November 2023

θ-angle physics of 2-color QCD

Fixed baryon charge and Near Conformal Dynamics

Based on [1] and [2]
in collaboration with J. Bersini, F. Sannino and M. Torres

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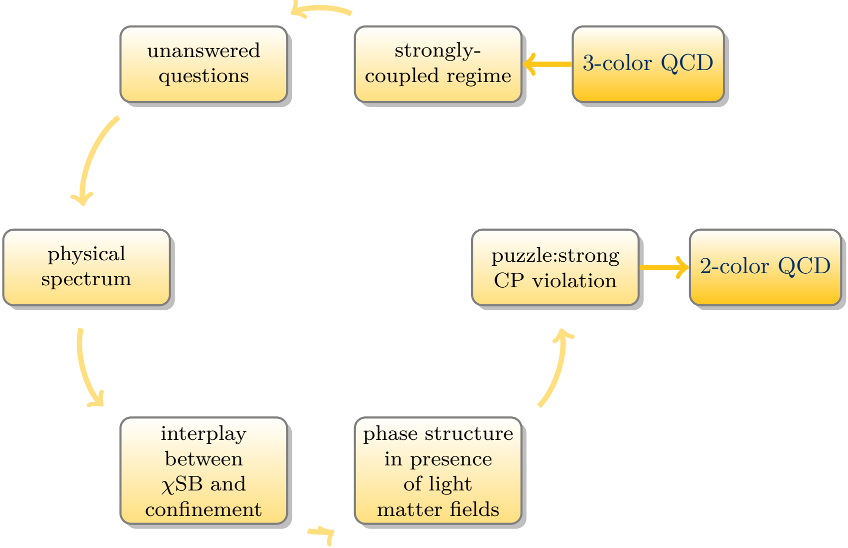
unanswered questions
strongly-coupled regime
3-color QCD
physical spectrum
puzzle:strong CP violation
2-color QCD
interplay between χSB and confinement
phase structure in presence of light matter fields

What we are going to talk about?

goal: we want to study QCD at finite baryon density

why?: we want to enrich and know more about QCD thermodynamics

problem: Finite density QCD cannot be efficiently studied on lattice due to the sign problem: the determinant of the Dirac operator is not real

solution: 2-color QCD: no sign problem due to the pseudo-reality of the quark representations

What is the main novelty of our work?

OLD:

- 2-color QCD at fixed baryon charge for general N_f [3]
- 2-color QCD at fixed baryon and isospin charge for general N_f [4]

NEW [1]:

study of the 2-color QCD EFT at fixed baryon charge with the inclusion of the topological term

θ-angle physics

The analysis of vacuum structure in non-abelian gauge theories allows to add

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu} = \theta q(x) \quad \text{where} \quad q(x) \quad \text{is the topological charge} \tag{1}$$

- this term violates CP symmetry: strong CP problem

What is its effect?

Due to chiral transformations, the CP violation depends on the more physical

$$\bar{\theta} = \theta - \arg\det M \quad \text{where} \quad M \quad \text{is the quark mass.} \tag{2}$$

The experiments give: $\boxed{\bar{\theta} < 10^{-10}}$

In this presentation we will talk about

- 2-color QCD:
  - From fondamental theory to EFT
- 2-color QCD and θ-angle:
  - Vacuum structure
- Symmetry breaking pattern
- Solving the dynamics
  - Ground State Energy
- Near-conformal 2-color QCD
- Charging near-conformal 2-color QCD
- Backup slides

FROM FUNDAMENTAL THEORY TO THE EFT

QC_2D Lagrangian in the Chirally Broken Phase

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_f^{N_f} i\,\bar{\psi}_f \gamma^\mu D_\mu \psi_f = -\frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu}_a + i q_L^\dagger \bar{\sigma}^\mu D_\mu q_L + i q_R^\dagger \sigma^\mu D_\mu q_R \,, \qquad (3)$$

When $N_c = 2$ the fundamental representation is pseudo-real ($\tau_a^* = \tau_a^T = -\tau_2 \tau_a \tau_2$) and the quantity $\tilde{q} = i\sigma_2 \tau_2 q_R^*$ transforms as a left quark, thus

$$(q_L, q_R) \to (q, \tilde{q}) \equiv Q^T \,.$$

The global symmetry group

$$U(N_f)_L \times U(N_f)_R \quad \text{is enlarged to} \quad \boxed{U(2N_f) \sim SU(2N_f) \times U(1)_A}\,.$$

Introducing a quark mass term

$$\bar{\psi}\psi = q_R^\dagger q_L + q_L^\dagger q_R = \frac{1}{2} Q^T \sigma_2 \tau_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Q + \text{h.c.} \qquad (4)$$

the global group $SU(2N_f)$ is further broken to $Sp(2N_f)$.

Two-color QCD Lagrangian for N_f Dirac fermions at the fundamental level:

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu}_a + i\bar{\mathcal{Q}}\bar{\sigma}^\mu \left[\partial_\mu - i G^a_\mu \frac{\tau_a}{2}\right]\mathcal{Q} - \frac{1}{2} m_q \mathcal{Q}^T \tau_2 E \mathcal{Q} + \text{h.c.}\,, \tag{5}$$

Two-spinor field $\mathcal{Q} = \begin{pmatrix} q_L \\ i\sigma_2\tau_2 q_R^* \end{pmatrix}$, Dirac mass $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_{N_f}$

In the Chirally broken phase, the Goldstones' dynamics is described by the phenomenological Lagrangian

$$\mathcal{L}_{\text{eff}} = \nu^2 \text{tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] + m_\pi^2 \nu^2 \text{tr}[\mathcal{M}\Sigma + \mathcal{M}^\dagger \Sigma^\dagger]\,, \quad \mathcal{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1}_{N_f}\,. \tag{6}$$

where

$$\Sigma = e^{i\Pi/\nu}\,, \qquad \Pi = \pi^a T_a\,, \quad T_a = \text{broken generators}$$

and

$$\Sigma \;\to\; U\Sigma U^T \quad \text{with} \quad U \in SU(2N_f)\,.$$

Quantum Field Theories at fixed charge

Fixing a charge means to impose a constrain which breaks Lorentz invariance:

$$Q = \int d^{d-1}x\, j^0 = \bar{Q} \quad \Rightarrow \quad \hat{\mathcal{L}} = \mathcal{L} - \mu Q \tag{7}$$

where μ is the chemical potential. In the present case the fixed baryon charge is associated to the generator

$$B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbb{1}_{N_f} \,. \tag{8}$$

At the fundamental level this can be done introducing a long derivative:

$$\partial_\mu \;\to\; \partial_\mu - i\mu B_\mu \,, \quad B_\mu = \delta^0_\mu B \tag{9}$$

which at the effective level means

$$\partial_\mu \Sigma \to D_\mu \Sigma = \partial_\mu \Sigma - i\mu_B [B_\mu \Sigma + \Sigma B_\mu] \,. \tag{10}$$

The charged Lagrangian results to be

$$\mathcal{L}_{\mathrm{eff},\mu} = \nu^2 \mathrm{tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] + 4i\nu^2 \mu_B \mathrm{tr}[B\Sigma^\dagger \partial_0 \Sigma] + m_\pi^2 \nu^2 \mathrm{tr}[\mathcal{M}\Sigma + \mathcal{M}^\dagger \Sigma^\dagger] + 2\nu^2 \mu_B^2 \mathrm{tr}[\Sigma B \Sigma^\dagger B + B^2]\,. \tag{11}$$

In the end we add the topological sector, which at the fundamental level is given by:

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu} = \theta q(x)\,. \tag{12}$$

where the topological charge $q(x)$ is related to the axial anomaly

$$\partial_\mu J_5^\mu = 4 N_f q(x)\,. \tag{13}$$

At the effective level this corresponds to the term

$$\mathcal{L}_\theta = -a\nu^2 \left(\theta - \frac{i}{4} \mathrm{Tr}\{\log \Sigma - \log \Sigma^\dagger\}\right)^2\,. \tag{14}$$

