Bond Durations: Corporates vs. Treasuries

by

Holger Kraft
and
Claus Munk

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Department of Business and Economics
Faculty of Social Sciences
University of Southern Denmark
Campusvej 55
DK-5230 Odense M
Denmark

Tel.: +45 6550 3271
Fax: +45 6615 8790
E-mail: lho@sam.sdu.dk
http://www.sam.sdu.dk/depts/virkl/about.shtml

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Holger Kraft*

email: kraft@mathematik.uni-kl.de

Department of Mathematics, University of Kaiserslautern;
Anderson School of Management, University of California at Los Angeles

Claus Munk†

email: cmu@sam.sdu.dk

Department of Business and Economics, University of Southern Denmark

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†Corresponding author. Full address: Department of Business and Economics, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark. Claus Munk gratefully acknowledges financial support from the Danish Research Council for Social Sciences.

Bond Durations: Corporates vs. Treasuries

ABSTRACT: We compare the durations of corporate and Treasury bonds in the reduced-form, intensity based credit risk modeling framework. In the case where default risk is independent of default-free interest rates, we provide in each of the three most popular recovery regimes a sufficient condition under which the duration of the corporate bond is smaller than the duration of a similar Treasury bond. In the case where the default intensity and the recovery rate may depend on the default-free interest rate, we also provide a sufficient condition for the duration of a corporate bond to be smaller than the duration of the corresponding Treasury bond, assuming that recovery of market value applies. We illustrate our findings and offer more details in a specific setting in which default-free interest rates follow a Vasicek model and recovery of market value applies with a constant loss rate and a default intensity which is affine in the default-free short rate. While the unanimous conclusion of earlier papers is that the corporate bond has a smaller duration than the corresponding Treasury bond, we demonstrate in this setting that the duration of a corporate coupon bond can very well be greater than that of the similar Treasury bond.

KEYWORDS: interest rate risk, duration, default risk, intensity models

JEL-CLASSIFICATION: E43, G12

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1 Introduction

The duration of an asset is a measure of its interest rate risk. Although duration has a long history, it is still an important and widely used tool in the risk management of portfolios of interest rate sensitive assets. Most papers studying duration focus on default-free (Treasury) bonds, but for the many portfolio managers also investing in defaultable (corporate) bonds it is important to understand the sensitivity of defaultable bonds to interest rate changes. The few existing papers addressing the duration of corporate bonds either derive durations from relatively simple firm-value based models or estimate the empirical relation between changes in the prices of corporate bonds and changes in interest rates.

In this paper we study the duration of corporate bonds in the framework of modern reduced-form valuation models, where the default event is represented by a stopping time driven by an intensity process. Duration is measured as the percentage decrease in price caused by a marginal increase in the default-free short-term interest rate. The duration of a corporate bond is affected by the assumed recovery regime and the dependence of the default intensity rate and the recovery rate on the default-free interest rates. In the case where default risk is independent of default-free interest rates, we provide in each of the three most popular recovery regimes a sufficient condition under which the duration of the corporate bond is smaller than the duration of a similar Treasury bond. With recovery of market value or recovery of Treasury, the condition is very weak, while the sufficient condition with recovery of face value is more restrictive. In the case where the default intensity and the recovery rate may depend on the default-free interest rate, we also provide a sufficient condition for the duration of a corporate bond to be smaller than the duration of the corresponding Treasury bond, assuming that recovery of market value applies.

We illustrate our findings and offer more details in a specific setting in which default-free interest rates follow a Vasicek model and recovery of market value applies with a constant loss rate and a default intensity which is affine in the default-free short rate. We show that whether the duration of the corporate bond is greater or smaller than that of the corresponding Treasury bond is mainly determined by the sensitivity of the default intensity to the default-free short rate. Several empirical studies provide information on the magnitude of this sensitivity, e.g. Longstaff and Schwartz (1995), Duffee (1998), Jarrow and Yildirim (2002), and Bakshi, Madan, and Zhang (2006). Given the range of parameter estimates reported in those studies, the duration of a corporate coupon bond can either be greater than or smaller than the duration of a similar Treasury bond.

Let us briefly review the related literature. In the very simple Merton (1974) setting for corporate debt valuation Chance (1990) shows that the duration of a defaultable zero-coupon bond is smaller than the duration of the similar default-free zero-coupon bond. Fooladi, Roberts, and Skinner (1997) define and study a duration-style measure in a specific pricing model very different from the models applied today for the pricing of defaultable claims. Babbel, Merrill, and Panning (1997) set up a pricing model with the default-free short rate and the value of the issuing firm as state variables. They calibrate the model to data and derive an estimated relation between corporate bond prices and default-free interest rates. Using that relation they conclude that default risk shortens the duration. In a similar setting Acharya and Carpenter (2002) endogenize the default decision by the issuer and study, among other things, how the duration of the corporate bond depends on the firm value. They conclude that default risk reduces the duration of a bond. Longstaff and Schwartz (1995) make similar conclusions about duration in their valuation model for corporate bonds. In fact they also argue that the duration of a corporate bond may very well be negative.[1] While all these papers thus agree that corporate bonds have smaller durations than default-free bonds, our results show that the opposite may also be the case in some empirically relevant situations.

The rest of this paper is organized as follows. In Section 2 we define duration formally and provide some general results that will be useful in later sections. The case where default risk and interest rate risk is independent is studied in Section 3. For each of the three most popular recovery assumptions we derive a condition under which the duration of a corporate coupon bond will be smaller than the duration of a similar Treasury coupon bond. In Section 4 we allow for dependence between interest rate risk and default risk. Assuming recovery of market value, we derive a condition ensuring that the duration of a corporate bond is smaller than the duration of a Treasury bond. Section 5 studies a concrete model in which we illustrate our theoretical findings and offer additional insights. Finally, Section 6 concludes.

2 General Modeling Framework

2.1 Defining durations

We consider an arbitrage-free financial market allowing the existence of a risk-neutral probability measure. The market has an instantaneously riskless asset (a money market account) with r_t denoting the continuously compounded short-term default-free interest rate at time t (the short rate). We define the duration of any asset as minus the percentage price sensitivity with respect to the short rate, i.e. if V_t denotes the time t value of the asset, the duration is defined as

$$D_t^V = -\frac{\partial V_t}{\partial r}\frac{1}{V_t},$$

if the value V_t is differentiable with respect to the short rate r, as we will assume is the case in the following.[2] We want to compare the duration of a default-free coupon bond to the duration of a defaultable coupon bond promising payments identical to the default-free bond. All the bonds we consider are assumed to have a face value of 1. We focus on fixed-rate bullet bonds maturing at time t_n and either discrete, periodic coupon payments of q at time $t_1, t_2, \ldots, t_n$ or a continuous coupon at the rate of q throughout the interval $[t, t_n]$.

First, consider default-free bonds. The time t price of the default-free zero-coupon bond maturing at time $t_n \geq t$ is $\bar{P}_t^{t_n} = \mathrm{E}_t[e^{-\int_t^{t_n} r_u\, du}]$, where $\mathrm{E}_t[\,]$ is the expectation under some fixed

[1]In the firm-value framework an increase in default-free interest rate has two opposing effects on the value of corporate debt. First, it will decrease the present value of the future cash flow to the debt. Second, it will increase the (risk-neutral) expected rate of return on the assets of the issuing firm and, hence, lower the default probability and increase the expected cash flow to the debt. If the last term dominates, the corporate bond price will be increasing in the default-free short rate, corresponding to having a negative duration.

[2]The duration of a bond was originally defined as minus the derivative of its price with respect to its own yield-to-maturity, divided by the price. However, it is well-known that the application of that duration in risk management requires a flat zero-coupon yield curve which can only change in form of parallel shifts and that this is incompatible with dynamic arbitrage-free term structure models; see, e.g., Ingersoll, Skelton, and Weil (1978) and Cox, Ingersoll, and Ross (1979). Moreover, the original definition only makes sense for bonds, not for other interest rate sensitive assets. In the context of dynamic term structure models, the price sensitivity to the default-free short rate is a much more appropriate and tractable measure of interest rate risk and it is well-defined for a broad set of assets.

risk-neutral probability measure conditional on time t information. The time t price of the default-free bond with discrete coupons is then

$$\bar{P}_t^{q,t_n} = q \sum_{t_j > t} \bar{P}_t^{t_j} + \bar{P}_t^{t_n},$$

where the sum is over all $j \in \{1, \ldots, n\}$ with $t_j > t$, i.e. all future coupon payment dates. For the bond with continuous coupons the price is

$$\bar{P}_t^{q,t_n} = q \int_t^{t_n} \bar{P}_t^s \, ds + \bar{P}_t^{t_n}.$$

The duration of the default-free coupon bond, $\bar{D}_t^{q,t_n} = -\frac{\partial \bar{P}_t^{q,t_n}}{\partial r} \frac{1}{\bar{P}_t^{q,t_n}}$, follows from

$$-\frac{\partial \bar{P}_t^{q,t_n}}{\partial r} = q \sum_{t_j > t} \bar{P}_t^{t_j} \bar{D}_t^{t_j} + \bar{P}_t^{t_n} \bar{D}_t^{t_n} \tag{1}$$

and

$$-\frac{\partial \bar{P}_t^{q,t_n}}{\partial r} = q \int_t^{t_n} \bar{P}_t^s \bar{D}_t^s \, ds + \bar{P}_t^{t_n} \bar{D}_t^{t_n}, \tag{2}$$

respectively, where

$$\bar{D}_t^s = -\frac{\partial \bar{P}_t^s}{\partial r} \frac{1}{\bar{P}_t^s}$$

is the duration of the default-free zero-coupon bond maturing at time s. It follows that the duration of a default-free coupon bond is a weighted average of the durations of the default-free zero-coupon bonds maturing at the different payment dates of the coupon bond.

