allowing lobbies to keep responding to each other with counter-offers, their game eliminates the asymmetry and the resulting second mover advantage of the game investigated by Groseclose and Snyder. Several settings are considered depending upon the type of offers that lobbies can make to legislators (Up-front payments versus promises contingent upon the voting outcome) and upon the role played by budget constraints[9]. The difference in the budgets of the lobbies plays a critical role in determining which lobby is successful when lobbies are budget constrained, and the difference in their willingness to pay plays an important role when they are not budget constrained. When lobbies are budget constrained, their main result states that the winning lobby is the one whose budget plus half of the sum of the value that each legislator attaches to voting in favor of this lobby exceeds the corresponding magnitude calculated for the other lobby. In contrast, when lobbies are not budget constrained, what matters are the lobbies' valuations and the intensity of preferences of a particular "near-median" group of legislators. The lobby with a-priori minority support wins when its valuation exceeds the other lobby's valuation by more than a magnitude that depends on the preferences of that near-median group. With our terminology, we can say that their main results are motivated by the derivation of the victory threshold. Once the value of this threshold is known, the identity of the winner as well as the lobbying expenditures and the identity of bribed legislators follow. Note however that they limit their analysis to the simple majority game and are not in position to evaluate the intrinsic role of the simple game and the legislative power of legislators.

Note finally that our game would have the features of a Colonel Blotto game if the two lobbies make their offers simultaneously instead of sequentially. These games are notoriously difficult to solve and very little is known in the case of asymmetric players.

[9]These considerations which are irrelevant in the case of our two-round sequential game are important in their game.

2 The Model and the Game

In this section, we describe formally the main ingredients of the problem as well as the lobbying game which constitute our model of vote-buying by lobbyists.

The external forces that seek to influence the legislature are represented by two players, whom we call Lobby 0 and Lobby 1. Lobby 1 wants the legislature to pass a bill (change, proposal, reform) that would change some area of law. Lobby 0 is opposed to this bill and wants to maintain the status quo. Lobby 0 is willing to spend up to $W_0 > 0$ dollars to prevent passage of the bill while Lobby 1 is willing to pay up to W_1 dollars to pass the bill. Sometimes, we refer to these two policies in competition as being policies 0 and 1. We assume that $\Delta W \equiv W_1 - W_0 > 0$. While this assumption may receive different interpretations[10], we will assume here that the two lobbies represent faithfully the two opposite sides of the society on this binary social agenda and therefore that policy 1 is the socially efficient policy. We could consider that the two lobbies represent more private or local interests and that W_1 and W_0 ignore the implications of these policies on the rest of the society: In that case the reference to social optimality should be abandoned. Finally, we could consider instead the budgets B_1 and B_0 of the two lobbies, and assume that they are budget constrained, i.e., that $B_1 \leq W_1$ and $B_0 \leq W_0$. Under that interpretation, the ratio $\frac{W_1}{W_0}$ should be replaced by the ratio $\frac{B_1}{B_0}$. This ratio which is (by assumption) larger than 1 will be a key parameter in our equilibrium analysis. Depending upon the interpretation, it could measure the intensity of the superiority of the reform as compared to the status quo or the ex ante advantage of Lobby 1 over Lobby 0 in terms of budgets.

The legislature is described by a *simple game*[11], i.e., a pair $(N, \mathcal{W})$, where $N = \{1, \ldots, n\}$ is the set of legislators and $\mathcal{W}$, the set of *winning* coalitions, satisfies (i) $\emptyset \notin \mathcal{W} \ni N$ and (ii) $S \in \mathcal{W}$ and $S \subseteq T$ implies $T \in \mathcal{W}$. Sometimes, we identify a simple game $(N, \mathcal{W})$ with its corresponding TU game (N, V) defined by $V(S) = 1$ if $S \in \mathcal{W}$ and $V(T) = 0$ if $T \in 2^N \setminus \mathcal{W}$. The interpretation is the following. A bill is adopted if and only if the subset of legislators who voted for the bill forms a winning coalition. From that perspective, the set of winning coalitions describes the rules operating in the legislature to make decisions. A coalition C is *blocking* if $N \setminus C$ is not winning: At least one legislator from C is needed to form a winning

[10] As explained forcefully in Dekel, Jackson and Wolinsky (2006a,b), in general, the equilibrium predictions will be sensitive to the type of offers that can be made by the lobbies and whether they are budget constrained or not. As explained later, these considerations are not relevant in the case of our lobbying game.

[11] In social sciences (Shapley (1962)), it is sometimes called a committee or a voting game. In computer science, it is called a quorum system (Holzman, Marcus and Peleg (1997)) while in mathematics, it is called a hypergraph (Berge (1989), Bollobás (1986)). An excellent reference is Taylor and Zwicker (1999).

coalition. We will denote by $\mathcal{B}$ the subset of blocking coalitions[12]; from the definition, the status quo is maintained as soon as the set of legislators who voted against the bill forms a blocking coalition. The simple game is called *proper* if $S \in \mathcal{W}$ implies $N \backslash S \notin \mathcal{W}$. The simple game is called *strong* if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$ and *constant-sum* if it is both proper and strong, i.e., equivalently if $\mathcal{B} = \mathcal{W}$[13]. The simple game is called *symmetric* if $S \in \mathcal{W}$ implies $T \in \mathcal{W}$ for all $T \subseteq N$ such that $\#T = \#S$. The set of minimal (with respect to inclusion) winning (blocking) coalitions will be denoted $\mathcal{W}_m(\mathcal{B}_m)$. A legislator is a *dummy* if he is not a member of any minimal winning coalition, while a legislator is a *vetoer* if he belongs to all blocking coalitions. A group of legislators forms an *oligarchy* if a coalition is winning iff it contains that group, i.e., each member of the oligarchy is a vetoer and the oligarchy does not need any extra support to win (legislators outside the oligarchy are dummies). When the oligarchy consists of a single legislator, the game is called dictatorial.

In this paper, all legislators are assumed to be biased towards policy 1, i.e., all of them will vote for policy 1 against policy 0 if no other event interferes with the voting process. It is introduced here for the sake of simplicity as, otherwise, we would have to consider an additional parameter of differences among the legislators that we prefer to ignore for the time being. Indeed, in contrast to Banks (2000) and Groseclose and Snyder (1996), our assumption on the preferences of legislators rule out the existence of horizontal heterogeneity. However, legislators also value money and we introduce instead some form of vertical heterogeneity. Precisely, we assume that legislators may differ according to their willingness to depart from social welfare. The type of legislator i, denoted by $\alpha^i \geq 0$, is the minimal amount of dollars that he needs to receive in order to sacrifice one dollar of social welfare. Therefore if the policy adopted generates a level of social welfare equal to W, the payoff of legislator i if he receives a transfer t^i is

$$t^i + \alpha^i W.$$

To promote passage of the bill, Lobby 1 can promise to pay money to individual legislators conditional on their supporting the bill. Similarly, Lobby 0 can promise to pay money to individual legislators conditional on their opposing the bill. We denote by $t_0^i \geq 0$ and $t_1^i \geq 0$ the (conditional) offers made to legislator i by lobbies 0 and 1 respectively. The corresponding n-dimensional vectors will be denoted respectively by t_0 and t_1.

The timing of actions and events that we consider to describe the lobbying game is the following.

1. Nature draws the type of each legislator.

[12] In game theory, $(N, \mathcal{B})$ is often called the dual game.

[13] When the simple game is constant-sum, the two competing alternatives are treated equally.

2. Lobby 1 makes contingent monetary offers to individual legislators.

3. Lobby 0 observes the offers made by Lobby 1 and makes contingent monetary offers to individual legislators

4. Legislators vote.

5. Payments (if any) are implemented.

This game has $n+2$ players. A strategy for a lobby is a vector in $\mathbb{R}^n_+$. Each legislator can chose among two (pure) strategies: to oppose or to support the bill.

To complete the description of the game, it remains to specify the information held by the players when they act. In this paper, we have already implicitly assumed that the votes of the legislators are observable, i.e., open voting, and that the vector $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ of legislators' types is common knowledge and without loss of generality such that $\alpha^1 \leq \alpha^2 \leq \cdots \leq \alpha^n$. We refer to this informational environment as *political certainty*. It has two implications: First, the lobbies know the types of the legislators when they make their offers and second, each legislator knows the type of any other legislator when voting[14].

3 The Victory Threshold

In this section, we begin our examination of the *subgame perfect Nash equilibria* of the lobbying game. Hereafter, we will refer to them simply as equilibria. Our first objective is to calculate a key parameter of the game, that we call the *victory threshold*. Once calculated, this parameter leads to the following preliminary description of the equilibrium. Either, the ratio $\frac{W_1}{W_0}$ is larger than or equal to the victory threshold and then Lobby 1 makes an offer and wins the game, or $\frac{W_1}{W_0}$ is smaller than the victory threshold and then Lobby 1 does not make any offer and Lobby 0 wins the game. The victory threshold depends both upon the vector of types α and the simple game $(N, \mathcal{W})$. Given the second mover advantage, the victory threshold is larger than or equal to 1. Therefore, while necessary, $W_1 > W_0$ is not sufficient in general to guarantee the victory of Lobby 1. The victory threshold provides the smallest value of the relative difference leading to such victory.

