Ryu-Takayanagi formula and beyond

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Holography: a short historical view

In **1993-94** 't Hooft [8] and Susskind [14] made a bold statement about QG:

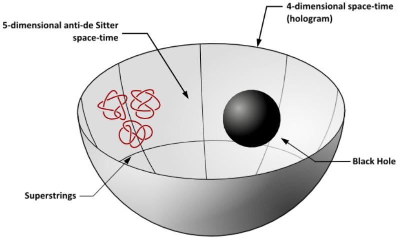
Statement

$$QG_{d+2} \underset{d.o.f.}{\rightleftarrows} QFT_{d+1}$$

An idea from the Bekenstein-Hawking [6, 2] formula

$$S_{BH} = \frac{\text{Area}(\Sigma)}{4G}$$

In **1997** Maldacena [11] made a concrete realization of such a statement, which then became



Statement

$$AdS_{d+2} \rightleftarrows CFT_{d+1}$$

AdS/CFT: a sketch

CFT

Conformal maps $x \to f(x)$ are those which that preserves angles

$$J^\mu_\nu(x) = \frac{\partial f^\mu(x)}{\partial x^\nu} = R^\mu_\nu(x)\Omega(x)$$

In QFT the prescription of conformality resides within

$$\langle\prod_{i=1}^{n}\pi_f(\mathcal{O}_i)(x_i)\rangle = \langle\prod_{i=1}^{n}\mathcal{O}_i(x_i)\rangle$$

The conformal group is locally isomorphic to

$$SO(d+1,2)$$

AdS

It is a solution for the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi G}(\mathcal{R} - 2\Lambda)$$

for $\Lambda < 0$.

It has a large isometry group

$$SO(d+1,2)$$

A possible global metric is given by

$$ds^2 = \frac{R^2}{\cos^2 r}(-dt^2 + dr^2 + sin^2 r\ d\Omega_d^2)$$

Visual AdS_3/CFT_2

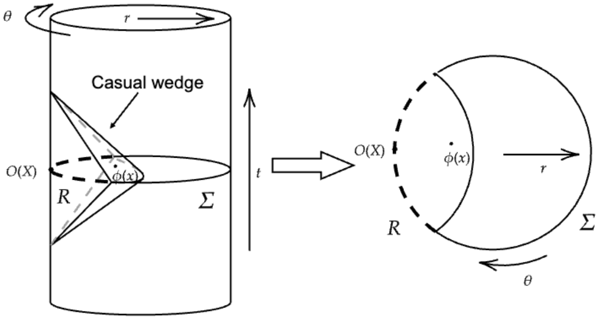


Figure: Example of bulk field reconstruction in the casual wedge.

Entanglement entropy

Some intuition about entanglement entropy

$$S(\rho) = -tr\rho \log \rho$$

considering a *discretized space*.

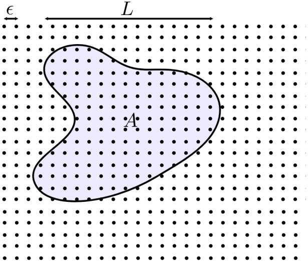


Figure: A region A of size L in a lattice system with lattice spacing ϵ [7].

The full Hilbert space

$$\mathcal{H} = \bigotimes_{\vec{x}} \mathcal{H}_{\vec{x}} \to \mathcal{H}_A \otimes \mathcal{H}_B$$

for a physical state[16], the entropy is

$$S(\rho_A) \propto \left(\frac{L}{\epsilon}\right)^{d-1}$$

with $\rho_A = tr_B \rho$.

Entanglement

Why do we even care about entanglement in QFT?
Isn't that just a correlation?

There are a few reasons why we should care:

- **Characterizing phases** [4, 9, 10] of many-body systems beyond Landau-Ginzburg paradigm
  - Confinement/deconfinement phase transition
  - Topological order
- **Reeh-Schlieder's theorem** [12]: entanglement is not just a property of the states but of the algebras of observables [15]
  - Ultraviolet divergence
- Microscopic understanding of the **black hole entropy**
- [5]...

Ryu-Takayanagi prescription

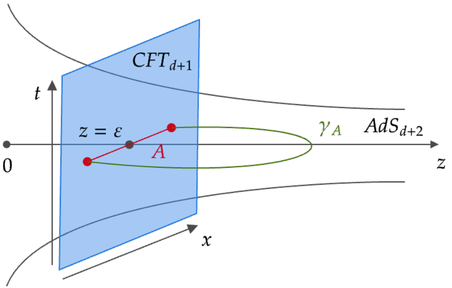


Figure: Schematic of the correspondence with metric: $ds^2 = R^2 \frac{-dx_0^2 + dz^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$.