The θ-angle physics of two-color QCD at fixed baryon charge

The complete Lagrangian for 2-color QCD at fixed baryon charge with global symmetry $\mathrm{SU}(2\mathrm{N_f})$ [3] and the θ-angle [5, 6] is thus:

$$\mathcal{L} = \mathcal{L}_{\partial\pi} + \mathcal{L}_{\mathrm{m}_\pi} + \mathcal{L}_{\mu} + \mathcal{L}_{\theta} \tag{15}$$

$$\mathcal{L}_{\pi} \quad \text{pions} \quad = \nu^2 \mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} + \mathrm{m}_\pi^2 \nu^2 \mathrm{Tr}\{\mathrm{M}\Sigma + \mathrm{M}^\dagger \Sigma^\dagger\}$$

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\end{aligned}$$

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$$\mathcal{L}_\mu \quad \text{baryon charge} = 4\mu\nu^2 \mathrm{Tr}\{\mathrm{B}\Sigma^\dagger \partial_0 \Sigma\} + 2\mu^2\nu^2 \left[\mathrm{Tr}\{\Sigma \mathrm{B}^\mathrm{T} \Sigma^\dagger \mathrm{B}\} + \mathrm{Tr}\{\mathrm{BB}\}\right]$$

$$\mathcal{L}_\theta \quad \theta\text{-angle} = -\mathrm{a}\nu^2 \left(\theta - \frac{\mathrm{i}}{4} \mathrm{Tr}\{\log \Sigma - \log \Sigma^\dagger\}\right)^2$$

2-COLOR QCD AND θ-ANGLE: VACUUM STRUCTURE

Vacuum Structure

The vacuum configuration for the complete Lagrangian

$$\mathcal{L} = \mathcal{L}_{\partial\pi} + \mathcal{L}_{m_\pi} + \mathcal{L}_\mu \tag{16}$$

is given by the following ansatz

$$\Sigma_c = \begin{pmatrix} 0 & 1_{N_f} \\ -1_{N_f} & 0 \end{pmatrix} \cos\varphi + i \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix} \sin\varphi \qquad \text{where} \qquad \mathcal{I} = \begin{pmatrix} 0 & -1_{N_f/2} \\ 1_{N_f/2} & 0 \end{pmatrix} . \tag{17}$$

We introduce the Witten variables α_i to consider the ground state of

$$\mathcal{L} = \mathcal{L}_{\partial\pi} + \mathcal{L}_{m_\pi} + \mathcal{L}_\mu + \mathcal{L}_\theta \tag{18}$$

$$\Sigma_0 = U(\alpha_i)\Sigma_c , \qquad U(\alpha_i) \equiv \mathrm{diag}[e^{-i\alpha_1}, \ldots, e^{-i\alpha_{N_f}}, e^{-i\alpha_1}, \ldots, e^{-i\alpha_{N_f}}] . \tag{19}$$

each phase α_i is an overall axial transformation for each left-right quark pair.

SYMMETRY BREAKING PATTERN

Pion kinetics, mass term and θ-angle

$$\mathcal{L} = \mathcal{L}_{\partial\pi} + \mathcal{L}_{m_\pi} + \mathcal{L}_\theta \tag{20}$$

$$\mathcal{L}_\pi \quad \text{pions' kinetics} = \nu^2 \mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} \tag{21}$$

$$\mathcal{L}_\pi \quad \text{pions' mass term} = m_\pi^2 \nu^2 \mathrm{Tr}\{M\Sigma + M^\dagger \Sigma^\dagger\} \tag{22}$$

$$\mathcal{L}_\theta \quad \theta\text{-angle} = -a\nu^2 \left(\theta - \frac{i}{4}\mathrm{Tr}\{\log\Sigma - \log\Sigma^\dagger\}\right)^2 \tag{23}$$

$$\Sigma_0 = U(\alpha_i)\Sigma_c, \quad \Sigma_c = \Sigma_M = \begin{pmatrix} 0 & 1_{N_f} \\ -1_{N_f} & 0 \end{pmatrix} = -\mathcal{M} \tag{24}$$

$$U(2N_f) \sim SU(2N_f) \times U(1)_A \to Sp(2N_f) \quad \text{(explicit)}$$

$$U(2N_f) \sim SU(2N_f) \times U(1)_A \rightsquigarrow Sp(2N_f) \quad \text{(spontaneous)}$$

$$\#\text{GBs} = (2N_f^2 - N_f - 1) \text{ massless or quasi-massless } (+1) \text{ S-massive particle (anomaly)}$$
$$= 2N_f^2 - N_f.$$

Pion kinetics, chemical potential and θ-angle

$$\mathcal{L} = \underline{\mathcal{L}_{\partial\pi}} + \underline{\mathcal{L}_{\mu}} + \underline{\mathcal{L}_{\theta}} \tag{25}$$

$$\begin{aligned} \underline{\mathcal{L}_{\partial\pi}} &= \nu^2 \mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} \\ \underline{\mathcal{L}_{\mu} \quad \text{baryon charge}} &= 4\mu\nu^2 \mathrm{Tr}\{B\Sigma^\dagger \partial_0 \Sigma\} + 2\mu^2\nu^2 \left[\mathrm{Tr}\{\Sigma B^T \Sigma^\dagger B\} + \mathrm{Tr}\{BB\}\right] \\ \underline{\mathcal{L}_{\theta} \quad \theta\text{-angle}} &= -a\nu^2 \left(\theta - \frac{i}{4}\mathrm{Tr}\{\log\Sigma - \log\Sigma^\dagger\}\right)^2 \end{aligned} \tag{26, 27}$$

$$\Sigma_0 = U(\alpha_i)\Sigma_c, \quad \Sigma_c = \Sigma_M \cos\varphi + i\Sigma_B \sin\varphi, \quad \Sigma_B = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}; \tag{28}$$

$$SU(2N_f) \times U(1)_A \rightsquigarrow Sp(2N_f) \to SU(N_f)_L \times SU(N_f)_R \times U(1)_B \rightsquigarrow Sp(N_f)_L \times Sp(N_f)_R.$$

$$\#GBs = (N_f^2 - N_f - 1) \text{ massless } (+1) \text{ massive S-particle } (+N_f^2) \text{ massive } (\propto \mu) = 2N_f^2 - N_f.$$

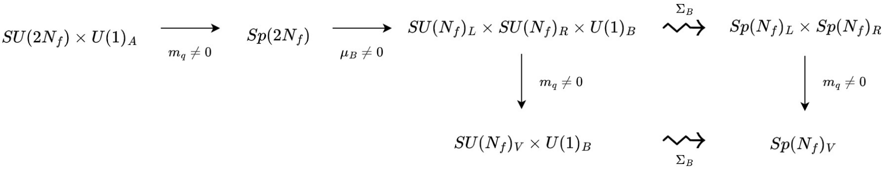
Pion kinetics, mass term, chemical potential and θ-angle

$$\mathcal{L} = \mathcal{L}_{\partial\pi} + \mathcal{L}_{\mathrm{m}_\pi} + \mathcal{L}_\mu + \mathcal{L}_\theta \tag{29}$$

$$\mathcal{L}_{\partial\pi} + \mathcal{L}_{\mathrm{m}_\pi} = \nu^2 \mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} + \mathrm{m}_\pi^2 \nu^2 \mathrm{Tr}\{\mathrm{M}\Sigma + \mathrm{M}^\dagger \Sigma^\dagger\}$$

$$\mathcal{L}_\mu \quad \text{baryon charge} = 4\mu\nu^2 \mathrm{Tr}\{\mathrm{B}\Sigma^\dagger \partial_0 \Sigma\} + 2\mu^2\nu^2 \left[\mathrm{Tr}\{\Sigma \mathrm{B}^{\mathrm{T}} \Sigma^\dagger \mathrm{B}\} + \mathrm{Tr}\{\mathrm{BB}\}\right] \tag{30}$$

$$\mathcal{L}_\theta \quad \theta\text{-angle} = -\mathrm{a}\nu^2 \left(\theta - \frac{\mathrm{i}}{4}\mathrm{Tr}\{\log\Sigma - \log\Sigma^\dagger\}\right)^2 \tag{31}$$



- Superfluid transition $\equiv$ Bose-Einstein (diquark) condensation $\rightarrow \frac{\mathrm{N_f^2 - N_f}}{2}$ massless GBs!

