The Modiclus and Core Stability

by

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The Modiclus and Core Stability

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Abstract

The modiclus, a relative of the prenucleolus, assigns a singleton to any cooperative TU game. We show that the modiclus selects a member of the core for any exact orthogonal game and for any assignment game that has a stable core. Moreover, by means of an example we show that there is an exact TU game with a stable core that does not contain the modiclus.

1 Introduction

The prenucleolus and the core are widely accepted solutions for cooperative transferable utility games. The prenucleolus selects a unique member of the core, whenever the core is nonempty. A further interesting solution, the modiclus, is a relative of the prenucleolus. The prenucleolus of a game is obtained by lexicographically minimizing the nonincreasingly ordered vector of excesses of the coalitions within the set of Pareto optimal

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payoff vectors. Analogously, the modiclus is obtained by lexicographically minimizing the non-increasingly ordered vector of differences of excesses. When comparing the definitions of the prenucleolus and the nucleolus, the excesses, i.e., the “dissatisfactions”, of the coalitions are replaced by the bi-excesses (differences of excesses) of the pairs of coalitions. The bi-excess between two coalitions S and T may be regarded as the envy of S against T. For the precise definition see Section 2.

The modiclus has many properties in common with the prenucleolus. For its nice behavior on the class of weighted majority games see Sudhölter (1996). Different from the prenucleolus the modiclus may not select a core element, even if the core is nonempty. E.g., the modiclus does not select a core member in the case of any asymmetric glove game. Instead it assigns the same amount to both the left-hand glove owners and the right-hand glove owners (see Sudhölter (2001)). Due to this kind of “equal treatment property” of groups of players the modiclus has an advantage over any selection of the core like the prenucleolus.

Glove games are assignment games as well as orthogonal games, which also allow for a canonical partition of the players into groups (see Section 4). In the present paper we shall show that the modiclus, when restricted to orthogonal games or assignment games, is a selection of the core, if the core is a (the unique) von Neumann-Morgenstern set.

We now briefly review the contents of the paper. Section 2 recalls definitions of some relevant solutions and of stability. In Section 3 we show that the modified least core (containing the modiclus) is a subset of the core of any assignment game whose core is stable. Section 4 is devoted to the discussion of orthogonal games. It is shown that any orthogonal game with a stable core is exact. Moreover, we deduce that the modified least core is contained in the core of any exact orthogonal game. Finally, Section 5 presents an example of an exact TU game that has a stable core - in fact, it has a large core - and that does **not** contain the modiclus.

2 Notation and Definitions

A (cooperative TU) *game* is a pair (N, v) such that $\emptyset \neq N$ is finite and $v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$. For any game (N, v) let $X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ denote the set of *Pareto optimal allocations* (*preimputations*). We use $x(S) = \sum_{i \in S} x_i$ $(x(\emptyset) = 0)$ for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$. For disjoint coalitions $S, T \in 2^N$ let $(x_S, x_T) = x_{S \cup T}$. For $x \in \mathbb{R}^N$ and $S \subseteq N$ let $e(S, x, v) = v(S) - x(S)$ denote the *excess* of S at x with respect to (N, v). For $X \subseteq \mathbb{R}^N$ let $\mathcal{N}((N, v); X)$ denote the *nucleolus* of (N, v) with respect to X, i.e. the set of members of X that lexicographically minimize the non-increasingly ordered vector of excesses of the coalitions (see Schmeidler (1969)). It is well known that the nucleolus with respect to $X(N,v)$ is a singleton, the unique element of which is called the *prenucleolus* of (N,v) and is denoted by $\nu(N,v)$. In order to define the modiclus of (N,v), let, for every pair $(S,T) \in 2^N \times 2^N$, the *bi-excesses* of (S,T) at x, $e^b(S,T,x,v)$, be given by

$$e^b(S,T,x,v) = e(S,x,v) - e(T,x,v).$$

The *modiclus* of (N,v) is the set of members of $X(N,v)$ that lexicographically minimize the non-increasingly ordered vector of bi-excesses of the pairs of coalitions. It is well known that the modiclus is a singleton, the unique element of which is called the *modiclus* of (N,v) and is denoted by $\psi(N,v)$ (see Sudhölter (1996)).