Next, consider defaultable bonds. Default risk is modeled by a default indicator $\mathbf{1}_{\{\tau > t\}}$, where the stopping time τ denotes the default time of the bond issuer. Let $h_\tau \in [0, 1]$ be the recovery of the bond, i.e. the payment received at the time of default or the value of the claims received in case of default. The time t value of a corporate zero-coupon bond with maturity t_n and zero recovery reads

$$P_t^{t_n} = \mathrm{E}_t \left[e^{-\int_t^{t_n} r_s \, ds} \mathbf{1}_{\{\tau > t_n\}} \right],$$

while with non-zero recovery we obtain

$$P_t^{t_n,h} = \mathrm{E}_t \left[e^{-\int_t^{t_n} r_s \, ds} \mathbf{1}_{\{t < \tau \leq t_n\}} h_\tau \right] + \mathrm{E}_t \left[e^{-\int_t^{t_n} r_s \, ds} \mathbf{1}_{\{\tau > t_n\}} \right] = H_t^{t_n,h} + P_t^{t_n},$$

where $H_t^{t_n,h} = \mathrm{E}_t \left[e^{-\int_t^{t_n} r_s \, ds} \mathbf{1}_{\{t < \tau \leq t_n\}} h_\tau \right]$ is the value of the recovery payment. The time t price of a defaultable coupon bond with coupon q and recovery h is given by

$$P_t^{q,t_n,h} = H_t^{t_n,h} + q \sum_{t_j > t} P_t^{t_j} + P_t^{t_n}$$

if coupons are paid at discrete points in time t_j and by

$$P_t^{q,t_n,h} = H_t^{t_n,h} + q \int_t^{t_n} P_t^s \, ds + P_t^{t_n}$$

if coupons are paid continuously. If recovery is zero, i.e. $h \equiv 0$ and thus $H_t^{t_n,h} \equiv 0$, we sometimes use the abbreviation $P_t^{q,t_n} = P_t^{q,t_n,0}$.

The durations of a corporate bond and the recovery payment are given by

$$D_t^{q,t_n,h} = -\frac{\partial P_t^{q,t_n,h}}{\partial r} \frac{1}{P_t^{q,t_n,h}}, \qquad \hat{D}_t^{t_n,h} = -\frac{\partial H_t^{t_n,h}}{\partial r} \frac{1}{H_t^{t_n,h}}.$$

Since the price of a corporate bond can be represented as $P_t^{q,t_n,h} = H_t^{t_n,h} + P_t^{q,t_n,0}$, the duration of the corporate bond equals the weighted average of the duration of the recovery payment and the duration of the bond with zero recovery:

$$\begin{aligned} D_t^{q,t_n,h} &= -\frac{\partial P_t^{q,t_n,h}}{\partial r} \frac{1}{P_t^{q,t_n,h}}, \\ &= -\frac{\partial H_t^{t_n,h}}{\partial r} \frac{1}{P_t^{q,t_n,h}} - \frac{\partial P_t^{q,t_n,0}}{\partial r} \frac{1}{P_t^{q,t_n,h}}, \\ &= (1 - w^h)\, \hat{D}_t^{t_n,h} + w^h\, D_t^{q,t_n,0}, \end{aligned} \tag{3}$$

where $1 - w^h = H_t^{t_n,h}/P_t^{q,t_n,h}$ and $w^h = P_t^{q,t_n,0}/P_t^{q,t_n,h}$ are weights. Since the no arbitrage assumption dictates that the corporate bond price $P_t^{q,t_n,h}$ is increasing in the recovery payment h, we get $w^h \in [0,1]$ implying $1 - w^h \in [0,1]$ as well.

The goal of this paper is to characterize the relation between the duration of a default-free bond, $\bar{D}_t^{q,t_n}$, and the duration of defaultable bond, $D_t^{q,t_n,h}$. It is thus crucial how $\bar{D}_t^{q,t_n}$ behaves compared to the duration of a corporate bond with zero recovery, $D_t^{q,t_n,0}$, and the duration of the recovery payment, $\hat{D}_t^{t_n,h}$. These are non-trivial question because although the prices $P_t^{q,t_n,h}$ and $H_t^{t_n,h}$ are monotonously increasing in h, the behavior of the corresponding durations is not obvious since the recovery payment h shows up both in the numerators and the denominators of the durations. We thus need to add structure to our model. Two properties, however, follow directly from the definition of the recovery payment:

Proposition 2.1 *(i) The duration of the recovery payment is not affected by scalar multiplications, i.e.* $\hat{D}_t^{t_n,\varepsilon h} = \hat{D}_t^{t_n,h}$ *for* $\varepsilon > 0$.

(ii) If the recovery process h is constant, then the duration of the recovery payment is independent of h, i.e. $\hat{D}_t^{t_n,h_1} = \hat{D}_t^{t_n,h_2}$ *for constants* $h_1,\, h_2 > 0$.

2.2 Durations and measures

As discussed above the duration of a default-free coupon bond is a weighted average of the durations of the default-free zero-coupon bonds maturing at the different payment dates of the coupon bond. The durations of default-free zero-coupon bonds of maturity up to t_n define a function $\bar{D}_t : [t, t_n] \to \mathbb{R}$ with value $\bar{D}_t^s$ for maturity s. We can think of $\bar{D}_t$ as a random variable on the set $[t, t_n]$. We will now show that we can write the duration of the default-free coupon bond as an expectation of this random variable, $\bar{D}_t^{q,t_n} = \mathrm{E}_{\bar{\nu}}[\bar{D}_t]$. First define the measure $\bar{\mu}$ on $[t, t_n]$ as

$$\bar{\mu}(ds) = q \sum_{t_j > t} \bar{P}_t^{t_j} \varepsilon_{t_j}(s) + \bar{P}_t^{t_n} \varepsilon_{t_n}(s)$$

for the case of discrete coupons and

$$\bar{\mu}(ds) = q \bar{P}_t^s ds + \bar{P}_t^{t_n} \varepsilon_{t_n}(s)$$

for the case of continuous coupons. Here, $\varepsilon_{t_j}(s)$ denotes the Dirac mass at t_j. Normalizing the measure by $\bar{\mu}([t, t_n]) = \bar{P}_t^{q,t_n}$ leads to a probability measure $\bar{\nu}(ds) = \bar{\mu}(ds)/\bar{\mu}([t, t_n])$. With either discrete or continuous coupons, we have

$$\int_t^{t_n} \bar{D}_t^s \, d\bar{\mu}(s) = -\frac{\partial \bar{P}_t^{q,t_n}}{\partial r},$$

and therefore the duration of the default-free coupon bond is

$$\bar{D}_t^{q,t_n} = \frac{\int_t^{t_n} \bar{D}_t^s \, d\bar{\mu}(s)}{\bar{\mu}([t,t_n])} = \int_t^{t_n} \bar{D}_t^s \, d\bar{\nu}(s) = \mathrm{E}_{\bar{\nu}}[\bar{D}_t].$$

In some interesting cases we will be able to express the duration of either the corporate coupon bond, $D_t^{q,t_n,h}$, or the duration of its recovery payment, $\hat{D}^{t_n,h}$, as the expectation $\mathrm{E}_\nu[\bar{D}_t]$ under a different probability measure ν on $[t,t_n]$. In the comparison between the duration of the corporate coupon bond and the duration of the default-free coupon bond we will then apply the following result.

Lemma 2.1 *Let ν_1 and ν_2 be probability measures on $[t,t_n]$ so that the density $\frac{d\nu_1}{d\nu_2} : [t,t_n] \to \mathbb{R}$ exists and is non-increasing. If $X : [t,t_n] \to \mathbb{R}$ is bounded, measurable, and non-decreasing, then $\mathrm{E}_{\nu_1}[X] \leq \mathrm{E}_{\nu_2}[X]$.*

Proof: Since any non-decreasing bounded, measurable real-valued function on $[t,t_n]$ can be approximated by a sum of indicator functions $\mathbf{1}_{[t',t_n]}$ with $t' \in [t,t_n]$, it is sufficient to prove the claim for $X(s) = \mathbf{1}_{[t',t_n]}(s)$, $s \in [t,t_n]$, i.e. to show that

$$0 \geq \mathrm{E}_{\nu_1}[\mathbf{1}_{[t',t_n]}] - \mathrm{E}_{\nu_2}[\mathbf{1}_{[t',t_n]}] = \mathrm{E}_{\nu_2}\left[\left(\frac{d\nu_1}{d\nu_2} - 1\right)\mathbf{1}_{[t',t_n]}\right] = \int_{[t',t_n]} \left(\frac{d\nu_1}{d\nu_2} - 1\right) d\nu_2.$$

Since $\frac{d\nu_1}{d\nu_2}$ is a density, it must be greater than or equal to 1 in some interval $[t,t^*]$. If $t^* \leq t'$, then $\frac{d\nu_1}{d\nu_2} \leq 1$ on $[t',t_n]$ and the result follows. If $t^* > t'$, we have

$$\begin{aligned}\int_{[t',t_n]} \left(\frac{d\nu_1}{d\nu_2} - 1\right) d\nu_2 &\leq \int_{[t,t']} \left(\frac{d\nu_1}{d\nu_2} - 1\right) d\nu_2 + \int_{[t',t_n]} \left(\frac{d\nu_1}{d\nu_2} - 1\right) d\nu_2 \\ &= \int_{[t,t_n]} \left(\frac{d\nu_1}{d\nu_2} - 1\right) d\nu_2 = 0,\end{aligned}$$

since ν_2 is a probability measure on $[t,t_n]$ and $\frac{d\nu_1}{d\nu_2}$ is a density. □

The monotonicity requirement on the density is the so-called Monotone Likelihood Ratio Condition which is also of importance in other fields of (financial) economics such as principal-agent problems; see, e.g., Rogerson (1985).