A coalition $T \subseteq N$ will be called *blocking*$^+$ if $S = T \setminus \{i\} \in \mathcal{B}_m$ for all $i \in T$. Let us denote by $\mathcal{B}^+_m$ the family of minimal blocking$^+$ coalitions. To prepare for the first proposition, let us examine intuitively the reaction $t_0 = (t^i_0)_{i \in N}$ of Lobby 0 to the vector of offers $t_1 = (t^i_1)_{i \in N}$ made by Lobby 1. The legislators can be partitioned into three groups. The first group

[14] The environment where the type α^i of legislator i is a private information, to which we refer as *political uncertainty*, is analyzed in Le Breton and Zaporozhets (2007) in the case where the two lobbies move simultaneously.

S_1 consists of the legislators i such that $t_0^i \leq t_1^i$. The second group S_2 consists of all the legislators i such that $t_1^i < t_0^i \leq t_1^i + \alpha^i \Delta W$. The third group S_3 consists of all the legislators i such that $t_0^i > t_1^i + \alpha^i \Delta W$.

Voting for the reform is a dominant strategy for the legislators from the first group while voting for the status quo is a dominant strategy for the legislators in the third group. The strategic interaction and the necessity to evaluate the probability of being pivotal only apply to the legislators from the second group. If a legislator does not consider himself to be pivotal, then it is optimal to vote for the status quo. Instead, if he considers his vote to be pivotal, then it is optimal to vote for the reform. We want the profile of votes from the legislators in that group to form a Nash equilibrium. Let S be the coalition of legislators being in the second or third group, i.e., $S = S_2 \cup S_3$. When is it the case that the profile where all the legislators in S vote for the status quo is a Nash equilibrium?

From what precedes, it is necessary and sufficient that no legislator i from S_2 considers his vote to be pivotal. This will be the case if $S \setminus \{i\} \in \mathcal{B}$. For any $S \in \mathcal{B}$ let

$$S_2 = \{i \in S \mid S \setminus \{i\} \in \mathcal{B}\}$$

and $S_3 = S \setminus S_2$. Let $\widehat{\mathcal{B}}_m$ be the family of minimal coalitions in $\mathcal{B}$ according to the order $\lhd$ defined as follows for all $S, S' \in \mathcal{B}$: $S \lhd S'$ iff $S \subseteq S'$ and $S_2 \subseteq S_2'$ (and, hence $S_3 \subseteq S_3'$).

The strategic optimal response of Lobby 0 is now easy to describe. From its perspective, the cheapest coalitions belong to the family $\widehat{\mathcal{B}}_m$. To any such coalition $S = S_2 \cup S_3$, the smallest cost is equal to:

$$\sum_{i \in S_2} t_1^i + \sum_{i \in S_3} \left(t_1^i + \alpha^i \Delta W\right)$$

It is interesting to see what coalitions are elements of $\widehat{\mathcal{B}}_m$. First, all the coalitions S in $\mathcal{B}_m$ belong to $\widehat{\mathcal{B}}_m$. They correspond to the case where $S_2 = \emptyset$. Their cost is therefore:

$$\sum_{i \in S} \left(t_1^i + \alpha^i \Delta W\right)$$

At the other extreme, all the coalitions S in $\mathcal{B}_m^+$ belong to $\widehat{\mathcal{B}}_m$. They correspond to the case where $S_3 = \emptyset$. Their cost is therefore:

$$\sum_{i \in S} t_1^i$$

This reasoning calls for two observations. We note first that in the case where the simple game is symmetric, we obtain: $\widehat{\mathcal{B}}_m = \mathcal{B}_m \cup \mathcal{B}_m^+$. Second, it is important to note that we have determined conditions under which there exists a Nash profile of votes leading to rejection of the reform. This does not mean of course that this Nash equilibrium is unique. For the sake of illustration, consider the case of a symmetric game for which the minimal size of a blocking coalition if b and let $S \in \mathcal{B}_m^+$ with the offer defined above. From above, we know that voting against the reform for all voters in S leads to a Nash equilibrium: The $b+1$ voters in S vote no if $t_0^i = t_1^i + \varepsilon$ where $\varepsilon > 0$ for all $i \in S$ and $t_0^i = 0$ otherwise. There are however other Nash equilibria. For instance take two legislators, say 1 and 2, out of the $b+1$ legislators and let them vote for the reform while the $b-1$ others keep voting against the reform. This profile of votes induces the reform as $b-1$ is not enough to block. It is a Nash equilibrium. The voters in S who keep voting against the reform play optimally as they are not pivotal. The voters 1 and 2 who vote in favor of the reform also vote optimally as they are pivotal, $t_1^1 + \alpha^1 \Delta W > t_0^1$ and $t_1^2 + \alpha^2 \Delta W > t_0^1$. This new Nash equilibrium calls for some coordination and there are $\frac{b(b+1)}{2}$ any such equilibria. The calculation of the cheapest offer is subordinated to the selection of this particular continuation equilibrium[15] which focuses on the worst case from the perspective of Lobby 1: Following its vector of offers, what is the worst Nash equilibria in the continuation game?

The "pessimistic" subgame-perfect equilibrium of this sequential version of the lobbying game can be easily described. Let $t_1 = (t_1^1, t_1^2, \ldots, t_1^n) \in \mathbb{R}_+^n$ be Lobby 1's offers. Lobby 0 will find profitable to make a counter-offer if there exists a coalition $S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m$ such that:

$$\sum_{i \in S_2} t_1^i + \sum_{i \in S_3} \left(t_1^i + \alpha^i \Delta W\right) < W_0$$

Indeed, in this case, there exists a vector $t_0 = (t_0^1, t_0^2, \ldots, t_0^n)$ of offers such that:

$$t_1^i + \alpha^i \Delta W < t_0^i \text{ for all } i \in S_3,\ t_1^i < t_0^i \text{ for all } i \in S_2 \text{ and } \sum_{i \in S} t_0^i < W_0.$$

Therefore, if Lobby 1 wants to make an offer that cannot be cancelled by Lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S_2} t_1^i + \sum_{i \in S_3} \left(t_1^i + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m$$

The cheapest offer t_1 meeting these constraints is solution of the following linear program:

$$\begin{array}{c} \min_{t_1 \in \mathbb{R}_+^n} \sum_{i \in N} t_1^i \\ \text{subject to the constraints} \\ \sum_{i \in S_2} t_1^i + \sum_{i \in S_3} (t_1^i + \alpha^i \Delta W) \geq W_0 \text{ for all } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m \end{array} \tag{1}$$

[15] There are also some mixed Nash equilibria.

Lobby 1 will find profitable to offer the optimal solution t_1^* of Problem (1) if the optimal value to this linear program is less than W_1. It is then important to be able to compute this optimal value. To do so, we first introduce the following definition from combinatorial theory.

Definition 1. Let $\mathcal{C}$ be a family of coalitions. For $i \in N$ define $\mathcal{C}^i = \{S \in \mathcal{C} \mid i \in S\}$.

1. A vector $\delta \in \mathbb{R}^{\#\mathcal{C}}$ is called a vector of *subbalancing coefficients* for $\mathcal{C}$ if

$$\begin{aligned} \sum_{S \in \mathcal{C}^i} \delta(S) &\leq 1 \text{ for all } i \in N \\ \text{and } \delta(S) &\geq 0 \text{ for all } S \in \mathcal{C}. \end{aligned}$$

2. The collection $\mathcal{C}$ is *balanced* if there exists $\delta \in \mathbb{R}^{\#\mathcal{C}}$, called vector of *balancing coefficients* for $\mathcal{C}$, such that

$$\begin{aligned} \sum_{S \in \mathcal{C}^i} \delta(S) &= 1 \text{ for all } i \in N \\ \text{and } \delta(S) &\geq 0 \text{ for all } S \in \mathcal{C}. \end{aligned}$$

(Hence, a balanced collection is nonempty.)

Let:

$$V(S) = \begin{cases} W_0 - \sum_{i \in S_3} \alpha^i \Delta W & \text{, if } S = S_2 \cup S_3 \in \widehat{\mathcal{B}}_m, \\ 0 & \text{, if } S \notin \widehat{\mathcal{B}}_m \end{cases} \tag{2}$$

The following result summarizes the equilibrium analysis of the sequential game.

Proposition 1.

1. If $W_1 \geq \sum_{S \in \widehat{\mathcal{B}}_m} \delta(S) V(S)$ for all vectors of subbalancing coefficients δ for $\widehat{\mathcal{B}}_m$, then there exists a subgame perfect equilibrium in which Lobby 1 makes an offer t_1^* selected among the optimal solutions to Problem (1), Lobby 0 does not make any offer and the bill is passed.

2. If $W_1 < \sum_{S \in \widehat{\mathcal{B}}_m} \delta(S) V(S)$ for at least one vector of subbalancing coefficients δ for $\widehat{\mathcal{B}}_m$, then, for any $S \in \widehat{\mathcal{B}}_m$ that satisfies $W^* \equiv \sum_{i \in S_3} \alpha^i \Delta W = \min\limits_{T \in \widehat{\mathcal{B}}_m} \sum_{i \in T_3} \alpha^i \Delta W$, $W^* < W_0$ and for any $0 < \varepsilon \leq W_0 - W^*$ there exists a subgame perfect ε-equilibrium in which Lobby 1 does not make any offer and Lobby 0 offers t_0^* where $t_0^{*i} = \alpha^i \Delta W + \frac{\varepsilon}{\#S}$ for all $i \in S_3$ and $t_0^{*j} = \frac{\varepsilon}{\#S}$ for all $j \in S_2$ so that the bill is not passed.