The *preventive power* of a coalition $S \subseteq N$ may be measured by $v^*(S) = v(N) - v(N \setminus S)$. The game (N,v^*) is the *dual game* of (N,v). For any $x \in X(N,v)$ and any pair (S,T) of coalitions,

$$e(N \setminus T, x, v^*) = v(N) - v(T) - x(N) + x(T) = -e(T,x,v).$$

Hence $e^b(S,T,x,v) = e(S,x,v) + e(N \setminus T, x, v^*)$. Thus, the modiclus lexicographically minimizes the vector of sums of excesses with respect to the game and its dual. We conclude that $\psi(N,v) = \psi(N,v^*)$. The following notation is useful. For $x \in \mathbb{R}^N$ denote $\mu(x,v) = \max\{e(S,x,v) \mid S \subseteq N\}$. The *modified least core* of (N,v), $\mathcal{MLC}(N,v)$, is defined by

$$\mathcal{MLC}(N,v) = \{x \in X(N,v) \mid \mu(x,v) + \mu(x,v^*) \leq \mu(y,v) + \mu(y,v^*) \text{ for all } y \in X(N,v)\}. \tag{2.1}$$

Hence, $\psi(N,v) \in \mathcal{MLC}(N,v)$ by definition.

The modiclus has some desirable properties (see Sudhölter (1996, 1997)). E.g., it selects a member of the core if the game is convex. Also, ψ satisfies the *strong null player property*, that is, if (N',v') and (N,v) are games such that $N \subseteq N'$ and $v'(S) = v(S \cap N)$ (the players of $N' \setminus N$ are null players), then $\psi(N',v')_N = \psi(N,v)$ and $\psi(N',v')_{N' \setminus N} = 0 \in \mathbb{R}^{N' \setminus N}$.

The *core* of (N,v), $\mathcal{C}(N,v)$, is defined by

$$\mathcal{C}(N,v) = \{x \in X(N,v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

A game is *balanced* if its core is nonempty (see Bondareva (1963) and Shapley (1967)). A game (N,v) is *totally balanced* if all subgames (S,v), $\emptyset \neq S \subseteq N$, are balanced.

A game (N,v) is *exact* if for any $S \subseteq N$ there exists $x \in \mathcal{C}(N,v)$ such that $v(S) = x(S)$.

Finally, we recall the definition of core stability. Let $x, y \in X(N,v)$ and $\emptyset \neq S \subseteq N$. Then x *dominates* y *via* S if $x(S) \leq v(S)$ and $x_i > y_i$ for all $i \in S$. Moreover, x *dominates* y if x dominates v via some coalition $\emptyset \neq S \subseteq N$. The core of (N,v) is *stable* if for any $y \in X(N,v)$ such that $y_i \geq v(\{i\})$ (y is an *imputation*) for all $i \in N$ there exists $x \in \mathcal{C}(N,v)$ that dominates y. Note that core stability is invariant under adding null players.

3 The Modiclus for Assignment Games with a Stable Core

Shapley and Shubik (1972) introduced assignment games. For finite sets S and T an *assignment* of (S,T) is a bijection $b : S' \to T'$ such that $S' \subseteq S$, $T' \subseteq T$, and $|S'| = |T'| = \min\{|S|, |T|\}$. We shall identify b with $\{(i, b(i)) \mid i \in S'\}$. Let $\mathcal{B}(S,T)$ denote the set of assignments. A game (N,v) is an *assignment game* if there exist a partition $\{P,Q\}$ of N and a nonnegative real matrix $A = (a_{ij})_{i\in P, j\in Q}$ such that

$$v(S) = \max_{b \in \mathcal{B}(S\cap P, S\cap Q)} \sum_{(i,j)\in b} a_{ij}.$$

Let (N,v) be an assignment game defined by the matrix A on $P \times Q$. As ψ satisfies the strong null player property and as core stability is invariant under adding null players, we shall assume in the sequel that $|P| = |Q| = p$. Also, we shall assume that $P = \{1, \ldots, p\}$ and $Q = \{1', \ldots, p'\}$. Finally, we shall assume that $v(N) = \sum_{i\in P} a_{ii'}$.

We say that A has a *dominant diagonal* if

$$a_{ii'} = \max_{j'\in Q} a_{ij'} = \max_{j\in P} a_{ji'} \text{ for all } i \in P.$$

According to Theorem 1 of Solymosi and Raghavan (2001) the assignment game (N,v) has a stable core if and only if A has a dominant diagonal.