RT formula [13]

$$S(\rho_A) = \min_{\gamma_A | \partial\gamma_A = \partial A} \frac{Area(\gamma_A)}{4G}$$

Concrete example AdS_3/CFT_2

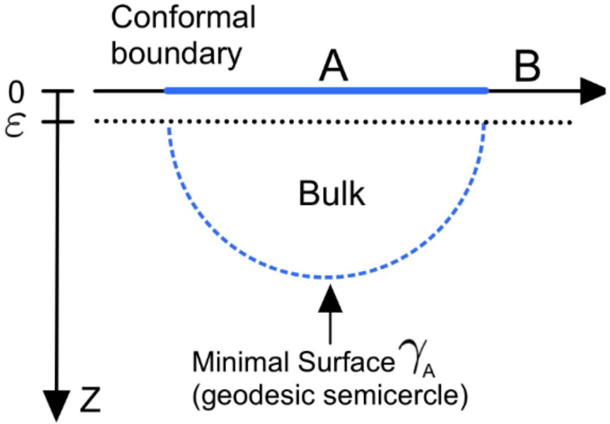


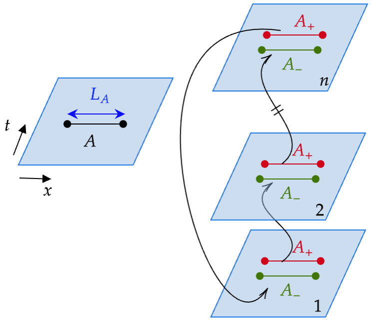
Figure: Minimal surfaces γ_A in the bulk for AdS_3.

Some calculations

CFT_2

Replica trick [4]

$$S(\rho_A) = \lim_{n\to 1} \frac{\log tr_A \rho_A^n}{1-n} = \frac{c}{3}\log\left(\frac{L_A}{\epsilon}\right)$$



AdS_3

Given the Poincarè line element

$$ds^2 = R^2 \frac{-dx_0^2 + dx^2 + dz^2}{z^2}$$

for $x_0 = const$ the geodesic action looks like

$$I = R\int d\xi \frac{\sqrt{x'(\xi)^2 + z'(\xi)^2}}{z(\xi)}$$

and its extremal solutions lead to

$$\frac{I_{ext}}{4G} \equiv \frac{Length(\gamma_A)}{4G} = \frac{R}{2G}\log\left(\frac{L_A}{\epsilon}\right)$$

CORRESPONDENCE CHECK! [3]

A Black hole case

CFT_2 at finite temperature [4]

Using the same methodology as before for $T = \beta^{-1}$

$$S(\rho_A) = \frac{c}{3} \log\left(\frac{\beta}{\pi\epsilon} \sinh \frac{\pi L_A}{\beta}\right)$$

BTZ black hole [1]

Given the Euclidean line element

$$ds^2 = (r^2 - r_+^2) d\tau^2 + \frac{R^2}{r^2 - r_+^2} dr^2 + d\phi^2$$

The geodesic distance can be found as

$$\cosh\left(\frac{Area(\gamma_A)}{R}\right)$$
$$= 1 + \frac{2\beta^2}{\epsilon^2} \sinh^2\left(\frac{\pi L_A}{\beta}\right)$$

hence obtaining

$$S(\rho_A) \simeq \frac{R}{2G} \log\left(\frac{\beta}{\pi\epsilon} \sinh \frac{\pi L_A}{\beta}\right)$$

What does it happen?

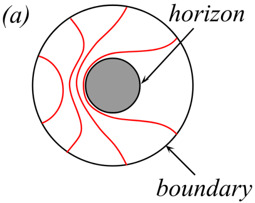


Figure: Minimal surfaces γ_A in the BTZ black hole for various sizes of A.

Take home messages

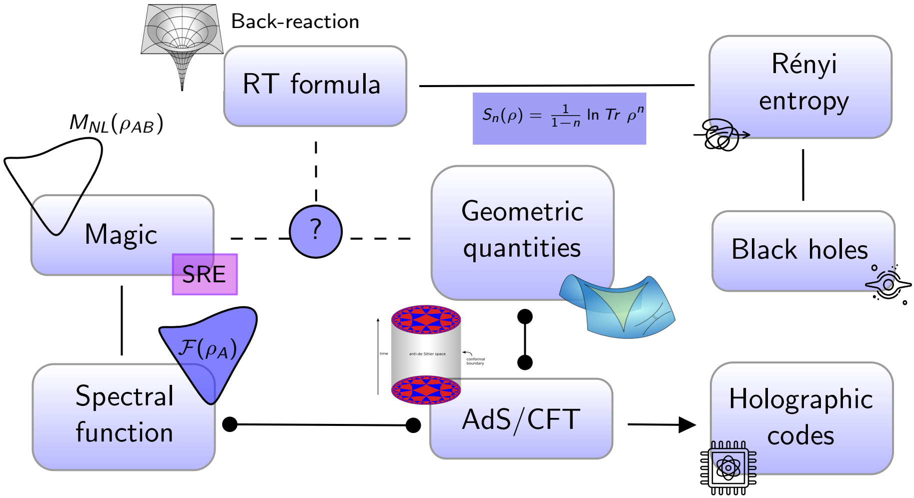
Key message I

It is possible to realize a mapping between quantum information quantities and geometrical gravitational quantities

Key message II

Entanglement plays a pivotal role as a key tool for understanding quantumness

A broad view


Back-reaction
RT formula
Rényi entropy
$S_n(\rho) = \frac{1}{1-n} \ln Tr\ \rho^n$
$M_{NL}(\rho_{AB})$
Magic
?
Geometric quantities
Black holes
SRE
$\mathcal{F}(\rho_A)$
Spectral function
time
anti-de Sitter space
conformal boundary
AdS/CFT
Holographic codes

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