Proof: Let $v^*\left(\widehat{\mathcal{B}},\alpha\right)$ be the optimal value of Problem (1). From the duality theorem of linear programming, $v^*\left(\widehat{\mathcal{B}},\alpha\right)$ is the optimal value of the following linear program:

$$\begin{gathered}\max_{\delta}\sum_{S\in\widehat{\mathcal{B}}_m}\delta(S)\left[W_0-\sum_{i\in S_3}\alpha^i\Delta W\right]\\ \text{subject to the constraints}\\ \sum_{S\in\widehat{\mathcal{B}}_m^i}\delta(S)\leq 1 \text{ for all } i\in N\\ \text{and } \delta(S)\geq 0 \text{ for all } S\in\widehat{\mathcal{B}}_m\end{gathered}$$

The conclusion follows. □

When $\alpha=0$, the determination of the cheapest offer for Lobby 1 simplifies to:

$$\begin{gathered}\min_{t_1\in\mathbb{R}_+^n}\sum_{i\in N}t_1^i\\ \text{subject to the constraints}\\ \text{and }\sum_{i\in S}t_1^i\geq W_0 \text{ for all } S\in\mathcal{B}_m\end{gathered}\tag{3}$$

It is immediate to see that the optimal value $v^*\left(\widehat{\mathcal{B}},\mathbf{0}\right)$ of (3) is proportional to W_0. Hereafter, it will denoted simply by $\gamma^*(\mathcal{B})W_0$ where $\gamma^*(\mathcal{B})$ is the hurdle factor as defined by Diermeier and Myerson (1999) which is the value of the problem:

$$\begin{gathered}\max_{\delta}\sum_{S\in\mathcal{B}}\delta(S)\\ \text{subject to the constraints}\\ \sum_{S\in\mathcal{B}^i}\delta(S)\leq 1 \text{ for all } i\in N\\ \text{and } \delta(S)\geq 0 \text{ for all } S\in\mathcal{B}\end{gathered}$$

It is straightforward to show that the value $v^*\left(\widehat{\mathcal{B}},\alpha\right)$ of (1) lies somewhere between $\gamma^*(\mathcal{B})W_0-\Delta W\sum_{i\in N}\alpha^i$ and $\gamma^*(\mathcal{B})W_0$. This is not surprising since this linear program has more constraints than the linear program attached to the procedural behavioral model and therefore $v^*\left(\mathcal{B},\alpha\right)$ is at least equal to the victory threshold derived in the procedural case.

From above, we deduce that if we are in Case 1 of Proposition 1, then:

$$\frac{W_1}{W_0}\geq\frac{\gamma^*(\mathcal{B})+\sum_{i\in N}\alpha^i}{1+\sum_{i\in N}\alpha^i}\tag{4}$$

The practical value of Proposition 1 is to reduce the derivation of the victory threshold to the exploration of the geometry of a convex polytope: the polytope of vectors of subbalancing coefficients. To use it efficiently, it may be appropriate to consider an arbitrary family of balanced coalitions, i.e., with edges not necessarily in $\widehat{\mathcal{B}}_m$. In the statement, we can trivially replace "$\sum_{S\in\widehat{\mathcal{B}}_m}\delta(S)\left[W_0-\sum_{i\in S_3}\alpha^i\Delta W\right]$ for all vectors of subbalancing coefficients δ for $\widehat{\mathcal{B}}_m$" by "$\sum_{S\subseteq N}\delta(S)V(S)$ for all vectors of balancing coefficients δ for 2^N". The first formulation is useful as soon as we are in position to characterize the vector of subbalancing coefficients attached to the family of coalitions $\widehat{\mathcal{B}}_m$, i.e., to the simple game[16]. This amounts essentially to explore the combinatorics of the simple game. A classification of simple games was first provided by von Neumann and Morgenstern (1944) and further explored by Isbell (1956, 1959). The second formulation takes advantage of the tremendous volume of research accomplished in cooperative game theory. Indeed, is well know since Bondareva (1963) and Shapley (1967) that a TU game has a nonempty core iff it is balanced. As pointed out by Shapley, this amounts to check the balancedness inequalities for the extreme points of the polytope of balanced collections of coalitions. He demonstrated that vector δ is an extreme point of the polytope of balanced collections iff the collection of coalitions $\{S\subseteq N\mid\delta(S)>0\}$ is minimal in terms of inclusion within the set of balanced collections of coalitions. A minimal balanced collection has at most n sets[17]. Peleg (1965) has given an algorithm for constructing the minimal balanced sets inductively. We illustrate the mechanical use of Proposition 1 through a sequence of simple examples.

Example 1. Consider the simple majority game with 3 legislators where $S\in\mathcal{B}_m$ iff $\#S=2$, i.e., $S=\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\mathcal{B}_m^+=\{N\}$. Besides the partitions, the unique minimal balanced family of coalitions is $\{\{1,2\},\{1,3\},\{2,3\}\}$ with the vector of balancing coefficients $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. From the ordering of the α^i and the proof of Proposition 1, we deduce that

$$\begin{aligned}
v^*(\mathcal{B},\alpha) &= \max\left\{W_0-\left(\alpha^1+\alpha^2\right)\Delta W,\ \frac{3W_0-2\left(\alpha^1+\alpha^2+\alpha^3\right)\Delta W}{2},W_0\ ,\ 0\right\}\text{ and}\\
\gamma^*(\mathcal{B}) &= \frac{3}{2}.
\end{aligned}$$

The first and last terms are never the largest; $v^*(\mathcal{B},\alpha)=W_0$ whenever $2\left(\alpha^1+\alpha^2+\alpha^3\right)\Delta W\geq W_0$. When $\alpha^1=\alpha^2=\alpha^3$, this happens when:

$$W_1\geq\frac{1+6\alpha^1}{6\alpha^1}W_0$$

We will examine later how to derive the optimal (offers) of Lobby 1 and in particular the personal characteristics of the legislators who are offered some positive amount. This will

[16] Holzman, Marcus and Peleg (1997) contains results on the polytope of balancing coefficients for an arbitrary proper and strong simple game.

[17] We refer the reader to Owen (2001) and Peleg and Sudhölter (2003) for a complete and nice exposition of this material.

depend obviously on two main features: α^i, i.e., his/her personal propensity to vote against social welfare and also its position in the family of coalitions. If legislator i is a dummy then, obviously, $t_1^i = 0$. But if he is not a dummy, then in principle all situations are conceivable: He may receive something in all optimal offers, in some of them or in none of them. It will be important to know the status of a legislator according to this classification in three groups.

Example 2. Consider the simple game with 4 legislators[18] where $S \in \mathcal{B}_m$ iff $S = \{1,2\}$, $\{1,3\}$, $\{1,4\}$ or $\{2,3,4\}$. According to Shapley (1967), besides the partitions, the minimal balanced families of coalitions are (up to permutations) the collections

$$\begin{aligned}&\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\},\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\},\\&\{\{1,2\},\{1,3\},\{2,3\},\{4\}\},\{\{1,2\},\{1,3,4\},\{2,3,4\}\}\end{aligned}$$

with the respective vectors of balancing coefficients $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)$, $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right)$ and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. We deduce from the proof of Proposition 1 that:

$$\begin{aligned}v^*(\widehat{\mathcal{B}}_m,\alpha) &= \max\left\{\begin{array}{c}\frac{4W_0-\left(3\alpha^1+\alpha^2+\alpha^3+\alpha^4\right)\Delta W}{3},\frac{5W_0-3\left(\alpha^1+\alpha^2+\alpha^3+\alpha^4\right)\Delta W}{3},\frac{2W_0-\left(2\alpha^1+\alpha^2+\alpha^3\right)\Delta W}{2},\\ \frac{3W_0-\left(2\alpha^1+2\alpha^2+\alpha^3+\alpha^4\right)\Delta W}{2},\ W_0-(\alpha^1+\alpha^2)\Delta W, W_0, 0\end{array}\right\},\\ \gamma^*(\mathcal{B}) &= \frac{5}{3}\end{aligned}$$

When $\alpha^1 = \alpha^2 = \alpha^3 = \alpha^4$, we obtain:

$$v^*(\widehat{\mathcal{B}}_m,\alpha) = \frac{1}{6}\ \max\left\{8W_0 - 12\alpha^1\Delta W, 10W_0 - 24\alpha^1\Delta W, 9W_0 - 18\alpha^1\Delta W, 6W_0\right\}$$

The representation of the different affine functions of α^1 that appear in the above expression leads to

$$v^*(\widehat{\mathcal{B}}_m,\alpha) = \begin{cases}\frac{10W_0-24\alpha^1\Delta W}{6} & \text{, if } 0 \le \alpha^1 \le \frac{W_0}{6\Delta W},\\ \frac{8W_0-12\alpha^1\Delta W}{6} & \text{, if } \frac{W_0}{6\Delta W} \le \alpha^1 \le \frac{7W_0}{12\Delta W},\\ W_0 & \text{, if } \alpha^1 \ge \frac{7W_0}{12\Delta W}.\end{cases}$$

Example 3. Consider the following simple game with 3 legislators and $S \in \mathcal{B}_m$ iff $S = \{1,2\}$ or $\{1,3\}$. The set of vectors of balancing coefficients has already been described in Example 1. We deduce easily that

$$\begin{aligned}v^*(\widehat{\mathcal{B}}_m,\alpha) &= \max\left\{W_0 - \alpha^1\Delta W, 0\right\} \text{ and}\\ \gamma^*(\mathcal{B}) &= 1\end{aligned}$$

[18] As demonstrated by von Neumann and Morgenstern ((1944), 52C), this is the unique proper and strong simple four-person game without dummies.