Theorem 3.1 *If (N,v) is an assignment game with a stable core, then $\mathcal{MLC}(N,v) \subseteq \mathcal{C}(N,v)$.*

Proof: Let (N,v) be an assignment game defined by the $P \times Q$ matrix A and let $x \in X(N,v)$. Then $\mu(x,v) \geq 0$. Moreover,

$$e(P,x,v^*)+e(Q,x,v^*) = v(N)-v(Q)-x(P)+v(N)-v(P)-x(Q) = 2v(N)-x(N) = v(N),$$

hence $\mu(x,v^*) \geq v(N)/2$. Now, let A have a dominant diagonal. The proof is complete as soon as we have shown the following claim:

$$\mathcal{MLC}(N,v) = \{x \in \mathcal{C}(N,v) \mid \mu(x,v^*) = v(N)/2\}.$$

In order to prove our claim it suffices to find a single preimputation $\widetilde{x} \in \mathcal{C}(N,v)$ that satisfies $\mu(\widetilde{x}, v^*) = v(N)/2$. Let $\widetilde{x} \in \mathbb{R}^N$ be defined by $\widetilde{x}_i = a_{ii'}/2 = x_{i'}$ for all $i \in N$. Then $\widetilde{x} \in X(N,v)$. Let $S \subseteq N$. Then

$$v(S) \leq \min\left\{\sum_{i \in S \cap P} a_{ii'}, \sum_{i' \in S \cap Q} a_{ii'}\right\},$$

because A has a dominant diagonal. Hence $v(S) \leq \widetilde{x}(S)$ and $\widetilde{x} \in \mathcal{C}(N,v)$. In order to show that $e(S, \widetilde{x}, v^*) \geq -v(N)/2$ we do not need the property that A has a dominant diagonal. Let $T = \{i \in S \cap P \mid i' \in S\} \cup \{i' \in S \cap Q \mid i \in S\}$. Then

$$v(S) \geq v(S \setminus T) + v(T) \geq v(S \setminus T) + \sum_{i \in T \cap P} a_{ii'}$$

and hence,

$$e(S, \widetilde{x}, v) \geq e(S \setminus T, \widetilde{x}, v) \geq -\widetilde{x}(S \setminus T) \geq -\sum_{i \in P} \frac{a_{ii'}}{2} = -\frac{v(N)}{2}.$$

q.e.d.

Theorem 3.1 shows that the modiclus of an assignment game is a member of the core provided the game has a stable core.

Note that core stability was not used to prove that $\mu(\widetilde{x}, v^*) = v(N)/2$. In fact Sudhölter (2001) shows that $\mu(x, v^*) = v(N)/2$ for any member of the modified least core of an arbitrary assignment game.

4 The Modiclus for Exact Orthogonal Games

Let (N,v) be a game. Kalai and Zemel (1982) showed that (N,v) is totally balanced if and only if it is a minimum of finitely many additive games, that is, there exist a finite sequence $(\lambda^\rho)_{\rho=1,\ldots,r}$ such that $\lambda^\rho \in \mathbb{R}^N$, $\rho = 1,\ldots,r$, and $v(S) = \min_{\rho=1,\ldots,r} \lambda^\rho(S)$ for all $S \subseteq N$. In view of the fact that all of our solutions are covariant under strategic equivalence, we may assume that $\min_{\rho=1,\ldots,r} \lambda^\rho_i = 0$ for all $i \in N$, that is, (N,v) is 0-normalized. A normalized totally balanced game (N,v) is *orthogonal* if there is a partition $\{N^\rho \mid \rho = 1,\ldots,r\}$ of N and $\lambda \in \mathbb{R}^N_+$ such that $v(S) = \min_{\rho=1,\ldots,r} \lambda(S \cap N^\rho)$. In this case we shall assume that $\lambda(N^1) \leq \cdots \leq \lambda(N^r)$. Also, we shall assume without loss of generality that $\lambda_i \leq \lambda(N^1) = v(N)$. The pair $(\{N_\rho \mid \rho = 1,\ldots r\}, \lambda)$ is called a representation of (N,v). A representation of an orthogonal game is "almost" unique. Indeed, λ is uniquely determined. Moreover, if (N,v) is not the flat game (that is $v(S) = 0$ for all $S \subseteq N$), then the partition is uniquely determined except that a null player may be a member of any element of the partition.

Let (N, v) be an orthogonal game and let $(\{N^\rho \mid \rho = 1, \ldots, r\}, \lambda)$ be a representation of (N, v).