GROUND STATE ENERGY

EOMs for the Witten variables

The Lagrangian evaluated on the vacuum ansatz reads

$$\mathcal{L}_\theta[\Sigma_0] = \nu^2\left[4\mathrm{m}_\pi^2 \mathrm{X}\cos\varphi + 2\mu^2 \mathrm{N_f}\sin^2\varphi - \mathrm{a}\bar{\theta}^2\right] \tag{32}$$

where for later convenience we introduced

$$\bar{\theta} = \theta - \sum_{\mathrm{i}}^{\mathrm{N_f}} \alpha_{\mathrm{i}}\,, \qquad \mathrm{X} = \sum_{\mathrm{i}}^{\mathrm{N_f}} \cos\alpha_{\mathrm{i}} \tag{33}$$

where $\bar{\theta}$ is the effective theta angle that enters physical observables
The equations of motion read

$$\sin\varphi\left(\mathrm{N_f}\cos\varphi - \frac{\mathrm{m}_\pi^2}{\mu^2}\mathrm{X}\right) = 0 \tag{34}$$

$$2\mathrm{m}_\pi^2 \sin\alpha_{\mathrm{i}}\cos\varphi = \mathrm{a}\bar{\theta}\,, \quad \mathrm{i} = 1, .., \mathrm{N_f} \tag{35}$$

GSE: ground state energy

$$E = -\nu^2 \left[4 m_\pi^2 X - a\bar{\theta}^2\right] , \quad \text{normal phase} \tag{36}$$

The superfluid phase is associated with diquark (baryon) condensation

$$E = -\nu^2 \left[2 \frac{N_f^2 \mu^4 + m_\pi^4 X^2}{N_f \mu^2} - a\bar{\theta}^2\right] , \quad \text{superfluid phase} \left(\cos\varphi = \frac{m_\pi^2}{N_f \mu^2} X\right) . \tag{37}$$

- $\theta = 0$: $X = N_f$ and the superfluid phase transition occurs at $\mu = m_\pi$
- $\theta \neq 0$: θ-dependence in both phases: the energy is minimized when X(normal phase) and X^2 (superfluid phase) is maximized

Solutions of the EOM for Witten variables: normal phase

the Witten variables are related to θ by the well-known equation

$$2m_\pi^2 \sin\alpha_i = a\bar{\theta} = a\left(\theta - \sum_i^{N_f}\alpha_i\right) \tag{38}$$

For the general solution we must have for any $\bar{\theta}$ fixed $\boxed{\sin\alpha_i = \sin\alpha_j}$

We solve in powers of m_π^2/a

At the leading order one needs to solve for $\bar{\theta} = 0$ and the angles α_i satisfy

$$\alpha_i = \begin{cases} \pi - \alpha, & i = 1, \dots, n \\ \alpha, & i = n+1, \dots, N_f \end{cases} \tag{39}$$

where α is the solution of the following modular equation

$$n(\pi - \alpha) + (N_f - n)\alpha = \theta \text{ Mod } 2\pi \ . \tag{40}$$

Periodicity of the solutions

The modulo comes from the fact that if a solution $\{\alpha_{\rm i}\}$ of eq.(38) is found, then it is possible to build another solution as follows

$$\alpha_1(\theta+2\pi)=\alpha_1(\theta)+2\pi\,,\qquad \alpha_{\rm i}(\theta+2\pi)=\alpha_{\rm i}(\theta)\,,\quad {\rm i}=2,\ldots,{\rm N_f}\,. \tag{41}$$

Physics depends only on $e^{-i\alpha_i}$: the dynamics is invariant under $\theta\to\theta+2\pi$

The solution of (40) is

$$\boxed{\alpha=\frac{\theta+(2{\rm k}-{\rm n})\pi}{({\rm N_f}-2{\rm n})},\quad {\rm k}=0,\ldots,{\rm N_f}-2{\rm n}-1\,,\quad {\rm n}=0,...,\left[\frac{{\rm N_f}-1}{2}\right]} \tag{42}$$

The range for k above emerges because for ${\rm k}\geq{\rm N_f}-2{\rm n}$ we repeat the solution for a given n.

More on the solutions

$$\boxed{\alpha = \frac{\theta + (2k - n)\pi}{(N_f - 2n)}, \quad k = 0, \ldots, N_f - 2n - 1\,, \quad n = 0, ..., \left[\frac{N_f - 1}{2}\right]} \tag{43}$$

the solutions with $n \neq 0$ spontaneously break $Sp(2N_f)$ because of the different phases for each flavour

the most general solution with $n = 0$ is

$$\boxed{U(\alpha_i) = e^{i\,\frac{\theta + 2\pi k}{N_f}}\, \mathbb{1}_{2N_f}} \tag{44}$$

CP symmetry

CP is conserved when

$$\bar{\theta} = \theta - \sum_{i=1}^{N_f} \alpha_i = 0 \qquad (45)$$

this happens if: $\bullet\theta = 0 \quad \bullet m_\pi^2 = 0$

- $\theta = \pi$ the Lagrangian is CP invariant and we have $X = \cos\left(\frac{(2k+1)\pi}{N_f}\right)$ which is maximized when $k = 0$ and $k = N_f - 1$, that is the vacua lie at [5]

$$U(\alpha_i) = e^{\frac{i\pi}{N_f}} \mathbb{1}_{2N_f}\,, \qquad U(\alpha_i) = e^{-\frac{i\pi}{N_f}} \mathbb{1}_{2N_f} \qquad (46)$$

The two solutions are related by a CP transformation $U \to U^\dagger$

CP is spontaneously broken by the vacuum(Dashen phenomenon [7–10])

Solutions of the EOM for Witten variables: superfluid phase

the EOM to solve in this case is

$$\frac{2m_\pi^4}{N_f \mu^2} \, X \sin\alpha_i = a\,\bar{\theta}\,, \qquad i = 1, \ldots, N_f \tag{47}$$

and we solve in expansion of $m_\pi^4/(a\mu^2)$

at leading order the solutions are the same of those for the normal phase

$$\boxed{\alpha = \frac{\theta + (2k - n)\pi}{(N_f - 2n)}, \quad k = 0, \ldots, N_f - 2n - 1\,, \quad n = 0, \ldots, \left[\frac{N_f - 1}{2}\right]} \tag{48}$$

$N_f = 2$

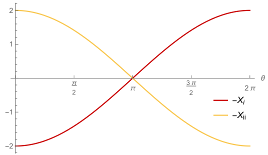
at leading order (in m_π^2/a or $m_\pi^4/(a\mu)$) the EOM is

$$\alpha_1 + \alpha_2 = \theta + 2k\pi \qquad \sin\alpha_1 = \sin(\theta + 2k\pi - \alpha_1) \tag{49}$$

and has solutions

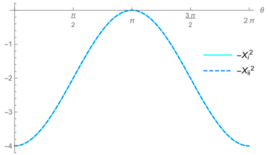
$$\bullet\{\alpha_1, \alpha_2\} = \left\{\frac{\theta}{2}, \frac{\theta}{2}\right\} \qquad \bullet\,\{\alpha_1, \alpha_2\} = \left\{\frac{\theta + 2\pi}{2}, \frac{\theta + 2\pi}{2}\right\} \tag{50}$$

Normal phase



the solutions cross at $\theta = \pi$

Superfluid phase



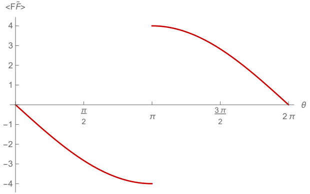
the energy is an analytic function of θ

CP breaking $N_f = 2$

CP order parameter: $$\left\langle F\tilde{F}\right\rangle \propto -\frac{\partial E}{\partial \theta} \tag{51}$$

Normal phase

• spontaneous symmetry breaking:

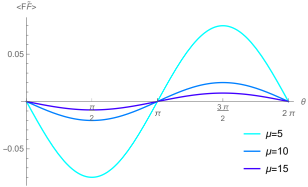


• explicit breaking of CP symmetry:

$$\bar{\theta} = \frac{2m_\pi^2}{a}\sin\frac{\theta}{2} \stackrel{\theta=\pi}{=} \frac{2m_\pi^2}{a} + \mathcal{O}\left(\frac{m_\pi^6}{a^3}\right)$$