If we reverse the order of plays between the two lobbies, then it suffices to replace $\mathcal{B}$ by $\mathcal{W}$ in all the statements above. Using the same technique, we would compute $v^*(\mathcal{W}, \alpha)$ and $\gamma^*(\mathcal{W})$. We will call these two numbers respectively the *dual victory threshold* and the *dual integral factor.*

4 Complements and Extensions

Proposition 1 constitutes an important element of the toolkit to determine the victory threshold. In this section, we continue the exploration of the problem having in mind to add more elements in the toolkit. In the first subsection, we show that in the special case where $\alpha = 0$, our problem is strongly connected to one of the most famous problems in the combinatorics of sets. We elaborate on the relationship with this branch of applied mathematics and show how to take advantage of this body of knowledge to get a better understanding of our own questions, on top of which is the determination of the hurdle factors attached to a simple game. In the second subsection, we illustrate the use of this branch of mathematics through a selected sample of examples. In the third subsection, we show that, quite surprisingly, the set of equilibrium offers to the legislators made by the first mover lobby coincides with the per capita least core of the simple game. We show that this per capita least core coincides with the least core (and therefore contains the nucleolus) when $\alpha = 0$.

4.1 Fractional Matchings and Coverings

The main purpose of this subsection is to connect our problem to the covering problem which is considered to be one of the most famous problems in the combinatorics of sets. As pointed out by Füredi (1988), "the great importance of the covering problem is supported by the fact that apparently all combinatorial problems can be reformulated as the determination of the covering number of a certain hypergraph". A *hypergraph* is an ordered pair $H = (N, \mathcal{H})$ where N is a nonempty finite set of n vertices and $\mathcal{H}$ is a nonempty collection of nonempty subsets of N called edges. The *rank* of H is the integer $r(H) \equiv \max\{\#E : E \in \mathcal{H}\}$. If every member of $\mathcal{H}$ has r elements, we call H *r-uniform.* An r-uniform hypergraph H is called *r-partite* if there exists a partition $\mathcal{P}$ of N such that $\#(S \cap E) = 1$ holds for all $E \in \mathcal{H}$ and all $S \in \mathcal{P}$. The *degree* of the hypergraph H, denoted $D(H)$, is the number $\max_{i \in N} \#\mathcal{H}^i$ (for the notation $\mathcal{H}^i$ see Definition 1). A hypergraph is D-regular if $\#\mathcal{H}^i = D(H)$ for all $i \in N$. Given positive integers k and s, a hypergraph is *k-wise s-intersecting* if the cardinality of the intersection of any k edges is at least s; *s-intersecting* and *intersecting* are used for 2-wise s-intersecting and 1-intersecting, respectively. A hypergraph $(N, \mathcal{H})$ is an *r-clique* if it is intersecting and if there does not exist $S \in 2^N \setminus \mathcal{H}$ such that $\#S = r$ and $(N, \mathcal{H} \cup \{S\})$ is an intersecting hypergraph, that is, an r-clique is a maximal intersecting family of rank r.

Given a positive integer k, a *k-cover* of H is a vector $t \in \{0, 1, \ldots, k\}^n$ such that

$$\sum_{i \in S} t^i \geq k \text{ for all } S \in \mathcal{H}. \tag{5}$$

A *k-matching* of H is a vector $\delta \in \{0, \ldots, k\}^{\mathcal{H}}$ such that

$$\sum_{S \in \mathcal{H}^i} \delta(S) \leq k. \tag{6}$$

A 1-cover (1-matching) is simply called a *cover* (*matching*) of H. Note that the covers of H may simply be identified with a subsets of N that have nonempty intersections with any edge[19]. Similarly, the matchings of H may be identified with collections of pairwise disjoint members of $\mathcal{H}$. A k-cover t^* minimizing $\sum_{i \in N} t^i$ subject to the constraints (5) is called an *optimal* k-cover and $\gamma_k^*(H) \equiv \sum_{i \in N} t^{*i}$ is called the *k-covering number*. A k-matching δ^* maximizing $\sum_{S \subseteq N} \delta(S)$ subject to the constraints (6) is called an *optimal* k-matching and $\mu_k^*(H) \equiv \sum_{S \subseteq N} \gamma^*(S)$ is called the *k-matching number*. Hence, $\gamma_1^*(H)$ is the minimum cardinality of the covers and is called the *covering number* of H while $\mu_1^*(H)$ is the maximum cardinality of a matching and is called the *matching number* of H. A hypergraph H with $\#\mathcal{H} \geq 2$ is *γ-critical* if each of its subfamilies has a smaller covering number, i.e., $\gamma_1^*((N, \mathcal{H} \setminus \{E\})) < \gamma_1^*(H)$ for all $E \in \mathcal{H}$.

A *fractional cover* of H is a vector $t \in \mathbb{R}^n$ such that

$$\sum_{i \in S} t^i \quad \geq \quad 1 \text{ for all } S \in \mathcal{H} \tag{7}$$

$$\text{and } t^i \quad \geq \quad 0 \text{ for all } i \in N. \tag{8}$$

A *fractional matching* of H is a vector $\delta \in \mathbb{R}^{\#\mathcal{H}}$ such that

$$\sum_{S \in \mathcal{H}^i} \delta(S) \quad \leq \quad 1 \text{ for all } i \in N \tag{9}$$

$$\text{and } \delta(S) \quad \geq \quad 0 \text{ for all } S \in \mathcal{H}. \tag{10}$$

A fractional cover t^* minimizing $\sum_{i \in N} t^i$ subject to the constraints (7) and (8) is called an *optimal* fractional cover and $\gamma^*(H) \equiv \sum_{i \in N} t^{*i}$ is called the *fractional covering number*. A fractional matching δ^* maximizing $\sum_{S \in \mathcal{H}} \delta(S)$ subject to the constraints (9) and (10) is called an *optimal* fractional matching and $\mu^*(H) \equiv \sum_{S \subseteq N} \delta^*(S)$ is called the *fractional matching number*.

[19]Indeed, if t is a cover of H, then $T = \{i \in N \mid t^i = 1\}$ satisfies $T \cap E \neq \emptyset$ for all $E \in \mathcal{H}$ and, vice versa, if $T \subseteq N$ with $T \cap E \neq \emptyset$ for all $E \in \mathcal{H}$, then $\chi_T \in \mathbb{R}^n$ defined by $\chi_T^i = 1$ if $i \in T$ and $\chi_T^j = 0$ if $j \in N \setminus T$ is a cover of H.

4.2 Hurdle, Dual Integral and Uniform Hurdle Factors

It follows immediately from these definitions that the hurdle factor $\gamma^*(\mathcal{B})$ is the fractional covering number of $H = (N, \mathcal{B})$ while the dual hurdle factor $\gamma^*(\mathcal{W})$ is the fractional covering number of $H = (N, \mathcal{W})$. If, in contrast to what has been assumed in the preceding section, money is available in indivisible units, then the appropriate parameter becomes $\gamma^*_{W_0}(\mathcal{C})$ where the integer W_0 is the value of policy 0 for Lobby 0 (when $\mathcal{C} = \mathcal{B}$, i.e., when Lobby 0 is the follower) expressed in monetary units. The case where $W_0 = 1$ is of particular interest as it describes the situation where Lobby 0 has a single unit of money to spend in the process. The problem is now purely combinatorial: To whom of the legislators should Lobby 1 spend one unit to prevent Lobby 0 from targeting a unique pivotal legislator[20]? Hereafter, the integer $\gamma^*_1(\mathcal{B})$ will be called the *integral hurdle factor* and the integer $\gamma^*_1(\mathcal{W})$ will be called the *integral dual hurdle factor*. While we will focus mostly on the divisible case, it is interesting to note the implications of indivisibilities on the equilibrium outcome of the lobbying game. The following developments apply equally to both hurdle factors and we will often use the symbol $\mathcal{H}$ without specifying whether $\mathcal{H} = \mathcal{B}$ or $\mathcal{H} = \mathcal{W}$. For an arbitrary hypergraph H, we have the inequalities:

$$\mu^*_1(H) \leq \frac{\mu^*_k(H)}{k} \leq \mu^*(H) = \gamma^*(H) \leq \frac{\gamma^*_k(H)}{k} \leq \gamma^*_1(H). \tag{11}$$

We deduce immediately from these inequalities that the value of the hurdle factor[21] increases with the "degree" of indivisibilities; indivisibilities act as additional integer constraints in the linear program describing the determination of the optimal fractional matchings and coverings. The relationships between these numbers are intricate and their investigation is an active subject of research in the theory of hypergraphs. For the sake of illustration, we report below some of the most significant results. The calculation of the covering number of an arbitrary hypergraph is an NP-hard problem in contrast to the determination of the fractional covering number which amounts to solve a linear program without any integer constraints. In addition to those discussed in Le Breton and Zaporozhets (2009), the examples presented below arise from the theory of simple games.

It is often assumed that the simple game $(N, \mathcal{W})$ is proper, i.e., the set of minimal winning coalitions is an intersecting family. In such case, it is clear that the pattern of intersections of winning coalitions plays some role in the determination of the integral and fractional hurdle

[20]To support that interpretation, we need however to assume that a legislator who is indifferent breaks the tie in direction of Lobby 0.

[21]It has been demonstrated by Chung, Füredi, Garey and Graham (1988) that for any rational number x, there exists a, hypergraph $H = (N, \mathcal{H})$ such that $\mu^*(\mathcal{H}) = x$.

factors. Since a cover is a set which intersects every edge, we deduce that any set in $\mathcal{W}$ is a cover. This implies that the dual integral hurdle factor is then smaller than $\min_{E\in\mathcal{W}} \#E$ and that *Lobby 0 will have to bribe a subset of legislators no larger than the size of the smallest winning coalition.* Lobby 0 may have to pay less like for instance in the case where all minimal winning coalitions contain a prescribed subset of legislators (the vetoers); in such case, $\gamma_1^*(\mathcal{W}) = 1$. This simple game displays a severe asymmetric treatment of legislators and we may wonder if some lower bounds on could be obtained if we impose more structure on $\mathcal{W}$.