Remark 4.1 *The orthogonal game (N, v) is exact if and only if $\lambda(N^\rho) = v(N)$ for every $\rho = 1, \ldots, r$. Indeed, if $\lambda(N^\rho) = v(N)$ for all $\rho = 1, \ldots, r$, then (N, v) is exact. In order to show the opposite direction let (N, v) be exact and let $\rho \in \{1, \ldots, r\}$. Then there exists $x \in \mathcal{C}(N, v)$ such that $x(N \setminus N^\rho) = v(N \setminus N^\rho) = 0$. Hence $x_i = 0$ for all $i \in N \setminus N^\rho$ and $x_j = \lambda_j$ for all $j \in N^\rho$.*

Lemma 4.2 *If an orthogonal game has a stable core, then it is exact.*

Proof: Let (N, v) be an orthogonal game represented by $(\{N^\rho \mid \rho = 1, \ldots, r\}, \lambda)$. If (N, v) is not exact, then $\lambda(N^r) > v(N)$. Let $\alpha = \frac{v(N)}{\lambda(N^r)}$ and let $y \in \mathbb{R}^N$ be defined by $y_{N \setminus N^r} = 0 \in \mathbb{R}^{N \setminus N^r}$ and $y_i = \alpha \lambda_i$ for all $i \in N^r$. Then $y(N) = v(N)$ and $y_j \geq 0$ for all $j \in N$. Hence y is an imputation. Assume, on the contrary, that y is dominated by some $x \in \mathcal{C}(N, v)$ by some nonempty coalition S. Then $\lambda(S \cap N^\rho) > 0$ for every $\rho = 1, \ldots, r$, because otherwise $v(S) = 0$. Let $S^r = S \cap N^r$. Then $x(S^r) > y(S^r) = \alpha\lambda(S^r)$. Thus, $v(S) > \alpha\lambda(S^r)$. Two cases may be distinguished. If $v(N \setminus S^r) = v(N)$, then

$$v(N) = x(N) = x(N \setminus S^r) + x(S^r) > v(N) + \alpha\lambda(S^r) > v(N),$$

which is impossible. If $v(N \setminus S^r) < v(N)$, then $v(N \setminus S^r) = \lambda(N^r \setminus S^r)$. Thus,

$$v(N) = x(N) = x(N \setminus S^r) + x(S^r) > \lambda(N^r \setminus S^r) + \alpha\lambda(S^r) \geq \alpha\lambda(N^r) = v(N),$$

which is also impossible. **q.e.d.**

The following example presents an orthogonal exact game that does not have a stable core.

Example 4.3 *Let $N = \{1, \ldots, 5\}$, let $\lambda = (2, 1, 1, 1, 1)$, let $N^1 = \{1, 2\}$, let $N^2 = \{3, 4, 5\}$ and let (N, v) be the orthogonal game represented by $(\{N^1, N^2\}, \lambda)$. Then (N, v) is exact. Moreover, $\mathcal{C}(N, v) = convh\{(2, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$, where "convh" denotes "convex hull". Let $y = (1, 1, 0, 1/2, 1/2)$. Then y is an imputation. Also, the $e(S, y, v) > 0$ just for $S = \{1, 3, 4\}$ and for $S = \{1, 3, 5\}$. Therefore, y is not dominated by any member of the core.*

It should be noted that Example 4.3 may be generalized (see Biswas, Parthasarathy, and Potters (1999), p. 6).

Theorem 4.4 *If (N, v) is an exact orthogonal game, then $\mathcal{MLC}(N, v) \subseteq \mathcal{C}(N, v)$.*

Proof: Let (N,v) be represented by $(\{N^\rho \mid \rho=1,\ldots,r\},\lambda)$ and let $x\in X(N,v)$. Then $\mu(x,v)\geq 0$. Moreover,

$$\sum_{\rho=1}^{r} e(N^k,x,v^*) = rv(N)-v(N) = (r-1)v(N) =: r\mu^*.$$

Hence $\mu(x,v^*)\geq \mu^*$. We claim that

$$\mathcal{MLC}(N,v)=\{x\in\mathcal{C}(N,v)\mid \mu(x,v^*)=\mu^*\}.$$

In order to prove our claim it suffices to find one $\widehat{x}\in\mathcal{C}(N,v)$ such that $\mu(\widehat{x},v^*)=\mu^*$. Let $\widehat{x}=\frac{1}{r}\lambda$. Then $\widehat{x}\in\mathcal{C}(N,v)$. Let $S\subseteq N$. It remains to show that $e(S,\widehat{x},v)\geq -\mu^*$. Let $\widehat{\rho}\in\{1,\ldots,r\}$ be such that $v(S)=\lambda(S\cap N^{\widehat{\rho}})$. Let $T=(N\setminus N^{\widehat{\rho}})\cup(S\cap N^{\widehat{\rho}})$. Then $e(S,\widehat{x},v)\geq e(T,\widehat{x},v)$, because $v(T)=v(S)$ and $S\subseteq T$. However,