The duration of a default-free zero-coupon bond will be decreasing in maturity in most dynamic term structure models, e.g. the one-factor Vasicek and Cox-Ingersoll-Ross (CIR) models as well as the two-factor Hull-White extension of the Vasicek model. Assuming that the condition on the duration of default-free zero-coupon bonds is satisfied, in order to apply the lemma it remains to study the density $d\nu/d\bar{\nu}$ between the two relevant measures.

2.3 Modeling the recovery payment

In the literature mainly three recovery regimes are studied:

(i) Recovery of market value (RMV): the value of the corporate bond immediately following a default is some fraction δ_τ of the market value of the corporate bond immediately before default. In this case $h_\tau = (1-l_\tau)P_{\tau-}^{q,t_n}$, where $l_s \in [0,1]$ is the fractional loss in market value in case of default at time s. From Duffie and Singleton (1999), the time t value of a promised unit payment at time t_n is

$$P_t^{t_n,l} = \mathrm{E}_t\left[e^{-\int_t^{t_n}(r_u + \lambda_u l_u)\,du}\right],$$

where λ is the intensity process of the default time.

(ii) Recovery of Treasury (RT): immediately upon default, the bond holder receives a fraction δ_τ of default-free, but otherwise identical, bonds. In this case $h_\tau = \delta_\tau \bar{P}_\tau^{q,t_n}$, cf. Jarrow and Turnbull (1995).

(iii) Recovery of face value (RF): immediately upon default, the bond holder receives a fraction k_τ of the face value of the bond and no further payments. In this case $h_\tau = k_\tau$, cf., e.g., Lando (1998).

As long as k, δ, and l are allowed to be stochastic processes, these assumptions are equivalent, but differences occur if they are assumed to be constant. In this paper, we concentrate on the analysis on the RMV assumption, but we will also discuss the RT and the RF assumption if interest rate risk and default risk are independent.

3 Independence

To analyze the duration of a corporate bond, we make the simplifying assumption in this section that interest rate risk and default risk are independent, i.e. r and τ are assumed to be independent. For zero-coupon bond prices this assumption implies that

$$P_t^s = \mathrm{E}_t\left[e^{-\int_t^s r_u\,du}\mathbf{1}_{\{\tau>s\}}\right] = Q_t^s \bar{P}_t^s,$$

where $Q_t^s := \mathrm{E}_t\left[\mathbf{1}_{\{\tau>s\}}\right]$ denotes the survival probability of the firm. From this representation it is clear that the following proposition holds.

Proposition 3.1 *If default risk is independent of interest rate risk, then the durations of a default-free zero coupon bond and the corporate zero-coupon bond with zero-recovery coincide, i.e.* $\bar{D}_t^s = D_t^s$.

Next, we consider coupon bonds, first for the zero-recovery case and then for various recovery assumptions.

3.1 Coupon Bond with Zero Recovery

If default risk and interest rate risk are independent, then in the case of discrete coupon payments

$$P_t^{q,t_n} = q\sum_{t_j>t} \bar{P}_t^{t_j} Q_t^{t_j} + \bar{P}_t^{t_n} Q_t^{t_n}$$

implying

$$-\frac{\partial P_t^{q,t_n}}{\partial r} = q\sum_{t_j>t} \bar{P}_t^{t_j} Q_t^{t_j} \bar{D}_t^{t_j} + \bar{P}_t^{t_n} Q_t^{t_n} \bar{D}_t^{t_n}$$

and in the case of continuous coupon payments

$$P_t^{q,t_n} = q\int_t^{t_n} \bar{P}_t^s Q_t^s\,ds + \bar{P}_t^{t_n} Q_t^{t_n}$$

implying

$$-\frac{\partial P_t^{q,t_n}}{\partial r} = q\int_t^{t_n} \bar{P}_t^s Q_t^s \bar{D}_t^s\,ds + \bar{P}_t^{t_n} Q_t^{t_n} \bar{D}_t^{t_n}.$$

Due to the survival probabilities satisfying $Q_t^{t_j} \leq 1$, we have both

$$\bar{P}_t^{q,t_n} \geq P_t^{q,t_n} \quad \text{and} \quad -\frac{\partial \bar{P}_t^{q,t_n}}{\partial r} \geq -\frac{\partial P_t^{q,t_n}}{\partial r}.$$

Therefore, it is not obvious which of the durations $\bar{D}^{q,t_n}$ and D^{q,t_n} is greater. The following theorem shows that the duration of the corporate bond is indeed smaller than or equal to the duration of the default-free bond under the assumptions of this section. The proof is based on an application of Lemma 2.1.

As discussed in Section 2.2, the duration of the default-free coupon bond can be written as $\bar{D}_t^{q,t_n} = \mathrm{E}_{\bar{\nu}}[\bar{D}_t]$. For the corporate coupon bond define the measure μ on $[t, t_n]$ by

$$\mu(ds) = q \sum_{t_j > t} \bar{P}_t^{t_j} Q_t^{t_j} \varepsilon_{t_j}(s) + \bar{P}_t^{t_n} Q_t^{t_n} \varepsilon_{t_n}(s),$$

if coupons are paid discretely, and

$$\mu(ds) = q \bar{P}_t^s Q_t^s ds + \bar{P}_t^{t_n} Q_t^{t_n} \varepsilon_{t_n}(s),$$

if coupons are paid continuously. In both cases define the probability measure ν by $\nu(ds) = \mu(ds)/\mu([t,t_n])$. With either discrete or continuous coupons, we have

$$\int_t^{t_n} \bar{D}_t^s \, d\mu(s) = -\frac{\partial P_t^{q,t_n}}{\partial r}$$

so that the duration of the defaultable coupon bond is

$$D_t^{q,t_n} = \frac{\int_t^{t_n} \bar{D}_t^s \, d\mu(s)}{\mu([t,t_n])} = \mathrm{E}_\nu[\bar{D}_t].$$

The measures ν and $\bar{\nu}$ are equivalent and thus there exists a density $d\nu/d\bar{\nu}$ on $[t, t_n]$ given by

$$Z := \frac{d\nu}{d\bar{\nu}} = \frac{\bar{\mu}([t,t_n])}{\mu([t,t_n])} Q_t. \tag{4}$$

Since the survival probabilities Q_t^s, and therefore Z_s, are non-increasing in s on $[t, t_n]$, the following proposition results from an application of Lemma 2.1.

Theorem 3.1 *If default risk is independent of interest rate risk and the duration $\bar{D}_t^s$ of default-free zero-coupon bonds is non-decreasing in maturity s, then the duration of a corporate coupon bond with zero-recovery is smaller than the duration of the corresponding default-free coupon bond, i.e. $D_t^{q,t_n} \leq \bar{D}_t^{q,t_n}$.*

We emphasize that under the independence assumption the proposition holds for any arbitrage-free interest rate model with differentiable bond prices and non-decreasing durations of default-free zeroes and any specification of the survival probabilities, which implies that survival probabilities Q_t^s are non-increasing in s.

3.2 Coupon Bond with Recovery of Market Value

As mentioned above the recovery of market value (RMV) assumption implies a recovery value of $h_\tau = (1 - l_\tau) P_{\tau-}^{q,t_n,l}$, where l is the loss rate. We assume that interest rate risk r is independent of default risk λ and recovery risk l. Setting $Q_t^{s,l} = \mathrm{E}[e^{-\int_t^s l_u \lambda_u \, du}]$, it follows from the results of Duffie and Singleton (1999) that

$$P_t^{q,t_n,l} = q \sum_{t_j > t} \bar{P}_t^{t_j} Q_t^{t_j,l} + \bar{P}_t^{t_n} Q_t^{t_n,l}$$

and

$$-\frac{\partial P_t^{q,t_n,l}}{\partial r} = q \sum_{t_j > t} \bar{P}_t^{t_j} Q_t^{t_j,l} \bar{D}_t^{t_j} + \bar{P}_t^{t_n} Q_t^{t_n,l} \bar{D}_t^{t_n}$$

or

$$P_t^{q,t_n,l} = q \int_t^{t_n} \bar{P}_t^s Q_t^{s,l}\, ds + \bar{P}_t^{t_n} Q_t^{t_n,l}$$

and

$$-\frac{\partial P_t^{q,t_n,l}}{\partial r} = q \int_t^{t_n} \bar{P}_t^s Q_t^{s,l} \bar{D}_t^s\, ds + \bar{P}_t^{t_n} Q_t^{t_n,l} \bar{D}_t^{t_n}.$$

Again the payment streams of the corporate bonds induce measures μ^l and ν^l such that $D^{q,t_n,l} = \mathrm{E}_{\nu^l}[\bar{D}_t]$, and the relevant density is

$$\frac{d\nu^l}{d\bar{\nu}} = \frac{\bar{\mu}([t,t_n])}{\mu^l([t,t_n])} Q_t^{\cdot,l},$$

where $Q_t^{\cdot,l}$ is interpreted as a random variable on $[t,t_n]$ with possible values $Q_t^{s,l}$. Therefore, the next proposition follows from an application of Lemma 2.1.