Superfluid phase

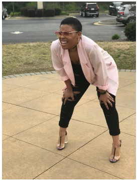
• NO spontaneous symmetry breaking:



• NO explicit breaking of CP symmetry:

$$\bar{\theta} = \frac{m_\pi^4}{a\mu^2}\sin\theta \stackrel{\theta=\pi}{=} 0$$

$N_f = 2$: more details



- $\theta = \pi$ the effective mass $m_\pi^2(\theta) \sim m_\pi^2 \left|\cos\left(\frac{\theta}{2}\right)\right|$ vanishes up to correction of order $\left(\frac{m_\pi^2}{a}\right)$
- mass term disappears from the Lagrangian and the global flavor symmetry is again SU(4)
- massless Goldstones when $SU(4) \rightsquigarrow Sp(4)$ [6]

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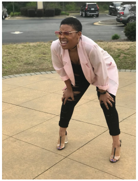

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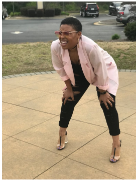
there is no chiral symmetry restoration in the fundamental Lagrangian: apparent paradox

$N_f = 2$: more details



- $\theta = \pi$ the effective mass $m_\pi^2(\theta) \sim m_\pi^2 \left|\cos\left(\frac{\theta}{2}\right)\right|$ vanishes up to correction of order $\left(\frac{m_\pi^2}{a}\right)$
- mass term disappears from the Lagrangian and the global flavor symmetry is again SU(4)
- massless Goldstones when SU(4) $\rightsquigarrow$ Sp(4) [6]

there is no chiral symmetry restoration in the fundamental Lagrangian: apparent paradox

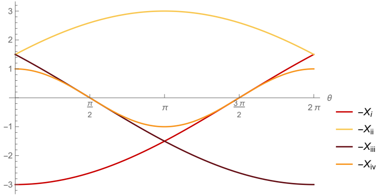
solved by realising that SU(4) is still broken by higher order mass terms in the effective Lagrangian also for $a \to \infty$ [5, 6]

$\mathrm{N_f} = 3$

solutions: $\mathrm{n} = 0 \implies \mathrm{k} = 0, 1, 2$ and $\mathrm{n} = 1 \implies \mathrm{k} = 0$

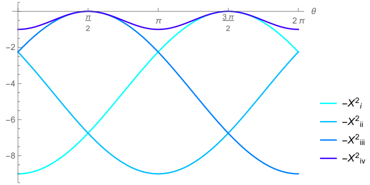
$$\text{i.}\left\{\frac{\theta}{3}, \frac{\theta}{3}, \frac{\theta}{3}\right\}, \quad \text{ii.}\left\{\frac{\theta+2\pi}{3}, \frac{\theta+2\pi}{3}, \frac{\theta+2\pi}{3}\right\}, \quad \text{iii.}\left\{\frac{\theta+4\pi}{3}, \frac{\theta+4\pi}{3}, \frac{\theta+4\pi}{3}\right\}, \quad \text{iv.}\{\theta-\pi\,, \theta-\pi\,, 2\pi-\theta\}$$

Normal phase



the solutions i. and iii. cross at $\theta = \pi$

Superfluid phase

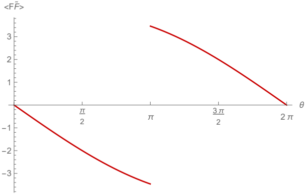


the solutions i.,ii. and iii. cross at $\theta = \pi/2, 3\pi/2$

CP breaking $\mathrm{N_f} = 3$

Normal phase

• SSB of CP at $\theta = \pi$:

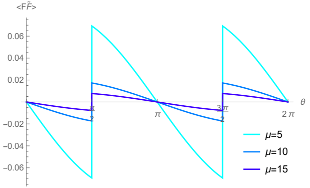


$$\bar{\theta} = \frac{\sqrt{3}m_\pi^2}{a} - \frac{m_\pi^4}{\sqrt{3}a^2} - \frac{m_\pi^6}{6\sqrt{3}a^3} + \left(\frac{m_\pi^8}{a^4}\right)$$

explicit breaking of CP symmetry

Superfluid phase

• NO SSB of CP at $\theta = \pi$ but at $\pi/2, 3\pi/2$:



$$\bar{\theta} = 0$$

NO explicit breaking of CP symmetry

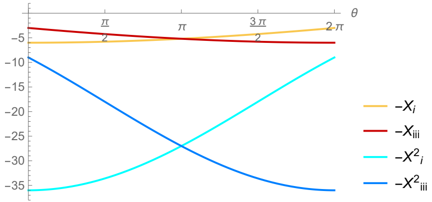
★ two novel phase transitions at $\theta = \pi/2$ and $\theta = 3\pi/2$ in the superfluid phase

$N_f = 6$

$$\text{Solutions i-vi}: \alpha_1 = \alpha_2, \cdots = \alpha_6 = \frac{\theta + 2\pi k}{6}, \qquad k = 0, \ldots, 5$$

$$\text{Solutions vii-ix}: \alpha_1 = \alpha_2 = \cdots = \alpha_5 = \frac{\theta - \pi + 2\pi k}{4}, \quad \alpha_6 = \pi - \alpha_1, \qquad k = 0, \ldots, 3$$

$$\text{Solutions x-xii}: \alpha_1 = \alpha_2 = \cdots = \alpha_4 = \frac{\theta - 2\pi + 2\pi k}{2}, \quad \alpha_5 = \alpha_6 = \pi - \alpha_1, \qquad k = 0, 1 . \quad (52)$$



- same energy dependence on θ in both phases
- SSB of CP symmetry at $\theta = \pi$
- explicit breaking of CP symmetry at $\theta = \pi$

General N_f

Solutions of the EOMs are generally not periodic of 2π for θ

The periodicity condition can be satisfied only if at least two solutions cross. Consider

$$U = e^{-i\alpha} \, \mathbb{1}_{2N_f} \tag{53}$$

and ask when two different solutions of the equation of motion can have the same energy. This corresponds to requiring

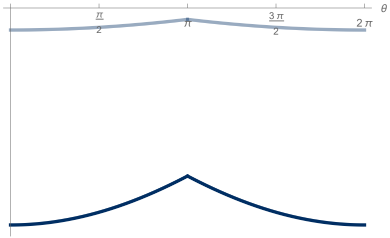
$$\cos\left(\frac{\theta + 2\pi k_1}{N_f}\right) = \cos\left(\frac{\theta + 2\pi k_2}{N_f}\right) , \qquad \text{normal phase} \tag{54}$$

$$\cos^2\left(\frac{\theta + 2\pi k_1}{N_f}\right) = \cos^2\left(\frac{\theta + 2\pi k_2}{N_f}\right) , \qquad \text{superfluid phase} \tag{55}$$

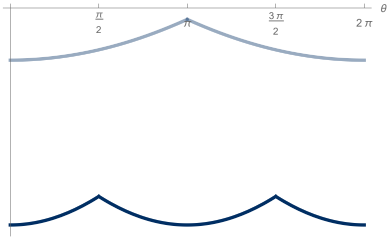
- Both conditions are satisfied when $\boxed{k_1 = -\frac{\theta}{\pi} - k_2 + N_f}$
- near $\theta = 0$ the ground state is $k_1 = 0$

θ-dependence of the energy [1, 5, 6]

N_f even



N_f odd



normal phase

$$0 \leq \cos \frac{\theta}{N_f} \leq \pi \leq \cos \frac{\theta+2\pi(N_f-1)}{N_f} \leq 2\pi$$