$$e(T,\widehat{x},v)=\lambda(S\cap N^{\widehat{\rho}})-\widehat{x}(T)=\lambda(S\cap N^{\widehat{\rho}})-\frac{r-1}{r}v(N)-\frac{1}{r}\lambda(S\cap N^{\widehat{\rho}})\geq -\frac{r-1}{r}v(N)=-\mu^*.$$

q.e.d.

By Theorem 4.4 and Lemma 4.2, the modiclus is a member of the core of any orthogonal game that is exact or that has a stable core. Section 7 of Rosenmüller and Sudhölter (2003) gives some conditions on the representation which guarantee that the modiclus of an exact orthogonal game coincides with the barycenter $\widehat{x}$ of the involved extreme points of the core, as defined in the proof of Theorem 4.4. Also, it is shown that $\psi(N,v)=\nu(N,v)=\widehat{x}$ if $\nu(N^\rho)=v(N)/r$ for all $\rho=1,\ldots,r$.

Remark 4.5 *Let (N,v) be an exact orthogonal game represented by $(\{N^1,N^2\},\lambda)$ and let $\widehat{x}\in\mathbb{R}^N$ be defined as in the proof of Theorem 4.4, that is, $\widehat{x}=\lambda/2$. Let $S\subseteq N$. Then $min_{\rho=1,2}\lambda(S\cap N^\rho)+\max_{\rho=1,2}\lambda(S\cap N^\rho)=\lambda(S)$. Also,*

$$e(N\setminus S,\widehat{x},v)=\min_{\rho=1,2}(\lambda(N^\rho)-\lambda(S\cap N^\rho))-\widehat{x}(N)+\widehat{x}(S)=\widehat{x}(S)-\max_{\rho=1,2}\lambda(S\cap N^\rho).$$

Hence $e(S,\widehat{x},v)-e(N\setminus S,\widehat{x},v)=2\widehat{x}(S)-\lambda(S)=0$. Hence, the excess of any coalition coincides with the excess of the complement coalition. It is straightforward to deduce that this fact implies that $\nu(N,v)=\psi(N,v)=\widehat{x}$.

Remark 4.6 *Solymosi and Raghavan (2001) show that every exact assignment game has a stable core. Hence, for every assignment game and every orthogonal game the modiclus selects a member of the core provided the game is exact or it has a stable core. In Section 5 it is shown that these results cannot be generalized to arbitrary exact games with a stable core.*

5 The Modified Least Core of an Exact 16-Person Game

By means of an example we show that there is an exact TU game with a stable core that does not contain the modiclus.

Let $N = \{1, \ldots, 16\}$ and let

$$N^1 = \{1,2,3\}, \quad N^2 = \{4,5,6\}, \quad N^3 = \{7,\ldots,16\},$$
$$S^1 = \{1,2,4\}, \quad S^2 = \{1,3,5\}, \quad S^3 = \{2,3,6\},$$
$$T^1 = \{1,5,6\}, \quad T^2 = \{2,4,5\}, \quad T^3 = \{3,4,6\}.$$

We shall now define a nonempty compact polyhedral set $\mathcal{C} \subseteq \mathbb{R}^N$ which will turn out to be the core of an exact game: Let $x \in \mathbb{R}^N$. Then $x \in \mathcal{C}$ iff

$$x(S) \geq -27 \text{ for all } S \subseteq N, \tag{5.1}$$
$$x_i \geq -1 \text{ for all } i \in N^1, \tag{5.2}$$
$$x_j \geq -3 \text{ for all } j \in N^2, \tag{5.3}$$
$$x(S^k) \geq 1 \text{ for all } k \in N^1, \text{ and} \tag{5.4}$$
$$x(N) = 0. \tag{5.5}$$