Proposition 3.2 *If default risk and recovery risk are independent of interest rate risk and the duration $\bar{D}_t^s$ of default-free zero-coupon bonds is non-decreasing in maturity s, then under RMV the duration of a corporate coupon bond is smaller than the duration of the corresponding default-free coupon bond, i.e. $D_t^{q,t_n,l} \le \bar{D}_t^{q,t_n}$.*

Clearly, the question arises whether $D_t^{q,t_n,l}$ increases with the recovery rate. From our considerations so far it would be sufficient if, for loss processes l and $\tilde{l}$ with $0 \le l \le \tilde{l} \le 1$, the ratio

$$\frac{d\nu^{\tilde{l}}}{d\nu^l} = \frac{\mu^l([t,t_n])}{\mu^{\tilde{l}}([t,t_n])} \frac{Q_t^{\cdot,\tilde{l}}}{Q_t^{\cdot,l}} = \frac{\mu^l([t,t_n])}{\mu^{\tilde{l}}([t,t_n])} \frac{\mathrm{E}_t[e^{-\int_t^{\cdot} \tilde{l}_u \lambda_u\, du}]}{\mathrm{E}_t[e^{-\int_t^{\cdot} l_u \lambda_u\, du}]} \tag{5}$$

is decreasing on $[t,t_n]$. Without loss of generality we can assume $\tilde{l} = 1$ because one can always set $\hat{\lambda} = \tilde{l}\lambda$ and $\hat{l} = l/\tilde{l} \in [0,1]$. Unfortunately, in general the ratio (5) does not possess the desired property as the following example demonstrates.

Example: Choose $t < T_0 < T_1 < t_n$ with $T_1 = 2T_0$ and set $l_u = 0$ for $u \in [t,T_0]$ and $l_u = 1$ for $u \in [T_0,T_1]$. Then we get

$$\frac{\mathrm{E}_t[e^{-\int_t^{T_1} \lambda_u\, du}]}{\mathrm{E}_t[e^{-\int_t^{T_1} l_u \lambda_u\, du}]} = \frac{\mathrm{E}_t[e^{-\int_t^{T_0} \lambda_u\, du}]\mathrm{E}_t[e^{-\int_{T_0}^{T_1} \lambda_u\, du}] + \mathrm{Cov}[e^{-\int_t^{T_0} \lambda_u\, du}, e^{-\int_{T_0}^{T_1} \lambda_u\, du}]}{\mathrm{E}_t[e^{-\int_{T_0}^{T_1} l_u \lambda_u\, du}]}.$$

In general we cannot assume that the covariance is negative (e.g. choose $\lambda_{2u} = \lambda_u$ for $u \in [t,T_0]$). Hence, if the covariance is strictly positive, it follows that

$$\frac{\mathrm{E}_t[e^{-\int_t^{T_1} \lambda_u\, du}]}{\mathrm{E}_t[e^{-\int_t^{T_1} l_u \lambda_u\, du}]} > \frac{\mathrm{E}_t[e^{-\int_t^{T_0} \lambda_u\, du}]}{\mathrm{E}_t[e^{-\int_t^{T_0} l_u \lambda_u\, du}]}$$

and the ratio is not decreasing. □

3.3 Recovery of Treasury

Following Jarrow and Turnbull (1995), the recovery process under recovery of treasury (RT) is modeled as $h_\tau = \delta_\tau \bar{P}_{\tau-}^{q,t_n}$, where δ denotes the fraction of a similar default-free bond which the bond holder receives upon default. As in (3) we have

$$D_t^{q,t_n,\delta} = (1 - w^\delta)\, \hat{D}_t^{t_n,\delta} + w^\delta\, D_t^{q,t_n,0}, \tag{6}$$

To draw conclusions about the relation between default-free and defaultable bonds, it is thus crucial how large the durations $\hat{D}_t^{t_n,\delta}$ and $D_t^{q,t_n,0}$ are, compared to the duration of default-free bond, $\bar{D}_t^{q,t_n}$. By Theorem 3.1, the independence assumption yields $D_t^{q,t_n,0} \leq \bar{D}_t^{q,t_n}$. Therefore, we will now concentrate on the duration of the recovery payment. Assuming that the default time has an intensity process λ, it follows from Lando (1998) that

$$H_t^{t_n,\delta} = \mathrm{E}_t\left[\int_t^{t_n} e^{-\int_t^s (r_u+\lambda_u)\,du}\delta_s \bar{P}_s^{q,t_n}\lambda_s\,ds\right].$$

Under the assumption that interest rate risk is independent of default *and recovery risk* we obtain

$$H_t^{t_n,\delta} = \int_t^{t_n} \mathrm{E}_t\left[e^{-\int_t^s r_u\,du}\bar{P}_s^{q,t_n}\right]\mathrm{E}_t\left[\delta_s e^{-\int_t^s \lambda_u\,du}\lambda_s\right]\,ds = \bar{P}_t^{q,t_n} I_t^{t_n}$$

where $I_t^{t_n} = \int_t^{t_n} \mathrm{E}_t\left[\delta_s e^{-\int_t^s \lambda_u\,du}\lambda_s\right]\,ds$.

Proposition 3.3 *Assume that interest rate risk is independent of default and recovery risk and that recovery of treasury applies. Then the duration of the recovery payment equals the duration of the corresponding default-free bond, i.e.* $\hat{D}_t^{t_n,\delta} = \bar{D}_t^{q,t_n}$.

Proof: The duration of $H_t^{t_n,\delta} = \bar{P}_t^{q,t_n} I_t^{t_n}$ can be calculated as

$$\hat{D}_t^{t_n,\delta} = -\frac{\partial H_t^{t_n,\delta}}{\partial r}\frac{1}{H_t^{t_n,\delta}} = -\frac{\partial \bar{P}_t^{q,t_n}}{\partial r}\frac{I_t^{t_n}}{H_t^{t_n,\delta}} = -\frac{\partial \bar{P}_t^{q,t_n}}{\partial r}\frac{1}{\bar{P}_t^{q,t_n}}\frac{\bar{P}_t^{q,t_n} I_t^{t_n}}{H_t^{t_n,\delta}} = \bar{D}_t^{q,t_n}.$$

□

This result has the following implications for the duration of corporate bonds.

Proposition 3.4 *Assume that default risk and recovery risk are independent of interest rate risk and that the duration* $\bar{D}_t^s$ *of default-free zero-coupon bonds is non-decreasing in maturity* s. *Then under RT the following is valid:*

(i) The duration of a corporate coupon bond with recovery δ *is smaller than the duration of the corresponding default-free coupon bond, i.e.* $D_t^{q,t_n,\delta} \leq \bar{D}_t^{q,t_n}$.

(ii) The duration of a corporate bond is increasing in the recovery rate, i.e. in the fraction δ.

Proof: Equation (6) and Proposition 3.3 imply

$$D_t^{q,t_n,\delta} = (1-w^\delta)\,\bar{D}_t^{q,t_n} + w^\delta\,D_t^{q,t_n,0} = \bar{D}_t^{q,t_n} - w^\delta\left(\bar{D}_t^{q,t_n} - D_t^{q,t_n,0}\right), \tag{7}$$

where $1-w^\delta = H_t^{t_n,\delta}/P_t^{q,t_n,\delta}$ and $w^\delta = P_t^{q,t_n,0}/P_t^{q,t_n,\delta} \in [0,1]$. By Theorem 3.1, we have $D_t^{q,t_n,0} \leq \bar{D}_t^{q,t_n}$. Hence, (i) follows from (7).

To prove (ii), note that w^δ is decreasing in δ because $P_t^{q,t_n,0}$ is independent of δ and, due to the no arbitrage requirement, $P_t^{q,t_n,\delta}$ is increasing in δ. Since $D_t^{q,t_n,0} \leq \bar{D}_t^{q,t_n}$ the claim follows from (7). □

We emphasize that the previous proposition does not nest the RMV assumption as a special case. We can incorporate an RMV-style recovery payment of $h_\tau = (1-l_\tau)P_{\tau-}^{q,t_n,l}$ into the RT-framework by setting $\delta_\tau = (1-l_\tau)P_{\tau-}^{q,t_n,l}/\bar{P}_{\tau-}^{q,t_n}$ but if l is independent of the short rate, δ will not be so and the proposition above does not apply.

Two points should be addressed: Firstly, in contrast to the previous section, it is now possible to show that the duration of a defaultable bond is increasing in the recovery rate (at least under an additional independence assumption). Secondly, note that even if a defaultable bond is actually not exposed to default risk, i.e. $\delta = 1$, its duration does not coincide with the duration of a corresponding default-free bond. This is due to the fact that upon default the investor receives the present value of the bond. From this point of view, a defaultable bond with $\delta = 1$ can be interpreted as a default-free bond which is callable and this callability is modeled via the intensity λ. The duration of such a bond is obviously smaller than the duration of a non-callable default-free bond.

3.4 Recovery of Face Value

Assuming recovery of face value (RF) means that $h_\tau = k_\tau$, where k is the fraction of face value which the bondholder receives upon default. Since the face value is normalized to one, k equals the amount of money which is paid back to the bondholder. Again (3) yields

$$D_t^{q,t_n,k} = (1 - w^k)\, \hat{D}_t^{t_n,k} + w^k\, D_t^{q,t_n,0}.$$

As in the previous subsection, we will thus concentrate on the duration of the recovery payment and place our analysis in the Cox process framework by Lando (1998) implying

$$H_t^{t_n,k} = \mathrm{E}_t\left[\int_t^{t_n} e^{-\int_t^s (r_u+\lambda_u)\, du} k_s \lambda_s\, ds\right].$$

Therefore, $H_t^{t_n,k}$ can be interpreted as the price of a corporate bond with stochastic coupon $k\lambda$ which is paid continuously. Since the face value is not paid back as a lump sum, this coupon consists of an interest rate part and a redemption part. For this reason, we will compare the duration of the recovery payment with the durations of corporate bonds with continuous coupon payment.