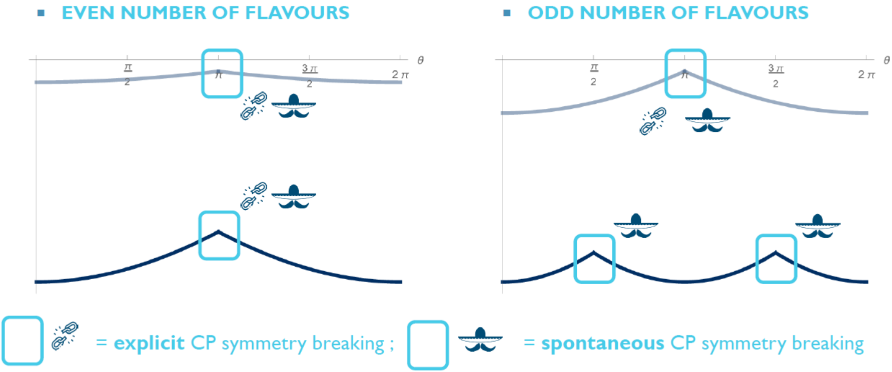
superfluid phase even N_f:

$$0 \leq \cos^2 \frac{\theta}{N_f} \leq \pi \leq \cos^2 \frac{\theta+2\pi(N_f-1)}{N_f} \leq 2\pi$$

superfluid phase odd N_f:

$$0 \leq \cos^2 \frac{\theta}{N_f} \leq \frac{\pi}{2} \leq \cos \frac{\theta+(N_f-1)\pi}{N_f} \leq \frac{3\pi}{2} \leq \cos^2 \frac{\theta-2\pi}{N_f} \leq 2\pi$$

θ-dependence of the energy [1, 5, 6]



two novel phase transitions at $\theta = \pi/2$ and $\theta = 3\pi/2$ in the superfluid phase

Take home messages

2-color QCD EFT at fixed baryon charge and global symmetry $SU(2N_f)$ in the presence of the θ-angle

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2-color QCD EFT at fixed baryon charge and global symmetry $SU(2N_f)$ in the presence of the θ-angle

normal phase:	θ-dependence of the energy is the same for even and odd N_f
superfluid phase:	the ground state energy has two minima for even N_f
	and three new minima for odd N_f
Dashen's phenomenon:	it happens at π for even N_f
	it happens at $\frac{\pi}{2}$ e a $\frac{3\pi}{2}$ for odd N_f

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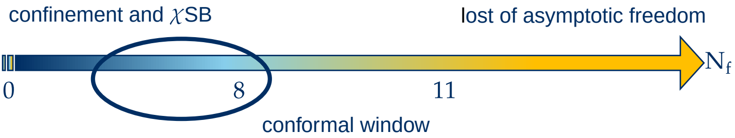


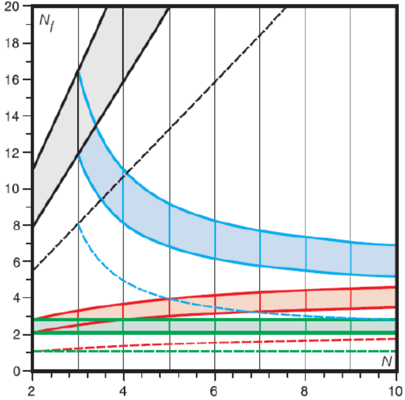
Future directions and interesting investigations

Recently in [11] the authors discovered the breaking of the conformal bound for dense QC_2D by lattice calculations.

- Can we say more when including the physics of the θ-angle?
- Suggestions? Ideas?

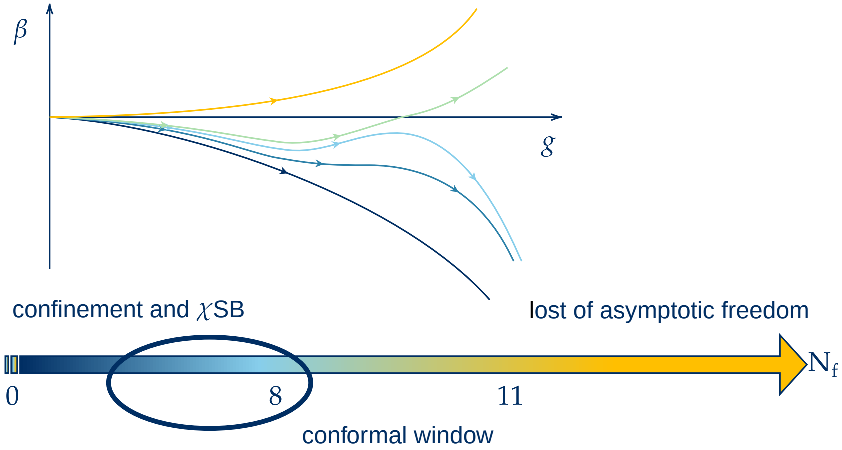
Interesting similar results found for QCD at finite isospin [12, 13]

Conformal Window



- IR dynamics of SU(N) theories depends non-trivially on N_f and N
- In the conformal window the theory features an IR fixed point
- Slightly below the lower edge the physics is still partly controlled by the fixed point: near-conformal dynamics

SU(2) : walking [14, 15]



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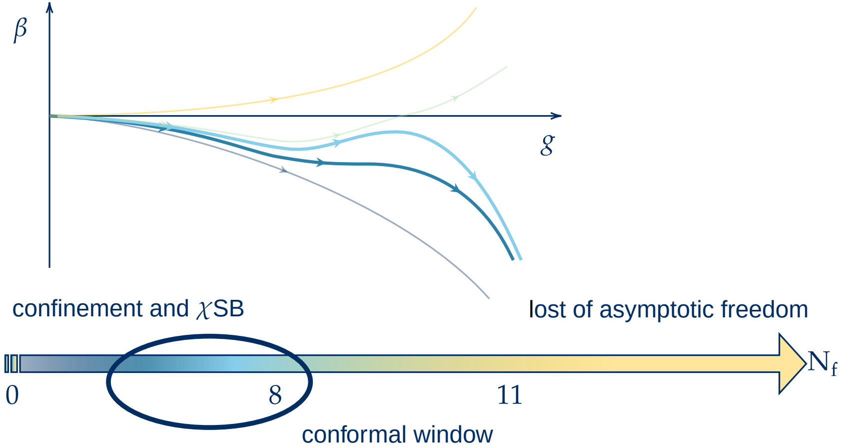

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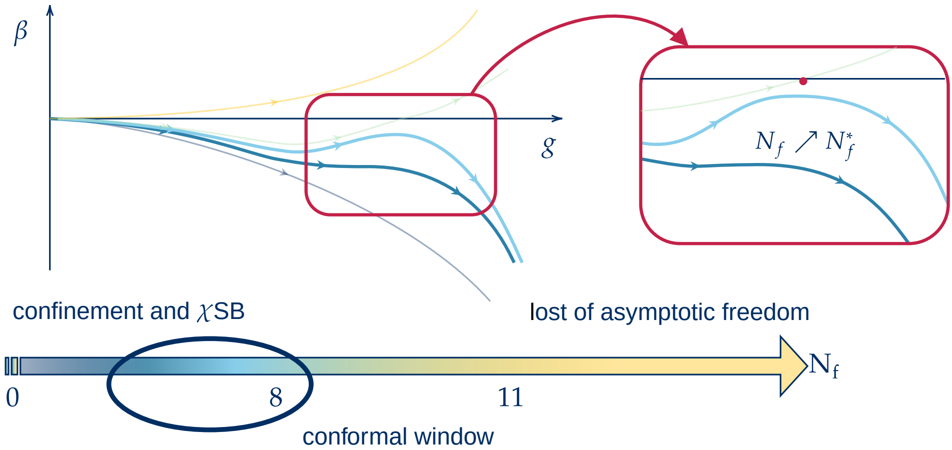

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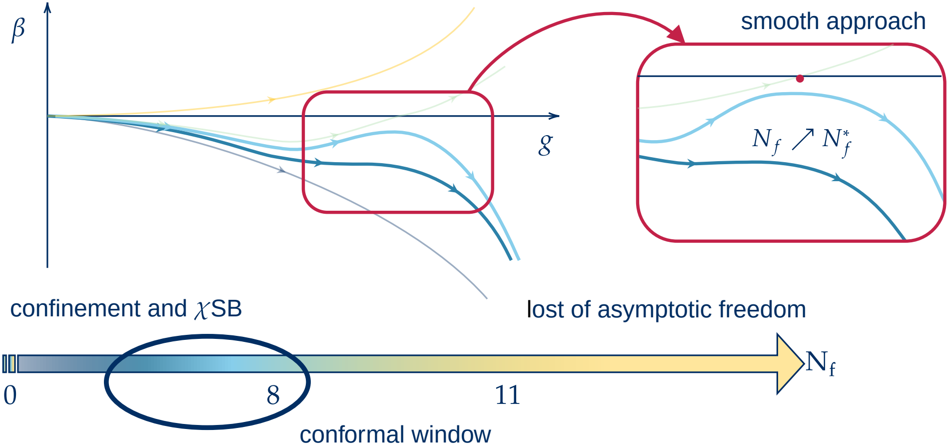