Indeed, $\mathcal{C}$ is a closed polyhedral set. Moreover, it is compact (see (5.1) and (5.5)) and nonempty. Let r be the number of extreme points of $\mathcal{C}$ and let λ^ρ, $\rho = 1, \ldots, r$, denote the extreme points. Then $\mathcal{C} = \text{convh}(\{\lambda^\rho \mid \rho = 1, \ldots, r\})$. Define (N, v) by

$$v(S) = \min_{\rho=1,\ldots,r} \lambda^\rho(S) \text{ for all } S \subseteq N.$$

Then (N, v) is exact (by (5.5)) and $\mathcal{C}(N, v) = \mathcal{C}$. In order to show that (N, v) has a stable core it suffices to verify that $\mathcal{C}$ is *large*, that is, if $y \in \mathbb{R}^N$ satisfies $y(S) \geq v(S)$ for all $S \subseteq N$, then there exists $\lambda \in \mathcal{C}$ such that $\lambda \leq y$. Indeed, according to Sharkey (1982) the core of a game is stable if it is large.

Lemma 5.1 *The game (N, v) has a large core.*

Proof: Let $y \in \mathbb{R}^N$ satisfy $y(S) \geq v(S)$ for all $S \subseteq N$. Let X denote the set of vectors $x \in \mathbb{R}^N$ that satisfy (5.1) – (5.4), $x(N) \geq 0$, and $x \leq y$. Then X is nonempty (because $y \in X$) and polyhedral. Hence X is compact. Let $\widehat{x} \in X$ be such that

$$\widehat{x}(N) \leq x(N) \text{ for all } x \in X. \tag{5.6}$$

It remains to show that $\widehat{x} \in \mathcal{C}$, that is, $\widehat{x}(N) = 0$. Assume, on the contrary, that $\widehat{x}(N) > 0$. Denote $N^- = \{i \in N \mid \widehat{x}_i \leq 0\}$. We first claim that

$$N^3 \subseteq N^- \text{ and } \widehat{x}(N^-) = -27. \tag{5.7}$$

Eq. (5.7) is shown by contradiction. If $\ell \in N^3 \setminus N^-$, then there exists $\epsilon > 0$ such that $\widehat{x} - \epsilon\chi_{\{\ell\}} \in X$. ($\chi_S \in \mathbb{R}^N$ denotes the indicator function of $S \subseteq N$.) If $\widehat{x}(N^-) > -27$, then there exists $\epsilon > 0$ such that $\widehat{x} - \epsilon\chi_{\{\ell\}} \in X$ for every $\ell \in N^3$. Hence, both cases are in contrast to Eq. (5.6).

Now the proof can be completed. By Eq. (5.7) and the assumption that $\widehat{x}(N) > 0$ there exists $S \subseteq N^1 \cup N^2$ such that $\widehat{x}(S) > 27$. We now claim that $\widehat{x}_i \leq 5$ for all $i \in N^1$ and $\widehat{x}_j \leq 3$ for all $j \in N^2$. Indeed, if $\widehat{x}_i > 5$ for some $i \in N^1$, then, in view of (5.2) – (5.4), there exists $\epsilon > 0$ such that $\widehat{x} - \epsilon\chi_{\{\ell\}} \in X$. A similar argument is valid if $\widehat{x}_j > 3$ for some $j \in N^2$. Both cases contradict Eq. (5.6). Hence, $\widehat{x}(S) \leq 3 \cdot 5 + 3 \cdot 3 = 24 < 27$ for all $S \subseteq N$ and the proof is complete. **q.e.d.**

It remains to verify the following result.

Lemma 5.2 $0 \in \mathbb{R}^N$ *is the unique member of the modified least core of* (N, v).

We postpone the proof of Lemma 5.2 and shall first, in order to determine the value of some coalitions, define 74 vectors of $\mathcal{C}$ as follows. For every $k \in N^1$ and $\ell \in N^3$ let

$$\lambda^0,\ \lambda^{S^k} \text{ (3 elements)},\ \lambda^{k\ell} \text{ (30 elements)},\ \lambda^{T^k\ell} \text{ (30 elements)},\ \lambda^{\ell} \text{ (10 elements)}$$

be defined by

$$\begin{aligned}
\lambda^0 &= \Big(-1,-1,-1,3,3,3,\underbrace{-\frac{3}{5},\ldots,-\frac{3}{5}}_{10\text{ times}}\Big),\\
\lambda_i^{S^k} &= \left\{\begin{array}{rl} 2 & \text{, if } i \in S^k \cap N^1,\\ -3 & \text{, if } i \in S^k \cap N^2,\\ -1 & \text{, if } i \in N^1 \setminus S^1\\ 0 & \text{, if } i \in (N^2 \setminus S^k) \cup N^3, \end{array}\right\},\\
\text{e.g., } \lambda^{S^1} &= (2,2,-1,-3,0,0,\underbrace{0,\ldots,0}_{10\text{ times}}),
\end{aligned}$$