There is an obvious trade-off between the number of minimal winning coalitions and the magnitude of the dual hurdle factor[22]. Suppose that to reflect equity among the legislators, all minimal winning coalitions are of the same size r, i.e., $\mathcal{W}_m$ is r-uniform. If we have many coalitions in $\mathcal{W}_m$, the hurdle factors are more likely to be large numbers. The hurdle factors are often very sensitive to the addition or the deletion of a coalition from $\mathcal{W}_m$. These patterns correspond to what has been defined above as γ-critical hypergraphs. How small can be a r-uniform hypergraph if we want the covering number to be at least equal to r?

To answer this question, let s be a positive integer. A set $T \subseteq N$ is an s-multicover of H is either $\#(E \cap T) \geq s$ or $T \supseteq E$ for all $E \in \mathcal{H}$. Hence, a 1-multicover of H is simply a cover of H. The *s-multicover number* of H is the smallest cardinality of an s-multicover of H. If H is s-intersecting, then an s-multicover T is called *nontrivial* if $\#(T \cap E) < \#E$ for all $E \in \mathcal{H}$. As defined earlier, an r-clique is an intersecting r-uniform hypergraph which does not have a non-trivial cover (i.e., a cover that has less than r elements). We can also consider the family $H_{(r,\gamma)}$ of r-uniform hypergraphs H which are intersecting and such that $\gamma_1^*(\mathcal{H}) = \gamma$ where γ is a given integer less than or equal to r. If we denote by $m(r)$ the minimum number of edges in an r-clique and by $n(r)$ the minimum number of edges for a hypergraph in $H_{(r,r)}$. As reported in Füredi (1988):

$$3r \leq m(r) \leq r^5 \text{ for all } r \geq 4$$

and

$$\frac{8}{3}r - 3 \leq n(r) \text{ for all } r \text{ and } n(q+1) \leq 4q\sqrt{q}\log q \text{ if } q \text{ is a prime power.}$$

This means that for all r there exist small r-cliques and small r-uniform hypergraphs with an integral hurdle factor equal to r but also that some minimal critical number of edges are necessary for the properties to be fulfilled. Füredi (1988) also reports many results describing some of the properties of critical hypergraphs and specific families of hypergraphs.

[22]See Idzik, Katona and Vohra (2001) for an exploration of the intersecting balanced families of sets.

4.3 Uniform Hurdle factor

Besides integer constraints (due to the indivisibility of money), we have not considered restrictions on the offers made by the lobbies to the legislators. In particular, the offers of a lobby can differ across legislators. In this subsection, we are going to sort out the implications of assuming that all the legislators receiving an offer from a lobby receive the same offer[23]. This means that from the perspective of any one of the two lobbies, the population of legislators is partitioned into two groups: those who receive an offer from that lobby and those who don't. Let T_1 denote the group of legislators receiving an offer from Lobby 1 and let s_1 be the amount of the offer to each member of T_1, i.e., $t_1^i = s_1 \chi_{T_1}$. Since this limitation applies equally to both, Lobby 0 and Lobby 1, the cheapest offer s_1 meeting these constraints is solution of the following linear program

$$\min_{(T_1, s_1) \in 2^N \times \mathbb{R}_+} s_1 \cdot \#T_1$$

$$\begin{gathered} \text{subject to the constraints} \\ s_1 \cdot \#S \geq W_0 \text{ for all } S \in \mathcal{B}_m \\ \text{and } S \cap T_1 \neq \emptyset \text{ for all } S \in \mathcal{B}_m. \end{gathered} \tag{12}$$

On one hand, the second set of constraints exclude the cases where $S \cap T_1 = \emptyset$. Indeed, in such case $t_1^i = 0$ for all $i \in S$ and Lobby 0 can easily bribe coalition S. On the other hand, if the inequality $s_1 \cdot \#S \geq W_0$ is violated, i.e., $t_1 \cdot \#S < W_0$, then there exists $s_0 > s_1$ such that $s_0 \cdot \#S < W_0$. An offer of an amount equal to s_0 to each of the legislators in S will be accepted by all legislators in $S \cap T_1$ and trivially by all those who are not in T_1. It is important to note that the constraint $s_1 \cdot \#S \geq W_0$ is less demanding than the constraint $s_1 \cdot \# (S \cap T_1) \geq W_0$ which would describe the situation where Lobby 0 is not constrained by the uniformity assumption. The solution of the above problem is strongly connected to the solution of the covering problem. Since it is linear in W_0, let $W_0 = 1$. First, we note immediately from the second set of constraints that the set T_1 must be a cover for the hypergraph $(N, \mathcal{B}_m)$. On the other hand, the tightest constraint in the first set of constraints are those attached to the smallest S in $\mathcal{B}_m$. If (T_1^*, s_1^*) is an optimal solution of (12), then we may deduce that

$$s_1^* = \frac{1}{\min\limits_{S \in \mathcal{B}_m} \#S}$$

[23]This assumption is made by Le Breton and Zaporozhets (2007) in their examination of the uncertainty setting. Morgan and Vardy (2007, 2008) refer to these offers as non-discriminatory vote buying.

and T_1^* is a minimal cover of $(N, \mathcal{B})$. Using our notations, we deduce that the value of the above linear program with integer constraints, called hereafter the *uniform hurdle factor* and denoted $\gamma_u^*(\mathcal{B})$, is equal to

$$\gamma_u^*(\mathcal{B}) = \frac{\gamma_1^*(\mathcal{B}_m)}{\min_{S \in \mathcal{B}_m} \#S}.$$

We would determine similarly the *dual uniform hurdle factor*, denoted $\gamma_u^*(\mathcal{W})$, as

$$\gamma_u^*(\mathcal{W}) = \frac{\gamma_1^*(\mathcal{W})}{\min_{S \in \mathcal{W}} \#S}$$

We have obtained, a quite surprising connection between the uniform hurdle factors and the integral hurdle factors. It provides an extra justification to compute the integral hurdle factors. Both uniform hurdle factors are smaller than their integral counterparts meaning that the uniformity constraint hurts less Lobby 1 than the indivisibility constraint. Note also that whenever $\mathcal{W}$ is a proper simple game, then:

$$\gamma_u^*(\mathcal{W}) \leq 1$$

If we consider the simple game of Example 2, we obtain that $\gamma_u^*(\mathcal{B}) = 1$ which is less than $\gamma^*(\mathcal{B}) = \frac{5}{3}$. Note also that in such case, if Lobby 0 was not constrained by uniformity, the factor would jump to 2 which is, as expected, larger than $\gamma^*(\mathcal{B})$.

4.4 Weighted majority Games

In this section, we focus on the class of weighted majority games. A simple game is a weighted majority game if there exists a vector $\omega = (\omega^1, \ldots, \omega^n; q)$ of $(n+1)$ nonnegative real numbers such that a coalition S is in $\mathcal{W}$ iff $\sum_{i \in S} \omega^i \geq q$ so that, by the definition of a simple game, (i) $q > 0$ and (ii) $\sum_{i=1}^n \omega^i \geq q$. Note that ω^i is the weight attached to legislator[24] i. The vector ω is called a *representation* of the simple game. It is important to note that the same game may admit several representations. A simple game is *homogeneous* if there exists a representation ω such that $\sum_{i \in S} \omega^i = \sum_{i \in T} \omega^i$ for all $S, T \in \mathcal{W}_m$.

Throughout, we assume that $N = \{1, \ldots, n\}$ with $n \geq 2$. Consider an arbitrary TU game (N, V) and let $x \in X_n \equiv \{y \in \mathbb{R}^n \mid \sum_{i=1}^n y^i = V(N), y^j \geq V(\{j\}) \forall j \in N\}$. Let $\theta(x)$ be the $2^n - 2$-dimensional vector[25] whose components are the numbers $V(S) - \sum_{i \in S} x^i$ for

[24] In most legislatures, legislators belong to political parties. Party discipline refers to the situation where any two legislators belonging to the same party vote similarly. Then, if two legislators i and j belonging to the same party are such that $\alpha^i = \alpha^j$, it is appropriate to assume that the players are the parties rather than the legislators themselves; in such case, ω^k denotes the number of legislators affiliated to Party k.

[25] This vector is called the vector of excesses attached to x.

$\emptyset \neq S \subsetneqq N$ arranged according to their magnitude, i.e., $\theta^i(x) \geq \theta^j(x)$ for $1 \leq i \leq j \leq 2^n - 2$. Similarly, let $\widehat{\theta}(x)$ be the $2^n - 2$ dimensional vector whose components are the numbers $\frac{V(S)-\sum_{i \in S} x^i}{\# S}$ for $\emptyset \neq S \subsetneqq N$ also arranged according to their magnitude, i.e., $\widehat{\theta}^i(x) \geq \widehat{\theta}^j(x)$ for $1 \leq i \leq j \leq 2^n - 2$. The *nucleolus* and the *per-capita nucleolus*[26] of (N, V) are the unique vectors $x^*, \widehat{x}^* \in X_n$ such that $\theta(x^*)$ and $\left(\widehat{\theta}(x^*)\right)$ are minimal, in the sense of the lexicographic order, of the sets $\{\theta(y) \mid y \in X_n\}$ and $\left\{\widehat{\theta}(y) \mid y \in X_n\right\}$, respectively. For the definition and uniqueness of x^* and $\widehat{x}^*$ see Schmeidler (1969), Justmann (1977), and Wallmeier (1983). The *least core* and the *per-capita least core*[27] are the subsets of X_n consisting of the vectors x such that $\theta_1(x) = \theta_1(x^*)$ and $\widehat{\theta}_1(x) = \widehat{\theta}_1(\widehat{x}^*)$, respectively. These polytopes will be denoted $LC(V, N)$ and $\widehat{LC}(V, N)$. Note that, by construction $x^* \in LC(V, N)$ and $\widehat{x}^* \in \widehat{LC}(V, N)$.