Under the assumption that interest rate risk is independent of default and recovery risk it follows that

$$H_t^{t_n,k} = \int_t^{t_n} \bar{P}_t^s \mathrm{E}_t\left[e^{-\int_t^s \lambda_u\, du} k_s \lambda_s\right] ds = \int_t^{t_n} \bar{P}_t^s K_t^s\, ds,$$

where $K_t^s = \mathrm{E}_t\left[e^{-\int_t^s \lambda_u\, du} k_s \lambda_s\right]$. Hence, the recovery payment induces the measure

$$\mu^H(ds) = \bar{P}_t^s K_t^s ds$$

and the probability measure $\nu^H(ds) = \mu^H(ds)/\mu^H([t,t_n])$. The duration of $H_t^{t_n,k}$ is then given by

$$\hat{D}_t^{t_n,k} = -\frac{\partial H_t^{t_n,k}}{\partial r}\frac{1}{H_t^{t_n,k}} = \frac{\int_t^{t_n} \bar{P}_t^s \bar{D}_t^s K_t^s\, ds}{\int_t^{t_n} \bar{P}_t^s K_t^s\, ds} = \mathrm{E}_{\nu^H}[\bar{D}_t].$$

The measure ν^H can be compared with the measures $\bar{\nu}$ and ν of a default-free coupon bond and of a corporate coupon bond with zero recovery. Consequently, the following densities need to be analyzed:

$$\begin{aligned} \frac{d\nu^H}{d\bar{\nu}} &= \frac{\bar{\mu}([t,t_n])}{\mu^H([t,t_n])}\frac{K_t}{q}\mathbf{1}_{\{\cdot \neq t_n\}}, \\ \frac{d\nu^H}{d\nu} &= \frac{\mu([t,t_n])}{\mu^H([t,t_n])}\frac{K_t}{qQ_t}\mathbf{1}_{\{\cdot \neq t_n\}}. \end{aligned} \tag{8}$$

Applying Lemma 2.1 again, we arrive at the following proposition.

Proposition 3.5 *Assume that default risk and recovery risk are independent of interest rate risk and that the duration $\bar{D}_t^s$ of default-free zero-coupon bonds is non-decreasing in maturity s. Assume that coupon payments are made continuously and that recovery of treasury applies. Then the following results hold:*

(i) If K_t is decreasing on $[t, t_n]$, then the duration of the recovery payment is smaller than the duration of a corresponding default-free coupon bond, i.e. $\hat{D}_t^{t_n,k} \leq \bar{D}_t^{q,t_n}$. Therefore, the duration of a defaultable coupon bond with recovery k is smaller than the duration of a corresponding default-free coupon bond, i.e. $D_t^{q,t_n,k} \leq \bar{D}_t^{q,t_n}$.

(ii) If K_t/Q_t is decreasing on $[t, t_n]$, then the duration of the recovery payment is smaller than the duration of the defaultable coupon bond with zero recovery, i.e. $\hat{D}_t^{t_n,k} \leq D_t^{q,t_n,0}$. Therefore, the duration of a defaultable coupon bond with recovery k is smaller than the duration of a corresponding defaultable bond with zero recovery, i.e. $D_t^{q,t_n,k} \leq D_t^{q,t_n,0} \leq \bar{D}_t^{q,t_n}$.

The time t cumulative distribution function of the default time τ is given by

$$F_t^\tau(s) = 1 - Q_t^s = 1 - \mathrm{E}_t[\mathbf{1}_{\tau>s}] = 1 - \mathrm{E}_t[e^{-\int_t^s \lambda_u \, du}]$$

and the corresponding density by

$$f_t^\tau(s) = \frac{\partial F_t^\tau(s)}{\partial s} = \mathrm{E}_t[e^{-\int_t^s \lambda_u \, du} \lambda_s].$$

The conditional hazard rate function of τ is defined as

$$\chi(t,s) = \frac{f_t^\tau(s)}{1 - F_t^\tau(s)} = \frac{f_t^\tau(s)}{Q_t^s}$$

Corollary 3.1 *Assume that the recovery process k is independent of interest rate and default risk.*

(i) If $f_t^\tau(s)\mathrm{E}_t[k_s]$ is decreasing in $s \in [t, t_n]$, then implication (i) of Proposition 3.5 holds.

(ii) If $\chi(t,s)\mathrm{E}_t[k_s]$ is decreasing in $s \in [t, t_n]$, then implication (ii) of Proposition 3.5 holds.

Clearly, for a constant recovery process k, it is sufficient if the density f_t^τ or the conditional hazard rate function $\chi(t, \cdot)$ are decreasing on $[t, t_n]$. We emphasize that these conditions are rather strong requirements. Since the densities (8) contain the indicator $\mathbf{1}_{\{\cdot \neq t_n\}}$, actually the redemption of the notional is ignored implying these strong conditions on the remaining part of the payment stream.

4 Dependence

From our derivations so far, it should be obvious that it is much more complicated to obtain results in a situation where default risk and interest rate risk are not independent. In general defaultable bonds can have smaller and larger duration than their default-free counterparts. Nevertheless, we will be able to derive a sufficient condition ensuring that the result from the previous section holds in the general situation as well.

4.1 Duration of Zero-Coupon Bonds with Zero Recovery

Throughout this subsection we place our considerations in a Cox process framework. Besides, we need to put more structure on the dynamics of the short rate. Let the dynamics of the short rate be given by the following SDE

$$dr_s = \alpha(s, r_s) \, ds + \beta(s, r_s) \, dW_s, \quad r_t = \bar{r},$$

where W is a (possibly) multi-dimensional Brownian motion. We assume this SDE possesses a unique solution $\{r_t\}_t$ and that the coefficients α and β can be differentiated with respect to t and r. The corresponding partial derivatives are denoted by α_t, α_r and β_t, β_r, respectively. Under the technical requirements on α and β which can be found in Protter (2005, pp. 311ff), the solution to the SDE can be differentiated with respect to the initial value $\bar{r}$ and the derivative $y_t \equiv \frac{\partial}{\partial \bar{r}} r_t$ satisfies the following linear SDE

$$dy_s = y_s[\alpha_r(s, r(s))\, ds + \beta_r(s, r_s)\, dW_s], \quad y_t = 1,$$

and thus

$$y_u = \exp\left(\int_t^u [\alpha_r(s, r(s)) - 0.5\beta_r(s, r_s)^2]\, ds + \int_t^u \beta_r(s, r_s)\, dW_s \right) \geq 0.$$

The time t prices of a default-free and a defaultable zero-coupon bond with maturities s are given by

$$\bar{P}_t^s = \mathrm{E}_t \left[e^{-\int_t^s r_u\, du} \right]$$

and

$$P_t^s = \mathrm{E}_t \left[e^{-\int_t^s (r_u + \lambda_u)\, du} \right], \tag{9}$$

respectively, where λ denotes the default intensity. We assume that the intensity λ is a function of the short rate and the state $\omega \in \Omega$. For notational convenience, we suppress this second dependency and assume that λ is differentiable with respect to r. The corresponding derivative is denoted by λ_r. Besides, to simplify notations, in the following we use $r = \bar{r}$. Taking derivatives with respect to the initial short rate leads to

$$-\frac{\partial \bar{P}_t^s}{\partial r} = \mathrm{E}_t \left[e^{-\int_t^s r_u\, du} \int_t^s y_u\, du \right] = \bar{P}_t^s \cdot \int_t^s \mathrm{E}_t^{Q^s} [y_u]\, du$$

and

$$-\frac{\partial P_t^s}{\partial r} = \mathrm{E}_t \left[e^{-\int_t^s (r_u + \lambda_u)\, du} \int_t^s y_u (1 + \lambda_r(r_u))\, du \right], \tag{10}$$

where Q^s denotes the s-forward measure. Therefore, the duration of a default-free zero is given by

$$\bar{D}_t^s = -\frac{\partial \bar{P}_t^s}{\partial r} \frac{1}{\bar{P}_t^s} = \int_t^s \mathrm{E}_t^{Q^s} [y_u]\, du \geq 0.$$

Note that, on the one hand, the derivative of the corporate bond price $\partial P_t^s / \partial r$ is negative if $\lambda_r > -1$. On the other hand, the default intensity is positive, i.e. $\lambda \geq 0$, and $y \geq 0$. Therefore, for $\lambda_r < 0$ we conclude

$$-\frac{\partial \bar{P}_t^s}{\partial r} \geq -\frac{\partial P_t^s}{\partial r}. \tag{11}$$

If, however, $\lambda_r > 0$ the relation need not hold. Consequently, the dependency of the intensity λ on the short rate r plays an important role. To calculate the durations of both bonds, we need to divide both derivatives by the respective bond price. If (11) holds, it is not clear which duration is greater because $\bar{P}_t^s \geq P_t^s$. Only if (11) is violated, is the duration of the corporate bond in any case greater than the duration of the default-free bond. But this is not the usual situation.