$$\lambda_i^{k\ell} = \left\{ \begin{array}{rl} 9 & \text{, if } i \in \{k, 7-k\}, \\ -1 & \text{, if } i \in (N^1 \cup N^2) \setminus \{k, 7-k\}, \\ -23 & \text{, if } i = \ell \\ 1 & \text{, if } i \in N^3 \setminus \{\ell\}, \end{array} \right\},$$

$$\text{e.g., } \lambda^{1\ 7} = (9, -1, -1, -1, -1, 9, -23, \underbrace{1, \ldots, 1}_{9 \text{ times}}),$$

$$\lambda_i^{T^k\ell} = \left\{ \begin{array}{rl} 6 & \text{, if } i \in T^k, \\ -1 & \text{, if } i \in (N^1 \cup N^2) \setminus T^k, \\ -24 & \text{, if } i = \ell \\ 1 & \text{, if } i \in N^3 \setminus \{\ell\}, \end{array} \right\},$$

$$\text{e.g., } \lambda^{T^1 7} = (6, -1, -1, -1, 6, 6, -24, \underbrace{1, \ldots, 1}_{9 \text{ times}}),$$

$$\lambda_i^{\ell} = \left\{ \begin{array}{rl} -1 & \text{, if } i \in N^1, \\ 6 & \text{, if } i \in N^2, \\ -24 & \text{, if } i = \ell \\ 1 & \text{, if } i \in N^3 \setminus \{\ell\}, \end{array} \right\},$$

$$\text{e.g., } \lambda^{7} = (-1, -1, -1, 6, 6, 6, -24, \underbrace{1, \ldots, 1}_{9 \text{ times}}).$$

It is straightforward to check that these 74 vectors are members of $\mathcal{C}$. Also, a careful inspection of the definitions of λ^0 and λ^{S^k} shows that $v(S) \leq 1$ for all $S \subseteq N$, that $v(S^k) = 1$ for all $k \in N^1$, and that

$$\mathcal{D} = \{S \subseteq N \mid v(S) \geq v(T) \ \forall\ T \subseteq N\}$$

is given by $\mathcal{D} = \{S^1, S^2, S^3\}$.

Let (N, v^*) be the dual game. As (N, v) is exact, v^* is given by

$$v^*(S) = \max_{\rho=1,\ldots,r} \lambda^\rho(S) \text{ for all } S \subseteq N.$$

We conclude that $v^*(S) \leq 27$ for all $S \subseteq N$. Let

$$\mathcal{D}(N, v^*) = \{S \subseteq N \mid v^*(S) \geq v^*(T) \ \forall\ T \subseteq N\} (= \{S \subseteq N \mid v(S) = 27\}).$$

A careful inspection of the definitions of $\lambda^{k\ell}$, $\lambda^{T^k\ell}$ and λ^ℓ shows that $\mathcal{D}^*$, defined by

$$\mathcal{D}^* = \Big\{(N^3 \setminus \{\ell\}) \cup P \Big| \ell \in N^3 \text{ and } \Big((P = \{k, 7-k\} \text{ or } P = T^k \text{ for some } k \in N^1) \text{ or } P = N^2\Big)\Big\}$$

satisfies $\mathcal{D}^* \subseteq \mathcal{D}(N, v^*)$.

We shall now repeat an useful characterization of the modiclus by "balanced collections of pairs of coalition". Let M be a finite nonempty set. A finite nonempty set $Z \subseteq \mathbb{R}^M$ is called *balanced* if there exist $\alpha_z > 0$, $z \in Z$, such that $\sum_{z \in Z} \alpha_z z = \chi_M$.

Theorem 5.3 (Sudhölter (1997), Theorem 2.2) *Let (M, w) be a game and let $x \in X(M, w)$. Then $x = \psi(M, w)$ if and only if for every $\beta \leq \mu(x, w) + \mu(x, w^*)$ the set*

$$Z(\beta, x, w) = \{\chi_S + \chi_T \mid S, T \subseteq M, e^b(S, T, x, w) \geq \beta\} \tag{5.8}$$

is balanced.