Near conformal dynamics of the theory

EFT of 2-color QCD at fixed baryon charge with global symmetry $SU(2N_f)$ [3] in the presence of the θ-angle [5, 6]

$$\mathcal{L} = \mathcal{L}_\pi + \mathcal{L}_\mu + \mathcal{L}_\theta \tag{56}$$

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$$\tilde{\mathcal{L}}_\theta = -\, e^{-4\sigma f} a\nu^2 \left(\theta - \frac{i}{4}\mathrm{Tr}\{\log\Sigma - \log\Sigma^\dagger\}\right)^2$$

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Charging the conformal window at nonzero θ-angle [2]

Dilaton-EFT of 2-color QCD with global symmetry $SU(2N_f)$ on non-trivial background

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$$\mathcal{M} = \mathbb{R} \times \mathcal{S}^3, \quad \mathrm{V}(\sigma) = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma\ \mathrm{e}^{-2\mathrm{f}\sigma} - \frac{\mathrm{m}_\sigma^2}{16\mathrm{f}^2}\left(4\mathrm{f}\sigma + \mathrm{e}^{-4\mathrm{f}\sigma} - 1\right) [16], \quad \underline{\tilde{\mathcal{L}}_\mathcal{M}} = \Lambda_0 \mathrm{e}^{-4\mathrm{f}\sigma} - \frac{\mathrm{R}^2}{12\mathrm{f}^2}\mathrm{e}^{-2\mathrm{f}\sigma}$$

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study of the ground state energy of the theory in the superfluid phase with semiclassical methods [17]

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study of the ground state energy of the theory in the superfluid phase with semiclassical methods [17]

state on the cylinder in the superfluid phase ⟷ operator with large global charge

approximate state/operator

$E_Q \, \mathcal{R} = \Delta_Q$ [18]

$$E^{\gamma\ll 1} = \frac{c_{4/3}Q^{4/3}}{\tilde{V}^{1/3}} + Q^{2/3}\tilde{V}^{1/3}\Bigg\{c_{2/3}\tilde{R} - \frac{{X_{00}}^2}{4\pi^2 N_f^3 c_{4/3}^4}\left(\frac{9m_\pi^2}{32\nu}\right)^2\Bigg[1-\gamma\Bigg(\frac{2}{3}\log Q - \frac{X_{10}}{X_{00}} - \log\left(\frac{32N_f\nu^2\pi^2 c_{4/3}\tilde{V}^{2/3}}{3}\right)\Bigg)\Bigg]\Bigg\} - \tilde{V}\log Q\Bigg\{\frac{16\pi^2}{9}N_f c_{2/3}c_{4/3}\nu^2 m_\sigma^2 - \frac{\gamma}{3\pi^2 N_f^4 c_{4/3}^5}\left(\frac{9m_\pi^2}{32\nu}\right)^2 \cdot \left[\frac{5}{8\pi^2 c_{4/3}^4 N_f^2}\left(\frac{9m_\pi^2}{32\nu}\right)^2 {X_{00}}^4 - c_{2/3}\tilde{R}N_f {X_{00}}^2 + \frac{9X_{00}X_{01}}{32c_{4/3}}\right]\Bigg\} + (Q^0)$$

$$E^{1-\gamma\ll 1} = \frac{c_{4/3}Q^{4/3}}{\tilde{V}^{1/3}} + c_{2/3}Q^{2/3}\tilde{R}\tilde{V}^{1/3} - \frac{9(1-\gamma){X_{00}}^2 m_\pi^4\tilde{V}\log Q}{64c_{4/3}^3 N_f^2} - \frac{16}{9}\pi^2 m_\sigma^2 N_f c_{2/3}c_{4/3}\nu^2\tilde{V}\log Q + (Q^0)\ ,$$

$$E^{\gamma\ll 1} = \frac{c_{4/3}Q^{4/3}}{\tilde{V}^{1/3}} + Q^{2/3}\tilde{V}^{1/3}\left\{c_{2/3}\tilde{R} - \frac{{X_{00}}^2}{4\pi^2 N_f^3 c_{4/3}^4}\left(\frac{9m_\pi^2}{32\nu}\right)^2\left[1-\gamma\left(\frac{2}{3}\log Q - \frac{X_{10}}{X_{00}} - \log\left(\frac{32N_f\nu^2\pi^2 c_{4/3}\tilde{V}^{2/3}}{3}\right)\right)\right]\right\} - \tilde{V}\log Q\left\{\frac{16\pi^2}{9}N_f c_{2/3}c_{4/3}\nu^2 m_\sigma^2 - \frac{\gamma}{3\pi^2 N_f^4 c_{4/3}^5}\left(\frac{9m_\pi^2}{32\nu}\right)^2 \cdot \left[\frac{5}{8\pi^2 c_{4/3}^4 N_f^2}\left(\frac{9m_\pi^2}{32\nu}\right)^2 {X_{00}}^4 - c_{2/3}\tilde{R}N_f {X_{00}}^2 + \frac{9X_{00}X_{01}}{32c_{4/3}}\right]\right\} + (Q^0)$$

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dove

$$c_{4/3} = \frac{3}{8}\left(\frac{\Lambda^2}{\pi N_f\nu^2}\right)^{2/3}, \quad c_{2/3} = \frac{1}{4f^2}\left(\frac{\pi^2}{N_f\nu^2\Lambda^4}\right)^{1/3}, \quad \tilde{R} = \frac{R}{6} \quad \text{and} \quad \tilde{V} = \frac{V}{2\pi^2}\,, \tag{58}$$

Take home messages

2-color QCD+non-zero baryon charge+ θ-angle

Take home messages

study of the vacuum structure of the theory as a function of the number of flavours

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GSE in inverse powers of the charge Q

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new phases for even and odd flavours[1]

Δ_Q of the lowest charge operator [2]

dressing of the EFT on the cilynder

GSE in inverse powers of the charge Q

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Thank you! ☺

Backup slides

Superfluid N_f odd

We have the solution $\boxed{k_1 = -k_2 + \frac{N_f}{2} - \frac{\theta}{\pi}}$ which can be realized for

$\alpha = \frac{\theta}{N_f}$ $\alpha = \frac{\theta - \pi}{N_f} + \pi$ $\alpha = \frac{\theta - 2\pi}{N_f}$

CP breaking

- Note that when $\mathrm{n} \neq 0$, the vacuum spontaneously breaks $\mathrm{Sp}(2\mathrm{N_f})$ because of the different phases for each quark flavour.
- CP is preserved when $\bar{\theta} = 0$. For equal mass quarks as considered here, this happens when $\mathrm{m}_\pi = 0$ or $\theta = 0$.
- For $\theta = \pi$ the Lagrangian possess CP symmetry but in the normal phase the latter is spontaneously broken by the vacuum [Dashen:1970et,DiVecchia:2013swa,Gaiotto:2017tne,DiVecchia:2017xpu] , leading to a strong θ-dependence near $\theta = \pi$.