To gain more insights about the duration of defaultable zero-coupon bonds, assume for the rest of this subsection that the intensity is an affine function of the short rate r. Then the derivative (10) can be rewritten as

$$
\begin{aligned}
-\frac{\partial P_t^s}{\partial r} &= (1+\lambda_r)\mathrm{E}_t\left[e^{-\int_t^s r_u+\lambda_u\,du}\int_t^s y_u\,du\right] \\
&= (1+\lambda_r)\mathrm{E}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du\right]\mathrm{E}_t\left[e^{-\int_t^s \lambda_u\,du}\right] \\
&\quad + (1+\lambda_r)\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du, e^{-\int_t^s \lambda_u\,du}\right] \\
&= -\frac{\partial \bar{P}_t^s}{\partial r}(1+\lambda_r)Q_t^s + (1+\lambda_r)\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du, e^{-\int_t^s \lambda_u\,du}\right].
\end{aligned}
$$

Therefore, the duration of the bond is given by

$$
D_t^s = (1+\lambda_r)\frac{\bar{P}_t^s Q_t^s}{P_t^s}\bar{D}_t^s + \frac{1+\lambda_r}{P_t^s}\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du, e^{-\int_t^s \lambda_u\,du}\right].
$$

Since $\bar{P}_t^s Q_t^s = P_t^s - \mathrm{Cov}_t[e^{-\int_t^s r_u\,du}, e^{-\int_t^s \lambda_u\,du}]$, this can be rewritten in the following way:

$$
\begin{aligned}
D_t^s &= (1+\lambda_r)\bar{D}_t^s \\
&\quad - \frac{1+\lambda_r}{P_t^s}\left(\bar{D}_t^s\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}, e^{-\int_t^s \lambda_u\,du}\right] - \mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du, e^{-\int_t^s \lambda_u\,du}\right]\right).
\end{aligned} \tag{12}
$$

Obviously the correlation between r and λ plays a decisive role whether the covariances are positive or negative. We see that the duration of the corporate bond is decreasing in both

- the covariance of the pathwise default-free bond price and the pathwise survival probability, and

- the covariance of the pathwise derivative of the default-free bond price with respect to the short rate and the pathwise survival probability.

Note that if short rate and intensity are uncorrelated, then both covariances and the derivative λ_r are zero and the durations coincide as was already stated in Proposition 3.1. Furthermore, we obtain the following result.

Proposition 4.1 *Assume that the intensity is an affine function of the short rate r. If the derivative y is deterministic, then $D_t^s = (1+\lambda_r)\bar{D}_t^s$.*

Proof: In this case, $\int_t^s y_u\,du = \int_t^s \mathrm{E}_t^{Q^s}[y_u]\,du = \bar{D}_t^s$. Substituting this relation into (12) gives the desired result. □

Note that the process y is deterministic in the Ho-Lee and the Vasicek models.

Under the assumption of Proposition 4.1 the duration of the defaultable zero-coupon bond is greater than the duration of the default-free bond if $\lambda_r > 0$, i.e. if interest rate risk and default risk are positively correlated, and smaller if the opposite is true. Furthermore, the fact that the covariances are canceling out in this special case gives us a strong hint that usually the covariances go in opposite directions. More precisely, for random variables $X = e^{-\int_t^s r_u\,du}$, $Y = e^{-\int_t^s \lambda_u\,du}$, $Z = -\int_t^s y_u\,du$ one gets

$$
\mathrm{Cov}_t[XZ, Y] = \mathrm{E}_t[Z]\mathrm{Cov}_t[X,Y] + \mathrm{Cov}_t[XY, Z] - \mathrm{E}_t[Y]\mathrm{Cov}_t[X,Z].
$$

Since $\mathrm{E}_t[Z] < 0$, at least the direct effect of $\mathrm{Cov}_t[X,Y]$ on $\mathrm{Cov}_t[XZ,Y]$ is negative. Furthermore,

$$
-\frac{\mathrm{Cov}_t[X,Z]}{\bar{P}_t^s} = \bar{D}_t^s - \int_t^s \mathrm{E}_t[y_u]\,du = \int_t^s \left(\mathrm{E}_t^{Q^s}[y_u] - \mathrm{E}_t[y_u]\right)du,
$$

implying that the sign of $\mathrm{Cov}_t[X,Z]$ depends on the absolute values of the expectations $\mathrm{E}_t^{Q^s}[y_u]$ and $\mathrm{E}_t[y_u]$. Besides, the sign of $\mathrm{Cov}_t[XY,Z]$ is in general also unknown. To summarize, these results indicate that without the independence assumption or without the assumptions of Proposition 4.1 there is no straight answer to the question of whether the duration of a defaultable zero-coupon bond is smaller or greater than the zero-coupon duration of the corresponding default-free bond.

4.2 Coupon Bond with Zero Recovery

Without the independence assumption the measure μ is given by

$$\mu(ds) = q \sum_{t_j > t} P_t^{t_j} \varepsilon_{t_j}(s) + P_t^{t_n} \varepsilon_{t_n}(s)$$

if coupons are paid at discrete points in time, and by

$$\mu(ds) = q P_t^s \, ds + P_t^{t_n} \varepsilon_{t_n}(s)$$

if coupons are paid continuously. The density (4) reads

$$Z := \frac{d\nu}{d\bar{\nu}} = \frac{\bar{\mu}([t,t_n])}{\mu([t,t_n])} \frac{P_t}{\bar{P}_t}. \tag{13}$$

As in the previous sections, we assume that default-free and defaultable zero-coupon bonds are differentiable so that their durations $\bar{D}_t$ and D_t are well-defined. This is for instance valid in suitable Cox process frameworks. The durations of default-free and defaultable coupon bonds are then given by

$$\bar{D}_t^{q,t_n} = \frac{\int_t^{t_n} \bar{D}_t^s \, d\bar{\mu}(s)}{\bar{\mu}([t,t_n])} = \mathrm{E}_{\bar{\nu}}[\bar{D}_t], \qquad D_t^{q,t_n} = \frac{\int_t^{t_n} D_t^s \, d\mu(s)}{\mu([t,t_n])} = \mathrm{E}_{\nu}[D_t].$$

The crucial difference to Section 3 is that in the case of defaultable bonds we take the expectation of D_t instead of $\bar{D}_t$. Under the independence assumption this makes no difference because, by Proposition 3.1, we have $D_t = \bar{D}_t$. In Subsection 4.1 we have seen that if we drop the independence assumption, then D_t may be greater or smaller than $\bar{D}_t$ depending on the correlation between interest rate and default risk. In the latter case, the following proposition holds true.

Proposition 4.2 *Assume that $D_t \leq \bar{D}_t$ and that the duration $\bar{D}_t^s$ of default-free zero-coupon bonds is non-decreasing in maturity s. If the density (13) is decreasing on $[t,t_n]$, then the duration of a corporate coupon bond with zero recovery is smaller than the duration of a corresponding default-free bond.*

Proof: If $D_t \leq \bar{D}_t$, we get $\mathrm{E}_\nu[D_t] \leq \mathrm{E}_\nu[\bar{D}_t]$. Besides, if the density is decreasing, similar arguments as in the Proof of Theorem 3.1 yield $\mathrm{E}_\nu[\bar{D}_t] \leq \mathrm{E}_{\bar{\nu}}[\bar{D}_t]$, which gives the desired result. □

We emphasize that this result holds for any arbitrage-free specification of the default-free interest rates and the default probabilities. Specifically, it is not necessary that the default time τ possesses an intensity. This is in sharp contrast to the result in the following Subsection 4.3.

The strength of the above result is that the whole problem is reduced to checking the relation between the durations of zero-coupon bonds as well as the ratio between their prices. To handle the case $D_t \geq \bar{D}_t$ in the same manner, the density (13) needs to be increasing implying that the duration of a corporate coupon bond with zero recovery is greater than the duration of a default-free bond. However, we are not aware of any situation where the above density is increasing. Since we have provided a sufficient condition only, this does not mean that the duration of corporate coupon bonds cannot be greater than the duration of default-free coupon bonds. An example will be given in a later section.

Let us briefly analyze under which conditions the density is decreasing on $[t, t_n]$. Since

$$P_t^s = \mathrm{E}_t\left[e^{-\int_t^s r_u\,du}\mathbf{1}_{\{\tau>s\}}\right] = \mathrm{E}_t\left[\mathbf{1}_{\{\tau>s\}}\right]\bar{P}_t^s + \mathrm{Cov}_t[\mathbf{1}_{\{\tau>s\}}, e^{-\int_t^s r_u\,du}],$$

we get

$$\frac{P_t^s}{\bar{P}_t^s} = Q_t^s + \frac{\mathrm{Cov}_t[\mathbf{1}_{\{\tau>s\}}, e^{-\int_t^s r_u\,du}]}{\bar{P}_t^s}.$$

Therefore, the covariance between default risk and interest rate risk plays a decisive role. For instance, if $\mathrm{Cov}_t[\mathbf{1}_{\{\tau>s\}}, e^{-\int_t^s r_u\,du}]$ is negative and non-increasing in s, then the density is decreasing, but again this is only a sufficient condition. Even if the covariance increases, the density (13) is decreasing as long as the survival probability Q_t^s decreases faster than $\mathrm{Cov}_t[\mathbf{1}_{\{\tau>s\}}, e^{-\int_t^s r_u\,du}]/\bar{P}_t^s$ increases.

4.3 Coupon Bond with Recovery of Market Value

In the RMV framework prices of corporate zero-coupon bonds are given by

$$P_t^{t_n,l} = \mathrm{E}_t\left[e^{-\int_t^{t_n}(r_s+l_s\lambda_s)\,ds}\right],$$

where l is the loss rate and λ is the default intensity. The measure μ^l is now given by

$$\mu^l(ds) = q\sum_{t_j>t} P_t^{t_j,l}\varepsilon_{t_j}(s) + P_t^{t_n,l}\varepsilon_{t_n}(s)$$

if coupons are paid at discrete points in time, and by

$$\mu^l(ds) = qP_t^{s,l}ds + P_t^{t_n,l}\varepsilon_{t_n}(s)$$

if coupons are paid continuously. The density (13) reads

$$Z := \frac{d\nu}{d\bar{\nu}} = \frac{\bar{\mu}([t,t_n])}{\mu^l([t,t_n])}\frac{P_t^{\cdot,l}}{\bar{P}_t}. \tag{14}$$

The following result follows analogously to Proposition 4.2.