Let $\mathcal{P}_n = 2^N \setminus \{\emptyset, N\}$. To any TU game V we attach the linear program

$$\begin{gathered} \min_{t \in \mathbb{R}^n} \textstyle\sum_{i \in N} t^i \\ \text{subject to the constraints} \\ \textstyle\sum_{i \in S} t^i \geq V(S) \text{ for all } S \in \mathcal{P}_n. \end{gathered} \tag{13}$$

Let $\gamma(V)$ be the value of this problem. Then V is balanced iff $V(N) \geq \gamma(V)$. Moreover, let

$$C^*(V) \equiv \min_{y \in X_n} \max_{S \in \mathcal{P}_n} \frac{V(S) - \sum_{i \in S} y^i}{\# S} \tag{14}$$

The following simple assertion holds.

Proposition 2. If (N, V) is a TU game, then $\gamma(V) = V(N) + nC^*(V)$.

Proof: Let $\varepsilon = \frac{\gamma(V)-V(N)}{n}$ and let t^* be an optimal solution of the linear program (13). Define $x = t - \epsilon \chi_N$ and observe that $\sum_{i \in N} x^i = V(N)$. Moreover,

$$\frac{V(S) - \sum_{i \in S} x^i}{\# S} = \frac{V(S) - \sum_{i \in S} t^i - \varepsilon \cdot \# S}{\# S} \leq \varepsilon \text{ for all } S \in \mathcal{P}_n$$

so that $C^*(V) \leq \varepsilon$.

[26] Strictly speaking, this is the (per-capita) prenucleolus. The nucleolus and the per-capita nucleolus are defined on the set of individually rational payoffs. If the cooperative game is zero-monotonic, i.e., if $V(S \cup \{i\}) - V(S) \geq V(\{i\})$ for all $i \in N$ and $S \subseteq N \setminus \{i\}$, the difference between the prenucleolus and the nucleolus vanishes. A simple game is always zero-monotonic unless $\{i\}, S \in \mathcal{W}$ for some $i \in N$ and $S \subseteq N \setminus \{i\}$. The per-capita prenucleolus may be different from the per-capita nucleolus even for a zero-monotonic weighted majority game: Let $n = 4$ and (N, V) be represented by $\omega = (1, 1, 1, 0; 2)$, i.e., $(N, \mathcal{W})$ arises from the simple 3-person majority game by just adding a null-player. The per-capita prenucleolus coincides with $\frac{1}{8}(3, 3, 3, -1)$ and, hence, assigns a negative amount to the nullplayer, whereas the per-capita nucleolus coincides with the prenucleolus and the nucleolus given by $\frac{1}{3}(1, 1, 1, 0)$.

[27] The notion of the (per-capita) least core was first introduced by Maschler, Peleg and Shapley (1979). The example in Footnote 26 shows that the per-capita least core may contain elements that are not individually rational even for zero-monotonic games. However, each payoff vector of the least core of a zero-monotonic game is individually rational.

To prove the opposite inequality let $y \in X_n$ such that

$$\max_{S \in \mathcal{P}_n} \frac{V(S) - \sum_{i \in S} y^i}{\# S} = C^*(V)$$

and define $z = y + C^*(V)\chi_N$. Then, for any $S \in \mathcal{P}_n$,

$$\sum_{i \in S} z^i = \sum_{i \in S} y^i + C^*(V) \cdot \# S \geq V(S)$$

so that

$$V(N) + nC^*(V) = \sum_{i \in N} z^i \geq \gamma(V) = V(N) + n\varepsilon$$

and, hence, $C^*(V) \geq \varepsilon$. $\square$

The argument is also quite instructive by itself as it demonstrates that the set of solutions of the linear program (13) above is strongly connected to the per-capita least core[28] of the cooperative TU game V. In the case of the determination of the optimal offer(s) by Lobby 1, the TU game V is defined by (2).

Remark 1. Proposition 1 remains valid if the TU game V defined by (2) is replaced by the TU game V' that differs from V only inasmuch as $\widehat{\mathcal{B}}_m$ is replaced by $\mathcal{B}$, i.e., the TU game V' is defined by

$$V(S) = \begin{cases} W_0 - \sum_{i \in S_3} \alpha^i \Delta W & , \text{if } S = S_2 \cup S_3 \in \mathcal{B}, \\ 0 & , \text{if } S \in 2^N \setminus \mathcal{B}. \end{cases} \tag{15}$$

Indeed, as $\widehat{\mathcal{B}}_m \subseteq \mathcal{B}$ and $V(S) = V'(S)$ for all $S \in \widehat{\mathcal{B}}_m$,

$$\begin{aligned} & \max \left\{ \sum_{S \in \widehat{\mathcal{B}}_m} \delta(S) V(S) \middle| \, \delta \text{ is a vector of subbalancing coefficients for } \widehat{\mathcal{B}}_m \right\} \\ \leq \; & \max \left\{ \sum_{S \in \mathcal{B}} \delta(S) V'(S) \middle| \, \delta \text{ is a vector of subbalancing coefficients for } \mathcal{B} \right\}. \end{aligned}$$

In order to show the opposite inequality, note that, for any $S \in \mathcal{B}$, there exists $\widehat{S} \in \widehat{\mathcal{B}}_m$ such that $\widehat{S} \subseteq S$ and $\widehat{S}_3 = S_3$. Now, if δ is a vector of subbalancing coefficients for $\mathcal{B}$, then we may define a vector $\widehat{\delta}$ of subbalancing coefficients for $\widehat{\mathcal{B}}_m$ by

$$\widehat{\delta}(T) = \sum \{\delta(S) \mid \widehat{S} = T\} \text{ for all } T \in \widehat{\mathcal{B}}_m.$$

[28]Strictly speaking it is the least core whenever the core of the game is empty. Here, we will focus almost exclusively onto that case.

As $V'(S) = V(\widehat{S})$ for all $S \in \mathcal{B}$,

$$\sum_{T \in \widehat{\mathcal{B}}_m} \widehat{\delta}(T) V(T) = \sum_{S \in \mathcal{B}} \delta(S) V(S).$$

When $\alpha = 0$, the above calculations can be further simplified. Indeed in such a case, the game V' is up to the multiplication by W_0, the simple game

$$V'(S) = \begin{cases} 1, & \text{if } S \in \mathcal{B}, \\ 0, & \text{otherwise.} \end{cases}$$

Applying (14) to V' yields

$$C^*(V') = \min_{y \in X_n} \max_{S \in \mathcal{P}_n} \frac{V'(S) - \sum_{i \in S} y^i}{\# S} = \min_{y \in X_n} \max \left\{ \max_{S \in \mathcal{B} \setminus \{N\}} \frac{1 - \sum_{i \in S} y^i}{\# S}, - \min_{T \in 2^N \setminus (\mathcal{B} \cup \{\emptyset\})} \frac{\sum_{i \in S} y^i}{\# S} \right\}.$$

This implies that the hurdle factor $\gamma^*(\mathcal{B})$ is equal to $1 + nC^*$. Let

$$C^{**} \equiv \max_{y \in \left\{ z \in \mathbb{R}^n_+ \middle| \sum_{i \in N} z^i = 1 \right\}} \min_{S \in \mathcal{B}_m} \sum_{i \in S} y^i.$$

Following the same line of arguments as above, it is easy to show that $\gamma^*(\mathcal{B}) = \frac{1}{C^{**}}$. Moreover, any $x \in \mathbb{R}^n_+$ that satisfies $\sum_{i \in N} x^i = 1$ and $\sum_{i \in S} x^i \geq C^{**}$ for all $S \in \mathcal{B}_m$, is an element of $LC(V', N)$, provided that V' is zero-monotonic. This means that in this case, the hurdle factor can be computed either via the least core or the per-capita least core. A similar statement is valid for the dual hurdle factor. From now on, we focus on the case where $\alpha = 0$.

In some cases it will be possible to order, partially or totally, the legislators according to *desirability* as defined by Maschler and Peleg (1966). Legislator $i \in N$ is at least as desirable as legislator $j \in N$ if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for all $S \subseteq N \setminus \{i, j\}$. Legislators i and j are symmetric or interchangeable if $S \cup \{j\} \in \mathcal{W}$ iff $S \cup \{i\} \in \mathcal{W}$ for all $S \subseteq N \setminus \{i, j\}$. Legislator i is said to be strictly more desirable than legislator j if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for all $S \subseteq N \setminus \{i, j\}$ and $S \cup \{i\} \in \mathcal{W}$ and $S \cup \{j\} \notin \mathcal{W}$ for some $S \subseteq N \setminus \{i, j\}$. Example 4 below shows that two symmetric legislators do not necessarily receive the same offer from Lobby 1 in all equilibria of the lobbying game.

Example 4

Consider the proper and strong weighted majority game that has the representation

$$(17, 9, 8, 6.5, 6.5, 5, 3, 2, 2; 30)$$

According to Krohn and Sudhölter (1995), the least core is the convex hull of the normalized vectors of weights that correspond to the weights of the representation except for players 4 and 5 who may receive 7/59 and 6/59 or symmetrically 6/59 and[29] 7/59.