Symmetry breaking pattern & Spectrum

$SU(2N_f) \times U(1)_A \overset{2N_f^2 - N_f}{\rightsquigarrow} Sp(2N_f)$

$a,\ m$ ↘ ↓ μ

$SU(N_f)_V \times U(1)_B$

superfluid phase ↓ $\frac{N_f^2 - N_f}{2}$

$Sp(N_f)_V$

$$\omega_1^2 = k^2 + \mu^2\,, \qquad \frac{1}{2}N_f(N_f+1)$$

$$\omega_2^2 = k^2 + \frac{m_\pi^4 X^2}{\mu^2 N_f^2}\,, \qquad \frac{1}{2}N_f(N_f-1) - 1$$

$$\omega_3^2 = k^2 + \frac{2\left(\mu^4 N_f^2 + 3m_\pi^4 X^2\right)}{N_f^2 \mu^2} + A\,, \qquad \bullet + \frac{1}{2}N_f(N_f-1)$$

$$\omega_4^2 = k^2 + \frac{2\left(\mu^4 N_f^2 + 3m_\pi^4 X^2\right)}{N_f^2 \mu^2} - A\,, \qquad \bullet + \frac{1}{2}N_f(N_f-1)$$

$$\omega_5^2 = k^2 + M_S^2\,, \qquad \bullet \quad 1$$

where

$$A = \frac{2}{N_f^2 \mu^2}\sqrt{\left(N_f^2 \mu^4 + 3m_\pi^4 X^2\right)^2 + 4N_f^2 \mu^2 m_\pi^4 k^2 X^2}\,, \tag{59}$$

$$M_S^2 = \frac{a\mu^4 N_f^3 + 2\mu^2 m_\pi^4 X^2}{2\mu^4 N_f^2 - 2m_\pi^4 X^2}\left(1 - \frac{m_\pi^4 X^2}{\mu^2 N_f^2}\right) \tag{60}$$

Large charge setup

We will consider our system on a manifold $\mathcal{M}$ with volume V and curvature R such that the underlying new scale of the theory is

$$\Lambda_{\mathrm{Q}} = (\mathrm{Q}/\mathrm{V})^{1/3} \tag{61}$$

where Q is the fixed baryon charge.
Concretely, we will take our manifold to be

$$\mathcal{M} = \mathbb{R} \times \mathrm{S}^{\mathrm{d}-1} \tag{62}$$

such that we can consider an approximate state-operator correspondence that implies

$$\Delta_{\mathrm{Q}} = \tilde{\mathrm{V}}^{1/3}\mathrm{E}_{\mathrm{Q}}\,, \qquad \mathrm{E}_{\mathrm{Q}} = \mu\mathrm{Q} - \mathcal{L} \tag{63}$$

where Δ_{Q} is the scaling dimension of the lowest-lying operator with baryon charge Q, E_{Q} is the ground state energy on $\mathbb{R} \times \mathrm{S}^{\mathrm{d}-1}$ at fixed charge, $\tilde{\mathrm{V}}^{1/3}$ is the radius of $\mathrm{S}^{\mathrm{d}-1}$.

Large charge expansion of the θ-angle physics

We double-expanded X first in γ and then also in $1/Q$ as follows

$$X = X_0 + X_1\gamma + (\gamma^2) \ , \qquad X_k = X_{k0} + \frac{X_{k1}}{Q^{2/3}} + \left(Q^{-4/3}\right) \ , \qquad \text{for } \gamma \ll 1$$

$$X = X_0 + X_1(1-\gamma) + ((1-\gamma)^2) \ , \qquad X_k = X_{k0} + \frac{X_{k1}}{Q^{4/3}} + \left(Q^{-2}\right) \ , \qquad \text{for } 1-\gamma \ll 1 \ .$$

where

$$X_{00} = N_f \cos\left(\frac{\theta + 2k\pi}{N_f}\right)$$

$$X_{01} = \frac{9m_\pi^4 \sin^2\left(\frac{\theta+2k\pi}{N_f}\right)\cos\left(\frac{\theta+2k\pi}{N_f}\right)}{8\ 2^{2/3}\pi^{4/3}a\ c_{4/3}^2}$$

$$X_{10} = 0$$

$$X_{11} = 0$$

$$\bar{\theta}_{00} = 0$$

$$\bar{\theta}_{01} = \frac{m_\pi^2 X_{00}\sin\left(\frac{\theta+2\pi k}{N_f}\right)}{aN_f}$$

$$\bar{\theta}_{10} = 0$$

$$\bar{\theta}_{11} = \frac{3m_\pi^2 \sin\left(\frac{2(\theta+2\pi k)}{N_f}\right)\log\left(\frac{8192\pi^2 c_{4/3}^3 N_f^3 v^6}{27Q^2}\right)}{32\ 2^{2/3}\pi^{4/3}a\ c_{4/3}^2}$$

EOMs

Evaluating the lagrangian (57) on the vacuum ansatz

$$\mathcal{L}_{\theta,\sigma}\left[\Sigma_0,\sigma_0\right] = -e^{-4f\sigma_0}\left(\Lambda^4 - \frac{m_\sigma^2}{16f^2}\right) - \frac{m_\sigma^2\left(4f\sigma_0 + e^{-4f\sigma_0} - 1\right)}{16f^2} - \frac{R\, e^{-2f\sigma}}{12f^2} + \\ + 4m_\pi^2\nu^2 X\cos\varphi\; e^{-f\sigma_0 y} + 2\mu^2 N_f\nu^2 e^{-2f\sigma_0}\sin^2\varphi - a\nu^2 e^{-4f\sigma_0}\bar{\theta}^2\,, \tag{64}$$

where

$$\bar{\theta} \equiv \theta - \sum_i^{N_f}\alpha_i\,, \qquad X \equiv \sum_i^{N_f}\cos\alpha_i\,, \qquad \Lambda^4 \equiv \Lambda_0^4 + \frac{m_\sigma^2}{16f^2}\,. \tag{65}$$

The respective equations of motion are

$$N_f\mu^2 e^{-2f\sigma}\cos\varphi - m_\pi^2 X e^{-f\sigma y} = 0 \tag{66}$$

$$a e^{-4f\sigma}\bar{\theta} - 2m_\pi^2\sin\alpha_i\cos\varphi e^{-f\sigma y} = 0\,, \qquad i = 1,..,N_f \tag{67}$$

$$\frac{Re^{-2f\sigma}}{6f} + 4af\nu^2 e^{-4f\sigma}Y^2 + 4f\Lambda_0^4 e^{-4f\sigma} - \frac{m_\sigma^2\left(1 - e^{-4f\sigma}\right)}{4f} + \\ -4f\mu^2 N_f\nu^2 e^{-2f\sigma}\sin^2\varphi - 4fm_\pi^2\nu^2 yX\cos\varphi e^{-f\sigma y} = 0 \tag{68}$$

$$4\mu N_f\nu^2 e^{-2f\sigma}\sin^2\varphi = \frac{Q}{V}\,. \tag{69}$$

Δ_Q

- $\gamma \ll 1$

$$\frac{\Delta_Q}{\Delta_Q^*} = 1 - \left(\frac{9m_\pi^2}{32\pi\nu}\right)^2 \frac{1 - \gamma \log\left(\frac{3\rho^{2/3}}{16(2\pi^2)^{1/3} c_{4/3}\nu^2 N_f}\right)}{4c_{4/3}^5 N_f} \cos^2\left(\frac{\theta + 2\pi k}{N_f}\right)\left(\frac{1}{2\pi^2\rho}\right)^{2/3}$$
$$+ \frac{\gamma}{c_{4/3}^6 N_f}\cos^2\left(\frac{\theta+2\pi k}{N_f}\right)\left(\frac{27m_\pi^4 \sin^2\left(\frac{\theta+2\pi k}{N_f}\right)}{256\ 2^{2/3}\pi^{4/3} a\ c_{4/3}^3 N_f^2} + \frac{5\left(\frac{9m_\pi^2}{64\pi\nu}\right)^2 \cos^2\left(\frac{\theta+2\pi k}{N_f}\right)}{6c_{4/3}^4 N_f} - \frac{c_{2/3}}{2}\left(\frac{\rho}{2\pi^2 Q}\right)^{2/3}\right)$$
$$\times \left(\frac{9m_\pi^2}{32\pi\nu}\right)^2 \left(\frac{1}{2\pi^2\rho}\right)^{4/3} \log Q - \frac{16}{9}\pi^2 c_{2/3}\nu^2 N_f m_\sigma^2 \left(\frac{1}{2\pi^2\rho}\right)^{4/3} \log Q$$

- $(1-\gamma) \ll 1$

$$\frac{\Delta_Q}{\Delta_Q^*} = 1 - \left(\frac{9m_\pi^4}{64c_{4/3}^4}(1-\gamma)\cos^2\left(\frac{\theta+2\pi k}{N_f}\right) + \frac{16}{9}\pi^2 c_{2/3}\nu^2 N_f m_\sigma^2\right)\left(\frac{1}{2\pi^2\rho}\right)^{4/3} \log Q$$