Proposition 4.3 *Assume that the duration of a defaultable zero-coupon bond is smaller than or equal to the duration of a default-free zero-coupon bond, $D_t^s \le \bar{D}_t^s$ for any $s \in [t, t_n]$, and that the duration $\bar{D}_t^s$ of default-free zero-coupon bonds is non-decreasing in maturity s. If the density (14) is decreasing on $[t, t_n]$, then under RMV the duration of a corporate coupon bond is smaller than the duration of a corresponding default-free bond.*

The crucial ratio in (14) can be rewritten as follows

$$\frac{P_t^{s,l}}{\bar{P}_t^s} = \mathrm{E}_t\left[e^{-\int_t^s l_u\lambda_u\,du}\right] + \frac{\mathrm{Cov}_t[e^{-\int_t^s l_u\lambda_u\,du}, e^{-\int_t^s r_u\,du}]}{\bar{P}_t^s}.$$

Although we are very well aware of the drawbacks of Gaussian models for the intensity, assume for the moment (as several authors do) that the integrals $X := -\int_t^s l_u\lambda_u\,du$ and $Y := -\int_t^s r_u\,du$ are jointly normally distributed with mean m_X and m_Y, standard deviation s_X and s_Y, as well as covariance s_{XY}. This is for example satisfied if l is constant and r as well as λ follow correlated

Ornstein-Uhlenbeck processes (a Vasicek-type model). Let $s_{X+Y}^2 = s_X^2 + s_Y^2 + 2s_{XY}$ be the variance of $X + Y$. Then

$$\begin{aligned}\mathrm{Cov}_t[e^X, e^Y] &= \mathrm{E}_t[e^{X+Y}] - \mathrm{E}_t[e^X]\mathrm{E}_t[e^Y] \\ &= e^{m_X + m_Y + 0.5 s_{X+Y}^2} - e^{m_X + 0.5 s_X^2 + m_Y + 0.5 s_Y^2} \\ &= e^{m_X + 0.5 s_X^2 + m_Y + 0.5 s_Y^2}(e^{s_{XY}} - 1),\end{aligned}$$

where

$$\begin{aligned}s_{XY} &= \mathrm{E}_t\left[\int_t^s r_u\,du \cdot \int_t^s l_u\lambda_u\,du\right] - \mathrm{E}_t\left[\int_t^s r_u\,du\right] \cdot \mathrm{E}_t\left[\int_t^s l_u\lambda_u\,du\right] \\ &= \int_t^s \int_t^s (\mathrm{E}_t[r_u l_v \lambda_v] - \mathrm{E}_t[r_u]\mathrm{E}_t[l_v\lambda_v])\,du\,dv = \int_t^s \int_t^s \mathrm{Cov}_t[r_u, l_v\lambda_v]\,du\,dv.\end{aligned}$$

In this particular situation, it is obvious that the the covariance between the short rate r_u and the adjusted default intensity $l_v\lambda_v$ plays a crucial role. More precisely, the density is decreasing if this covariance sufficiently negative. The condition "sufficiently negative" is needed because in a Gaussian model $\bar{P}_t^s$ and $Q_t^{s,l} = \mathrm{E}_t[e^{-\int_t^s l_u\lambda_u\,du}]$ are not always decreasing in s. In the following section, we will look at a specific model for the valuation of corporate bonds allowing for more precise statements about duration.

5 Example

Assume that the default-free short rate r_t follows the Ornstein-Uhlenbeck process

$$dr_t = (\theta - \kappa r_t)\;dt + \sigma_r\,dW_t \tag{15}$$

as in the famous Vasicek model so that the price of a default-free zero-coupon bond is

$$\bar{P}_t^T = e^{-\bar{A}(T-t) - \bar{B}(T-t)r_t}, \tag{16}$$

where

$$\begin{aligned}\bar{B}(\tau) &= \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right), \\ \bar{A}(\tau) &= \frac{1}{\kappa}\left(\theta - \frac{\sigma_r^2}{2\kappa}\right)\left(\tau - \bar{B}(\tau)\right) + \frac{\sigma_r^2}{4\kappa}\bar{B}(\tau)^2.\end{aligned}$$

The duration of this bond is $\bar{D}_t^T = \bar{B}(T-t)$, which is increasing in time to maturity.

Assume that recovery of market value applies and the default risk adjusted short rate is affine in the default-free short rate,

$$R_t \equiv r_t + \lambda_t l_t = k_0 + k_1 r_t. \tag{17}$$

Then it can be shown that the corporate zero-coupon bond price $P_t^T = \mathrm{E}_t[\exp\{-\int_t^T R_u\,du\}]$ becomes

$$P_t^T = e^{f(T-t)}\left(\bar{P}_t^T\right)^{k_1}, \tag{18}$$

where

$$f(\tau) = -k_0\tau + \frac{\sigma_r^2 k_1(k_1 - 1)}{2\kappa^2}\left(\tau - \bar{B}(\tau) - \frac{\kappa}{2}\bar{B}(\tau)^2\right).$$

The duration of the corporate bond is

$$D_t^T = k_1\bar{D}_t^T = k_1\bar{B}(T-t), \tag{19}$$

which is smaller than the duration of the default-free bond if $k_1 < 1$.

Equation (17) is satisfied in two cases:

(i) Constant loss rate, $l_t = L \geq 0$, and affine default intensity, $\lambda_t = \Lambda_0 + \Lambda_1 r_t$. Then $k_0 = \Lambda_0 L$, $k_1 = 1 + \Lambda_1 L$. Hence, the duration of the corporate zero-coupon bond will be smaller [larger] than the duration of the default-free zero-coupon bond of the same maturity if $\Lambda_1 < 0$ [if $\Lambda_1 > 0$]. The duration of the corporate bond will be negative if $\Lambda_1 < -1/L$, which is theoretically possible. Also note that the sign of the duration of the corporate bond does not depend on the *level* of the default probability (as the discussion in Longstaff and Schwartz (1995) suggests) but on the *interest rate sensitivity* of the default probability.

(ii) Constant default intensity, $\lambda_t = \Lambda > 0$, and affine loss rate, $l_t = L_0 + L_1 r_t$. Then $k_0 = \Lambda L_0$, $k_1 = 1 + \Lambda L_1$, and the duration of the corporate bond will be smaller [larger] than the duration of the default-free bond if $L_1 < 0$ [if $L_1 > 0$].

The empirical evidence on the key parameters is mixed and leads to different conclusions on the durations. Bakshi, Madan, and Zhang (2006) estimate a model empirically with $R_t = k_0 + k_1 r_t$ and report that k_1 is 1.018 for BBB-rated bonds (higher duration of the corporate bond) and 0.985 for A-rated bonds (lower duration of the corporate bond). They add a firm-specific distress variable S_t so that $R_t = k_0 + k_1 r_t + k_2 S_t$. For all the five concrete distress variables they consider, they find k_1 to be below one, namely in the range $[0.767, 0.910]$ for BBB bonds and in $[0.902, 0.966]$ for A bonds, leading to the corporate bond having a smaller duration than the default-free bond.

Jarrow and Yildirim (2002) assume a generalized Vasicek model, a constant loss rate, and a default intensity affine in the default-free short rate. They estimate the parameters using corporate default swap quotes for 22 individual companies. Their company-specific estimates of Λ_1 range from 1.3 to 26.9 basis points. This would imply that the duration of a zero-coupon bond issued by any of these companies would be greater than the duration of a default-free zero-coupon bond.

The yields of the default-free and the corporate zero-coupon bonds are

$$\bar{y}_t^{t+\tau} = \frac{\bar{A}(\tau)}{\tau} + \frac{\bar{B}(\tau)}{\tau} r_t, \quad y_t^{t+\tau} = k_1 \bar{y}_t^{t+\tau} - k_1 \frac{f(\tau)}{\tau},$$

respectively, so that the yield spread becomes

$$y_t^{t+\tau} - \bar{y}_t^{t+\tau} = (k_1 - 1)\bar{y}_t^{t+\tau} - k_1 \frac{f(\tau)}{\tau} = (k_1 - 1)\frac{\bar{A}(\tau)}{\tau} - k_1 \frac{f(\tau)}{\tau} + (k_1 - 1)\frac{\bar{B}(\tau)}{\tau} r_t.$$

As mentioned in the introduction several empirical studies, e.g. Longstaff and Schwartz (1995) and Duffee (1998), conclude that yield spreads are generally decreasing in default-free yields. Within the current setting, this will be the case if $k_1 < 1$, which is exactly when the duration of the corporate bond is smaller than the duration of the default-free bond.

Next, we investigate the duration of coupon bonds in this framework. The price of a default-free bond with a periodic coupon of q is

$$\bar{P}_t^{q,t_n} = q \sum_{t_j > t} \bar{P}_t^{t_j} + \bar{P}_t^{t_n},$$

from which it follows that the duration of the coupon bond is

$$\bar{D}_t^{q,t_n} = \sum_{t_j > t} \frac{q \bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} \bar{D}_t^{t_j} + \frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}} \bar{D}_t^{t_n} = \sum_{t_j > t} \bar{a}_t^{t_j} \bar{D}(t_j - t) = \sum_{t_j > t} \bar{a}_t^{t_j} \bar{B}(t_j - t),$$

where

$$\bar{a}_t^{t_j} = \begin{cases} q \frac{\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} & \text{for } t_j < t_n, \\ (1+q) \frac{\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} & \text{for } t_j = t_n. \end{cases}$$

Note that $\sum_{t_j>t} \bar{a}_t^{t_j} = 1$.