In this example, the violation of desirability relation applies to two legislators who are interchangeable and to a situation where the least core does not degenerate on the nucleolus. We will see now that this "pathological" behavior of legislator's prices do not extend to strict desirability. Peleg (1968) has demonstrated that for a proper and strong weighted majority game, any imputation in the least core is a representation. Since a representation assigns a bigger weight to a strictly more desirable player, it follows that the price offered to legislator i is larger than the price offered to legislator j in all equilibria, if legislator i is strictly more desirable than legislator j. This monotonicity property does not extend to weighted majority games which are not proper and strong as demonstrated by Example 5 below.

Example 5

Consider the following 6-person game with representation

$$(5, 5, 4, 3, 2, 2; 14)$$

taken from Kopelowitz (1967). We claim that the vector $x = (1, 1, 0, 1, 0, 0)/3$ belongs to the least core. According to Kopelowitz, the nucleolus is (4,4,3,2,1,1)/15 and, hence, it assigns 2/3 to the winning coalition {1,2,5,6}. Now, the vector x assigns to each winning coalition at least 2/3 and it is nonnegative. Hence, the maximal excess is 1/3 in both cases. However, player 3 is strictly more desirable than 4.

5 Applications

In this last section, we illustrate the techniques and notions introduced before by considering different families of simple games. First, we consider simple games where winning coalitions are large and therefore blocking coalitions are small. We show how to use some results from the theory of graphs to compute the hurdle factor(s) or to obtain approximation of these numbers. In a second part, we look at specific real world simple games described as vector-weighted majority games of low dimension and we also calculate the relevant parameters.

[29]The least core of each proper and strong weighted majority game with less than 9 legislators is a singleton so that in this case symmetric legislators receive the same offer.

5.1 Simple Games with Large Winning Coalitions (Small Blocking Coalitions)

When we consider the hypergraph of the minimal blocking coalitions of a simple game, the fractional and integral covering numbers are likely to be large numbers when its set of edges contains many small coalitions. This will happen as soon as in the simple game, a coalition is winning if it contains most of the players. The extreme case of such situation is unanimity according to which a coalition is winning if it contains all the legislators. In such case, any singleton is a blocking coalition and then $\mu_1^*(\mathcal{B}) = \gamma^*(\mathcal{B}) = \gamma_1^*(\mathcal{B}) = n$. The "closest" situation to unanimity is the case where each winning coalition contains at least $n-1$ legislators. This case has been extensively studied by several authors including Lucas (1966), Maschler (1963) and Owen ((1968), (1977)). We now consider the more general case that each winning coalition contains at least $n-2$ legislators and that for any three-person coalition $T \subseteq N$ there is $i \in T$ with $N \setminus (T \setminus \{i\}) \notin \mathcal{W}$. In such a case each minimal blocking coalition consists either of a single vetoer or it is a pair of legislators. Hence, in the particular subcase that each $(n-1)$-person coalition is winning, i.e., that vetoers are absent, the hypergraph $(N, \mathcal{B}_m)$ of minimal blocking coalitions is, in fact, an *ordinary* graph, and we may reconstruct from $\mathcal{B}$, via duality, the set $\mathcal{W}$ of winning coalitions: Indeed, $S \in \mathcal{W}_m$ if and only if either $S = N \setminus \{i\}$ for some $i \in N$ such that $\{i, j\} \in \mathcal{B}$ for all $j \in N \setminus \{i\}$ or $S = N \setminus \{i, j\}$ for some $i, j \in N$ such that $\{i, j\} \notin \mathcal{B}$.

In such a case, we can take advantage of the results established in the theory of graphs to derive information on the different hurdle factors. In that respect, it will also be useful to calculate the matching and fractional matching numbers to obtain lower bounds on the hurdle factor(s) via (11). The largest possible value of $\gamma^*(\mathcal{B})$ is $\frac{n}{2}$ which is realized, for instance, when the graph is complete. From the point of view of matchings[30], it correspond to what is called in graph theory as a *perfect matching*. If there is a perfect matching, we deduce from (11) that $\gamma^*(\mathcal{B}) = \frac{n}{2}$. If the graph is bipartite, Hall's theorem provides necessary and sufficient condition for the existence such a perfect matching. For an arbitrary graph, Tutte's beautiful theorem[31] also provides necessary and sufficient condition for the existence such a perfect matching.

When there is no perfect matching, we can still explore the set of maximum matchings and obtain a lower bound on $\gamma^*(\mathcal{B})$ through inequality (11). We know from Lovasz (1975)

[30]The results on matching theory to which we refer here can be found in Lovàsz and Plummer (2006). A friendly presentation is offered by Simeone (2006).

[31]It also follows from that theorem that $\mu_2^*(H) = 2\gamma^*(H) = \gamma_2^*(H)$.

that

$$\gamma^*(H) \leq \frac{\mu_1^*(H) + \gamma_1^*(H)}{2}$$

The celebrated Edmonds-Gallai structure theorem offers deep insights on the structure of any maximum matching. In order to make the best possible use of Proposition 1 in such case, it is important to characterize the family of balanced collections. It has been demonstrated by Balinski (1972) that δ is an extreme point of the polytope of fractional matchings iff there exists a collection Q of node-disjoint edges and odd cycles such that

$$\delta(\{i,j\}) = \begin{cases} 1 \text{ if } \{i,j\} \in Q, \\ \frac{1}{2} \text{ if } \{i,j\} \text{ belongs to an odd cycle of } Q, \\ 0 \text{ otherwise.} \end{cases}$$

This important result suggests to identify the partitions of N with the largest number of vertices either belonging to an odd cycle or a to a pair. The length of the longest odd cycle or the cumulate length of a disjoint family of odd cycles provide lower bounds for $\gamma^*(\mathcal{B})$. However, these questions are not easy from a computational perspective.

As already discussed, not all the pairs need to be blocking. Besides the fact that the legislators may differ in term of power, in some circumstances some "feasible" pairs can be simply ignored by Lobby 1 if they are unlikely to form. In such a case, the simple game not only describes the rules of the legislature but also incorporates information about characteristics of the legislators relevant to predict which potential blocking coalitions could form. If the population of legislators is partitioned according to several types like for instance gender, geography, ethnicity or ideology, some coalitions, corresponding to a particular mixing of the types, may be considered as unfeasible. A nice illustration is the case of a bicameral system where to be winning a coalition must contain all the members or at least one of the two chambers. In such case, the hypergraph of blocking coalitions consists of all pairs with one member in each chamber; in this bicameral illustration, the graph of blocking coalitions is simply the complete bipartite graph. Consider the case of a legislature with equal numbers of males and females and assume that a proposal is blocked if the coalition contains at least one female and one male. If all such coalitions are likely to form, the set of minimal blocking coalitions consists of all pairs composed with a male and a female. If in addition, legislators are also differentiated according to left and right[32], then it is reasonable to assume that only

[32]The construction of the relevant blocking pairs may of course be more complicated. Consider for instance the case where the legislators are located in a multidimensional Euclidean ideological space and let d_{ij} denote the distance between i and j. If we assume that two legislators act together iff their distance does not exceed some exogenous threshold ρ, the set of relevant minimal blocking coalitions is the set of pairs $\{i,j\}$ such that $d_{ij} \leq \rho$. An interesting case deserving further exploration is the case where there is a hierarchy among the legislators. It would correspond here to the case where the legislators could be ordered say from 1 to n in such a way that if $\{j,k\} \in \mathcal{B}$ for some $j < k$, then $\{i,k\} \in \mathcal{B}$ for all $i < j$.

pairs of legislators with the same ideology form. Let p_M and p_F, respectively, denote the proportions of left legislators in the male and female population and assume, for the sake of simplicity, that $p_M < \frac{1}{2}$ and $p_M + p_F = 1$. Using König's theorem on bipartite graphs, it is easy to show that

$$\mu_1^*(\mathcal{W}) = \mu^*(\mathcal{W}) = \gamma^*(\mathcal{W}) = \gamma_1^*(\mathcal{W}) = np_M.$$

This last example is peculiar as the long chain of inequalities (11) degenerates into a perfect equality[33]: The integral and fractional hurdle factors are equal and coincide themselves with the integral and fractional matching numbers. Calculating the matching number is quite easy as it amounts to find a partition of the set of legislators into the largest possible number of blocking coalitions. Therefore, when the above equalities hold, the calculation of the hurdle factor becomes very easy.

5.2 Vector Weighted Majority Games

Every simple game is a vector weighted majority games as defined by Taylor and Zwicker (1999). A simple game $(N, \mathcal{W})$ is a vector-weighted majority game if there exists a positive integer k, an assignment of k-tuple weights to the players $\left(w_j^i\right)_{1\leq j\leq k}$ for all $i \in N$, and a k-tuple quota $q = (q_j)_{1\leq j\leq k}$ such that for every coalition $S \subseteq N$, $S \in \mathcal{W}$ iff $\sum_{i\in S} w_j^i \geq q_j$ for all $j = 1, \ldots, k$. A simple game is said of *dimension* k if it can be represented using k-tuples as weights and quota but cannot be represented using $(k-1)$-tuples as weights and quotas. Hereafter we shall focus on vector-weighted majority games with a small dimension. This class includes all the weighted majority games (i.e., vector-weighted majority game of dimension 1) as defined earlier (like for instance the United nations Security Council) but also many important real world examples which are not weighted majority games like for instance the Canadian constitutional amendment scheme, the US legislative system, the European rule and voting by count and account. In this subsection we derive the hurdle factors for a sample of real world vector weighted majority games describing the decision making process of some important organizations.