Spectrum

$$\mathrm{SU}(2\mathrm{N_f}) \times \mathrm{U}(1)_\mathrm{A} \overset{2\mathrm{N_f^2}-\mathrm{N_f}}{\rightsquigarrow} \mathrm{Sp}(2\mathrm{N_f}) \longrightarrow \mathrm{SU}(\mathrm{N_f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{B} \overset{\frac{\mathrm{N_f^2}-\mathrm{N_f}}{2}}{\rightsquigarrow} \mathrm{Sp}(\mathrm{N_f})_\mathrm{V} \tag{70}$$

Having in mind the hierarchy of scales $\mathrm{m} \ll \sqrt{\mathrm{a}} \leq \mu \ll 4\pi\nu$, we focus on the spectrum of light modes

- $\frac{1}{2}\mathrm{N_f}(\mathrm{N_f}-1)$ massless Goldstones: $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} + \bullet$ of $\mathrm{Sp}(\mathrm{N_f})$
- 1 pseudo-Goldstone $\bullet$ of $\mathrm{Sp}(\mathrm{N_f})$ with mass $\propto \sqrt{\mathrm{a}}$

> the spectrum changes when (near)conformal dynamics is realized through the dilaton dressing

we expand around the vacuum solution as follows

$$\Sigma = e^{i\Omega}\Sigma_0 e^{i\Omega^t} \quad \text{where} \quad \Omega = \begin{pmatrix} \pi & 0 \\ 0 & -\pi^t \end{pmatrix} + \tilde{\beta} S \begin{pmatrix} 1_{\mathrm{N_f}} & 0 \\ 0 & 1_{\mathrm{N_f}} \end{pmatrix}, \quad \tilde{\beta} \equiv \frac{1}{\sqrt{2\mathrm{N_f}}}, \quad \pi = \sum_{a=0}^{\dim \frac{\mathrm{U}(\mathrm{N_f})}{\mathrm{Sp}(\mathrm{N_f})}} \pi^a T_a$$

Spectrum

$$\frac{\tilde{\mathcal{L}}}{4\nu^2 \sin^2\varphi\, e^{-2\sigma_0 f}} = \begin{pmatrix} \pi^0 & \hat{\sigma} & S \end{pmatrix} D^{-1} \begin{pmatrix} \pi^0 \\ \hat{\sigma} \\ S \end{pmatrix} + \sum_{a=1}^{\dim(\square)} \partial^\mu \pi^a \partial_\mu \pi^a \tag{71}$$

with the inverse propagator D^{-1} defined as

$$D^{-1} = \begin{pmatrix} \omega^2 - k^2 & i\omega\mu f\sqrt{2N_f} & 0 \\ -i\omega\mu f\sqrt{2N_f} & \frac{\omega^2 - k^2}{8\nu^2 \sin^2\varphi} - M_\sigma^2 & \frac{1}{2} I_{\hat{\sigma}S} \\ 0 & \frac{1}{2} I_{\hat{\sigma}S} & \frac{(\omega^2 - k^2)}{\sin^2\varphi} - M_S^2 \end{pmatrix}, \qquad I_{\hat{\sigma}S} = \frac{\sqrt{2} f \mu^2 m_\pi^4 \sqrt{N_f} XyZ}{m_\pi^4 X^2 - \mu^4 N_f^2 e^{2f\sigma_0(y-2)}} \tag{72}$$

where $Z \equiv \sum_{i=1}^{N_f} \sin\alpha_i$ and the Lagrangian masses for the dilaton-field and the S mode are given by

$$M_\sigma^2 = -\frac{f^2 \mu^2 N_f e^{-6f\sigma_0} \left(\nu^2 m_\pi^4 X^2 \left(y^2 - 2\right) e^{6f\sigma_0} + 2\mu^4 \nu^2 N_f^2 e^{2f\sigma_0(y+1)} - 4\Lambda^4 \mu^2 N_f e^{2f\sigma_0 y}\right)}{2\nu^2 \left(\mu^4 N_f^2 e^{2f\sigma_0(y-2)} - m_\pi^4 X^2\right)} \tag{73}$$

$$M_S^2 = \frac{a\mu^4 N_f^3 e^{2f\sigma_0(y-1)} + 2\mu^2 m_\pi^4 X^2 e^{4f\sigma_0}}{2\mu^4 N_f^2 e^{2f\sigma_0 y} - 2m_\pi^4 X^2 e^{4f\sigma_0}}\,. \tag{74}$$

Spectrum

conformal invariance dictates the existence of a massless mode with speed $v_G = \frac{1}{\sqrt{d-1}} = \frac{1}{\sqrt{3}}$ [19]

→ mixing between the singlet mode π_0 with the dilaton

→ change π_0 velocity from $v_G = 1 \to v_G = \frac{1}{\sqrt{3}}$

In the large-charge limit, the above reduces to

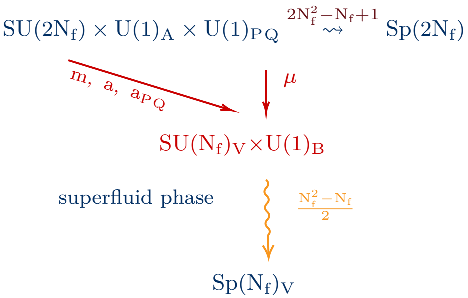
$$\gamma \ll 1: \quad \omega_2 = k\left[\frac{1}{\sqrt{3}} + \frac{\sqrt{3}\, X_{00}{}^2}{(2\pi^2)^{2/3} c_{4/3}^5 N_f^3}\left(\frac{9m_\pi^2}{128\pi\nu}\right)^2 \left(\frac{V}{Q}\right)^{2/3} + \dots\right] + \mathcal{O}(k^2)$$

$$(1-\gamma) \ll 1: \quad \omega_2 = k\left[\frac{1}{\sqrt{3}} + 1\left(\frac{2^{5/3} c_{2/3} \nu^2 m_\sigma^2}{3\sqrt{3}\pi^{2/3}} + \frac{9\sqrt{3} m_\pi^4 X_{00}{}^2}{128\sqrt[3]{2}\pi^{8/3} c_{4/3}^4 N_f^2}\right)\left(\frac{V}{Q}\right)^{4/3} + \dots\right] + \mathcal{O}(k^2)$$

Axion

We denote by ν_{PQ} the scale of $U(1)_{PQ}$ spontaneous symmetry breaking and by a_{PQ} the coefficient of the $U(1)_{PQ}$ anomalous term.

$$\mathcal{L}_{\hat{a}} = \nu^2 \mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} + \nu_{PQ}^2 \partial_\mu N \partial^\mu N^\dagger + 4\mu\nu^2 \mathrm{Tr}\{B\Sigma^\dagger \partial_0 \Sigma\} + m_\pi^2 \nu^2 \mathrm{Tr}\{M\Sigma + M^\dagger \Sigma^\dagger\} + 2\mu^2\nu^2 \left[\mathrm{Tr}\{\Sigma B^T \Sigma^\dagger B\} + \mathrm{Tr}\{BB\}\right] - a\nu^2 \left(\theta - \frac{i}{4}\mathrm{Tr}\{\log\Sigma - \log\Sigma^\dagger\} - \frac{i}{4} a_{PQ}(\log N - \log N^\dagger)\right)^2 . \tag{75}$$



$$D^{-1} = \begin{pmatrix} \frac{\omega^2 - k^2}{\sin^2\varphi} - M_S^2 & -\frac{a\sqrt{N_f}a_{PQ}}{4\sqrt{2}\nu_{PQ}\sin^2\varphi} \\ -\frac{a\sqrt{N_f}a_{PQ}}{4\sqrt{2}\nu_{PQ}\sin^2\varphi} & \frac{\omega^2 - k^2}{4\nu^2\sin^2\varphi} - M_{\hat{a}}^2 \end{pmatrix}, \tag{76}$$

where

$$M_S^2 = \frac{(a\mu^4 N_f + 2\mu^2 m_\pi^4)}{2\mu^4 - 2m_\pi^4} \tag{77}$$

$$M_{\hat{a}}^2 = \frac{a\mu^4 a_{PQ}^2}{16\nu_{PQ}^2(\mu^4 - m_\pi^4)} . \tag{78}$$