For a similar corporate coupon bond, recovery of market value implies a price of

$$P_t^{q,t_n} = q \sum_{t_j>t} P_t^{t_j} + P_t^{t_n},$$

and the duration becomes

$$D_t^{q,t_n} = \sum_{t_j>t} a_t^{t_j} D_t^{t_j} = k_1 \sum_{t_j>t} a_t^{t_j} \bar{B}(t_j - t),$$

where

$$a_t^{t_j} = \begin{cases} q \frac{P_t^{t_j}}{P_t^{q,t_n}} & \text{for } t_j < t_n, \\ (1+q) \frac{P_t^{t_j}}{P_t^{q,t_n}} & \text{for } t_j = t_n \end{cases}$$

so that $\sum_{t_j>t} a_t^{t_j} = 1$.

For concreteness assume that version (i) of the model applies, i.e. that the loss rate is given by a constant L and the default intensity is $\lambda_t = \Lambda_0 + \Lambda_1 r_t$. We assume that the current default-free short rate is $r_0 = 4\%$ and that the parameters of the short rate process are $\kappa = 0.15$, $\sigma_r = 0.01$, and $\theta = 0.007833$ resulting in an asymptotic zero-coupon yield of 5% and an upward-sloping yield curve. We consider 10-year bullet bonds with semi-annual coupons of 3%. For the corporate bond the loss in case of default is assumed to be $L = 40\%$. The fixed part of the default intensity is $\Lambda_0 = 0.025$ so that $k_0 = 0.01$. We vary Λ_1 from -0.5 to 0.5 corresponding to variations in default intensities (at the current short rate) from 0.005 to 0.045 and variations in k_1 from 0.8 to 1.2. Figure 1 shows yield spread curves for different values of Λ_1.

[Figure 1 about here.]

With the assumed parameters the duration of the 10-year Treasury coupon bond is 4.3099. Figure 2 shows how the duration of the 10-year corporate coupon bond varies with Λ_1. For $\Lambda_1 < 0$, i.e. $k_1 < 1$, the duration of a corporate zero-coupon bond is smaller than the duration of the similar default-free zero-coupon bond for any maturity, cf. (19), and we see from the figure that the duration of the corporate coupon bond is then also smaller than the duration of the Treasury coupon bond. This is consistent with Proposition 4.3. For $\Lambda_1 = 0$, the durations of a corporate and a default-free zero-coupon bond of the same maturity are identical. Nevertheless, the duration of the 10-year corporate coupon bond (4.2663) is smaller than the duration of the similar default-free bond. In fact, this is also true for slightly positive values of Λ_1 (up to approximately 0.027), which shows that even if the durations of corporate zero-coupon bonds are higher than the durations of default-free zero-coupon bonds, the duration of a corporate coupon bond may be smaller than the duration of a default-free coupon bond. For values of Λ_1 higher than 0.027, the durations of both corporate zero-coupon and corporate coupon bonds are higher than those of their default-free counterparts.

[Figure 2 about here.]

6 Conclusion

The literature on the duration of corporate bonds is very sparse despite the obvious applications of duration in interest rate risk management. The few existing papers conclude that the duration of a corporate bond is smaller than the duration of a similar default-free bond. In this paper we have offered a much more formalized comparison of the durations of defaultable and default-free bonds and we have arrived at a more balanced conclusion. We have provided general conditions under which the duration of a corporate bond is smaller than the duration of the corresponding default-free bond. In a concrete setting allowing for dependence between default-free interest rates on the one hand and default intensities and/or recovery rates on the other hand, we have shown that the conditions are not necessary for the duration of the corporate bond to be smaller, but we have also demonstrated that the duration of the corporate bond can very well be greater than that of the default-free bond.

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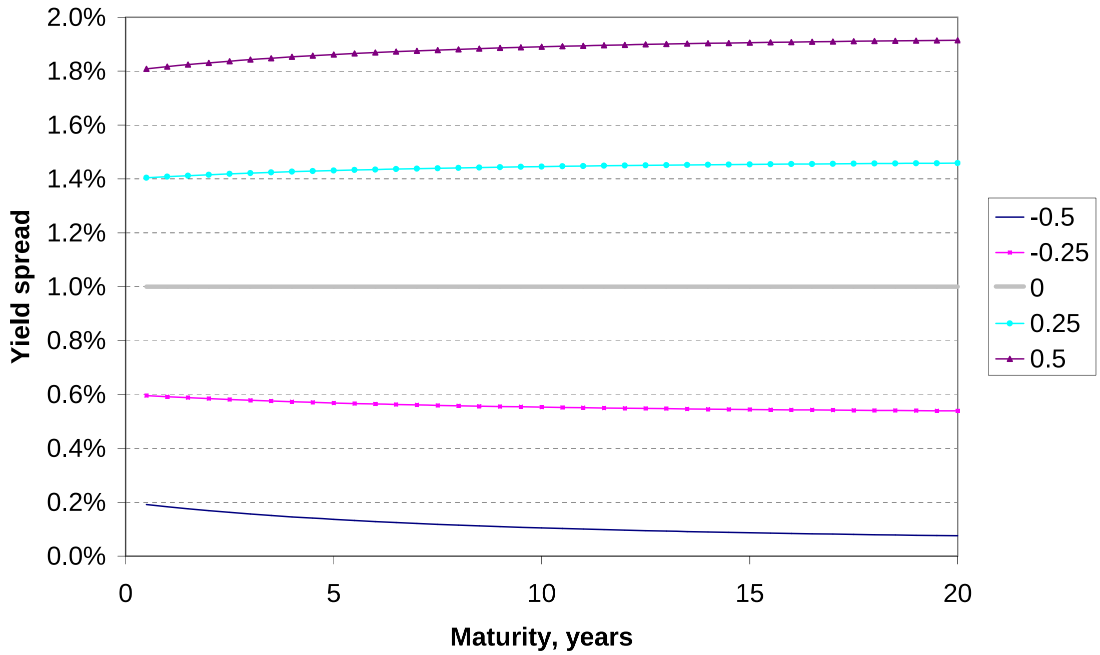

Figure 1: **Zero-coupon yield spreads.** The figure shows the difference between the yield of a corporate zero-coupon bond and a default-free zero-coupon bond as a function of time to maturity. The different curves are for different values of the parameter Λ_1 ranging from -0.5 (lower curve) to 0.5 (upper curve). The current default-free short rate is 4%. The parameters of the short rate process are $\kappa = 0.15$, $\sigma_r = 0.01$, and $\theta = 0.007833$ so that the asymptotic zero-coupon yield is 5%. The loss in case of default is $L = 0.4$ and the fixed part of the default intensity is $\Lambda_0 = 0.025$.

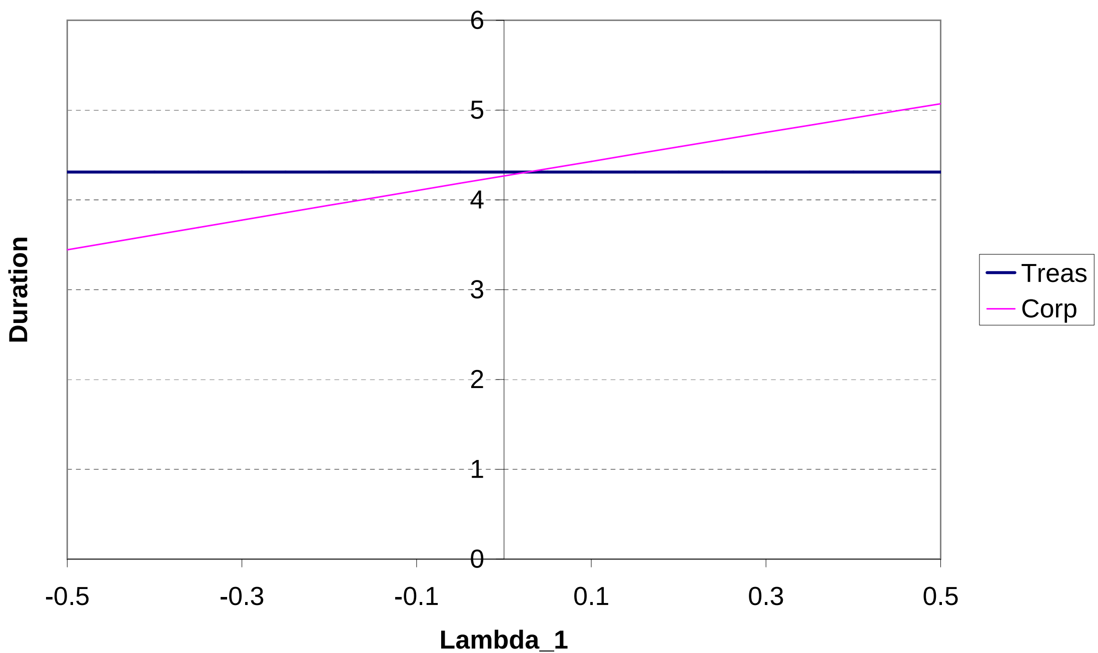

Figure 2: **The durations of 10-year bonds.** The figure shows the duration of a 10-year corporate coupon bond as a function of the interest rate sensitivity of the default intensity rate. The horizontal line shows the duration of the similar Treasury bond. The coupon is 6% per year with semi-annual payments. The current default-free short rate is 4%. The parameters of the short rate process are $\kappa = 0.15$, $\sigma_r = 0.01$, and $\theta = 0.007833$ so that the asymptotic zero-coupon yield is 5%. The loss in case of default is $L = 0.4$ and the fixed part of the default intensity is $\Lambda_0 = 0.025$.