Example 6 (The United Nations Security Council). The voters are the 15 countries that make up the security council, 5 of which are called permanent members whereas the other 10 are called nonpermanent members. Passage requires to total of at least 9 votes, subject to approval from any one of the 5 permanent members. It is easy to show that this simple game is a weighted majority game: Assigning a weight of 7 to each permanent

[33]A hypergraph for which this is true is called *normal* and a nice characterization has been obtained by Lovasz (1972).

member, a weight of 1 to any nonpermanent member and a quota equal to 39 provides a representation. If Lobby 1 acts to pass a reform (here a resolution), the problem of determination of the least core reduces to the minimization of $5x_1 + 10x_2$ with respect to $(x_1, x_2) \in \mathbb{R}^2_+$ under the constraints

$$x_1 \geq 1 \text{ and } 7x_2 \geq 1.$$

We deduce that the least core consists of the unique vector $\left(1, \frac{1}{7}\right)$ (which is the nucleolus) and that the hurdle factor $5 + \frac{10}{7}$ is approximately equal to 6.43.

If instead Lobby 0 acts to block a reform, the problem of determination of the least core reduces to the minimization of $5x_1 + 10x_2$ with respect to $(x_1, x_2) \in \mathbb{R}^2_+$ under the constraint

$$5x_1 + 4x_2 \geq 1.$$

Now we obtain that the least core consists of the unique vector $\left(\frac{1}{5}, 0\right)$ (which is the nucleolus) and that the dual hurdle factor is equal to 1. Here, only the permanent members receive an offer and with a hurdle factor equal to 1, lobbying expenditures by Lobby 1 remain moderate. We could wonder what would be the consequences of limiting somehow the veto power of the permanent members and/or changing the level of the qualified majority to pass a reform. For instance, suppose that passage requires to total of at least 9 votes, subject to approval of at least 3 permanent members. The constraints now becomes

$$3x_1 + 6x_2 \geq 1 \text{ and } 5x_1 + 4x_2 \geq 1.$$

In that case, if as above, Lobby 0 acts to block a reform, both permanent and nonpermanent members are likely to receive offers as the least core consists of the convex hull of the vectors $\left(\frac{1}{3}, 0\right)$ and $\left(\frac{1}{9}, \frac{1}{9}\right)$ and the dual hurdle factor $\frac{5}{3}$ is approximately equal to 1.66. Consider finally the case where passage requires to total of at least 10 votes, subject to approval of at least 3 permanent members. The constraints now becomes

$$3x_1 + 7x_2 \geq 1 \text{ and } 5x_1 + 5x_2 \geq 1.$$

It is straightforward to show that the least core consists of the unique vector $\left(\frac{1}{10}, \frac{1}{10}\right)$ (which is the nucleolus) and that the dual hurdle factor is equal to 1.50.

From 1954 to 1965, the simple game $(N, \mathcal{W})$ describing the council had 5 permanent members, 6 nonpermanent members and the qualified majority was equal to 7. Proceeding as above, we obtain that the hurdle factor $\gamma^* (\mathcal{B})$ was equal to 6.20 while $\gamma^* (\mathcal{W}) = 1$. The 1965 system is less vulnerable to lobbying than the 1954's one. It would be interesting to use this apparatus to evaluate some of the proposals to reform membership and voting rules of the United Nations Security Council. Many countries criticize the lack of representativeness of the current council. Among the proposals, we can find:

· The G4 proposal which ask the addition of 6 new permanent members without veto power and 4 new nonpermanent members.

· The African proposal which is similar to the G4 proposal except for the fact that it asks that the new permanent members also had a veto power and 5 new permanent members instead of 4.

· The "United for Consensus" proposal which simply asks for the addition of 10 new permanent members.

These proposals propose to increase the current size of 15 members to 25 or 26 members. In our setting being a permanent member without veto power is equivalent to be a nonpermanent member. No specification of the required qualified majority is provided but given the historical attachment to a supermajority requirement of $60-63\%$, we could expect a quota equal to 15. The first, second, and third proposal, respectively, leads to a hurdle factor $\gamma^*(\mathcal{B})$ equal to approximately 8.67, 12.25, and 8.67. A way to compromise between the first and second proposal could consist in offering to each pair (or triple) of new permanent members a veto power. To compromise with the third, we could increase the quota from 15 to 18. In general, a council composed of n_1 permanent members with regular veto voter, n_2 permanent members with veto voter offered to pairs, $n_3 = n - n_1 - n_2$ nonpermanent members, and a quota equal to q, where $n_1 + n_2 < q < n-1$, leads to a hurdle factor equal to

$$n_1 + \frac{n_2}{2} + \frac{n_3}{n_1 + n_2 + n_3 + 1 - q}.$$

Example 7 (Amending the Canadian Constitution). We consider first the impressive scheme for amending the Canadian constitution, proposed at the Victoria conference in 1971 (Straffin (1993)). The problem in designing a constitutional amendment scheme for Canada is that the Canadian provinces are very jealous of their constitutional prerogatives and extraordinarily diverse both in politics and in size. The provinces of Ontario and Québec together contained 64% of the Canadian population in 1970, whereas the four small "Atlantic" provinces together contained less than 10%. This extreme diversity suggests asymmetric treatment of the provinces in a constitutional amendment scheme, but exactly how to do it is a delicate matter. The Victoria scheme proposed that a constitutional amendment would have to be approved by:

· both Ontario (O) and Québec (Q), and

· at least two of the four Atlantic provinces (New Brunswick (NB), Nova Scotia (NS), Newfoundland (NF) and Prince Edward Island (PEI)) and

· British Columbia (BC) and at least one of the prairie provinces (i.e., Alberta (A), Saskatchewan (S), Manitoba (M)) or all three prairie provinces.

The three components $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ of this tricameral simple game $\mathcal{W}$ are easy to analyze: $\mathcal{W}_1$ is the unanimity game with two players, $\mathcal{W}_2$ is the symmetric game with four players and a quota equal to 2 and $\mathcal{W}_3$ is the apex game (example 2) with four players. We deduce that the hurdle factor $\gamma^*(\mathcal{W})$ of $\mathcal{W}$ is equal to $\min\left(\gamma^*(\mathcal{W}_1), \gamma^*(\mathcal{W}_2), \gamma^*(\mathcal{W}_3)\right) = \min\left(1, 2, \frac{5}{3}\right) = 1$. Similarly the hurdle factor of the dual game $\mathcal{B}$, $\gamma^*(\mathcal{B})$, is equal to $\gamma^*(\mathcal{B}_1) + \gamma^*(\mathcal{B}_2) + \gamma^*(\mathcal{B}_3) = 2 + \frac{4}{3} + \frac{5}{3} = 5$. For this last simple game, O and Q receive each 20 %, each of the four Atlantic provinces receives 6.67 %, BC receives 13.33 % while A, S and M receive each 6.67 %.

The constitutional amendment scheme which Canada finally adopted in 1982 was far less equitable than the Victoria scheme, in the sense that voting power as measured by either index does not approximate population at all (Kilgour (1983)). According to this scheme, to be approved, an amendment needs the support of at least two-thirds of the provinces that have, in the aggregate, according to the then latest general census, at least fifty percent of the population of all the provinces. The first principle implies that at least 7 of out the 10 provinces are needed to pass the amendment or equivalently any 4 provinces can block an amendment. On the basis of the 1981 census, no single province can block an amendment; the minimal blocking coalitions are $\{O, Q\}$, $\{O, BC, A\}, \{O, BC, M\}, \{O, BC, S\}$, $\{O, BC, NS\}$, and all 4-person coalitions that do not contain any of the foregoing coalitions. It can be verified that $\gamma^*(\mathcal{B}) = 3$. Here the least core of $G = (N, \mathcal{B})$ does not collapse on the nucleolus. The game $(N, \mathcal{W})$ is not strong and for instance there are imputations in the least core of G that do not strictly respect the desirability relation. Moreover, the desirability relation of G is complete, but G is not a weighted majority game.

6 Concluding Remarks

In this paper, we have examined the equilibrium behavior of two lobbyists playing sequentially to buy the votes of legislators. In doing so, we have highlighted the key role played by the hurdle factor which is a parameter of the simple game describing the decision making process in the legislature. When the hurdle factor is large, it is less likely to observe lobbying at equilibrium but when it happens, lobbying activities are more significant. We have pointed out the connection between the computation of the hurdle factor and the covering problems in graph theory. Among the applications, a special attention has been devoted to two cases: on the one hand, the case where minimal blocking coalitions are pairs of legislators and on the other hand, the case of vector weighted majority games.

This last topic calls for more applications of our methods. One important example of vector weighted majority game is provided by the rules of governance of the Current European Union and the Enlarged European Union. Another important class of games is the class of linear games with consensus (Carreras and Freixas (2004)), pioneered by Peleg (1992) under the heading "voting by count and account" (Peleg (1992)). The 1982 constitutional amendment scheme is part of this family. The general question of the computation of the hurdle factor and the characterization of the least core for such class of games is still unexplored. The account side appears in organization where the financial contributions of the members play a role in the determination of their voting rights like in the IMF where only the financial weight matters. This voting system leads to some under-representation of the developing and transition countries and have been criticized on several grounds. Some new voting rules have been suggested (Brauninger(2003), Hirokawa and Vlach (2006), Leech (2002), Morgan (2007), O'Neill and Peleg (2000), Rapkin and Strand (2006)). It would be useful to evaluate the hurdle factors and least cores of these new alternative schemes.

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