Moreover, the shall apply the following simple remark (Sudhölter (1997), Remark 2.8).

Remark 5.4 *Let Z be a balanced subset of $\mathbb{R}^M$ and $z \in \mathbb{R}^M$. If z is in the linear span of Z, then $Z \cup \{z\}$ is balanced.*

Proof of Lemma 5.2:

Step 1: First we shall show that $\psi(N, v) = 0$. Let $Z = \{\chi_S + \chi_T \mid S \in \mathcal{D},\ T \in \mathcal{D}^*\}$. Then Z is a subset of $Z(\mu(0, v) + \mu(0, v^*), 0, v)$ defined by Eq. (5.8). In view of Theorem 5.3 and Remark 5.4 it suffices to show that

(1) Z spans $\mathbb{R}^N$;

(2) Z is balanced.

ad (1): The set

$$\{\chi_{S^1} + \chi_T \mid T \in \mathcal{D}^*,\ N^3 \setminus T = \{7\}\} \cup \{\chi_{S^1} + \chi_T \mid T \in \mathcal{D}^*,\ N^2 \subseteq T\}$$

is a basis of $\mathbb{R}^N$ and it is contained in Z.

ad (2): Let π be any permutation of N such that

$$\pi(1) = 2,\ \pi(2) = 3,\ \pi(3) = 1,\ \pi(6) = 5,\ \pi(5) = 4, \text{ and } \pi(4) = 6.$$

Then π is a symmetry of (N, v) and of $\mathcal{D}^*$. Hence, in order to verify (2), it suffices to show that the set $\widetilde{Z} \subseteq \mathbb{R}^3$, defined by

$$\widetilde{Z} = \{(z\chi_{N^1}, z\chi_{N^2}, z\chi_{N^3}) \mid z \in Z\},$$

satisfies the following condition: There exist $\alpha_{\widetilde{z}} > 0$, $\widetilde{z} \in \widetilde{Z}$, such that

$$\sum_{\widetilde{z} \in \widetilde{Z}} \alpha_{\widetilde{z}} \widetilde{z} = (|N^1|, |N^2|, |N^3|) = (3, 3, 10).$$

If $S \in \mathcal{D}$, then $(|S \cap N^1|, |S \cap N^2|, |S \cap N^3|) = (2, 1, 0)$. Moreover,

$$\{(|T \cap N^1|, |T \cap N^2|, |T \cap N^3|) \mid T \in \mathcal{D}^*\} = \{(1, 1, 9), (1, 2, 9), (0, 3, 9)\}.$$

Hence,

$$\widetilde{Z} = \{(3, 2, 9), (3, 3, 9), (2, 4, 9)\}.$$

Now,

$$6 \cdot (3, 2, 9) + (3, 3, 9) + 3 \cdot (2, 4, 9) = 9 \cdot (3, 3, 10),$$

hence our Claim is shown.

Step 2: Let $x \in \mathcal{MLC}(N, v)$. It remains to show that $x = 0$. By Eq. 2.1 and Step 1, $\mu(x, v) + \mu(x, v^*) = \mu(0, v) + \mu(0, v) = 1 + 27$. Let $\alpha_z > 0$, $z \in Z(28, 0, v)$, be such that $\sum_{z \in Z(28,0,v)} \alpha_z z = \chi_N$. Then

$$\sum_{z \in Z(28,0,v)} x \cdot z = x(N) = 0.$$

Here $\cdot$ denotes the scalar product. Hence $e(S, x, v) + e(T, x, v) = 28$ for all $S \in \mathcal{D}, T \in \mathcal{D}^*$. Consequently, $x \cdot z = 0$ for all $z \in Z(28, 0, v)$. By Step 1, (2), $x = 0$. **q.e.d**

We conclude that $\psi = \psi(N, v) = 0 \in \mathbb{R}^N$. However, $0 \notin \mathcal{C}(N, v)$.

Remark 5.5 *Let*

$$x = \Big(\underbrace{\frac{1}{3}, \ldots, \frac{1}{3}}_{6 \ \textit{times}}, \underbrace{-\frac{1}{5}, \ldots, -\frac{1}{5}}_{10 \ \textit{times}}\Big).$$

Then $x \in \mathcal{C}(N, v)$ and $x(S^k) = v(S^k) = 1$ for $k \in N^1$. Hence x dominates ψ via any of the coalitions S^k of positive excess.

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