Moduli Spaces in CFT: Large Charge Operators

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G. Cuomo, L. Rastelli, A. Sharon

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Motivation and Summary

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- Large charge expansion in CFTs with global symmetries

$$\Rightarrow \quad \Delta_{\text{min}}(Q) \propto Q^{\frac{d}{d-1}}, \quad Q \gg 1$$

- What if conformal invariance is spontaneously broken and(!) the global charge too?

$$\Rightarrow \quad \Delta_{\text{min}}(Q) = \alpha_0 Q + \ldots \ , \quad Q \gg 1$$

- This result is <u>independent of the details of the theory</u> and of supersymmetry
- Main tools: NGBs EFT & large charge

Some background, part 1
QFTs at Large Charge

- Idea: physical systems characterised by large quantum numbers admit a semiclassical treatment

- Fixing the charge: $\mathscr{H} \quad \rightarrow \hat{\mathscr{H}} = \mathscr{H} + \underline{\mu} Q, \qquad Q = \int d^{d-1}x \; j_0(x)$

chemical potential

$$\text{g.s.:} \quad |0> \quad \text{s.t.} \quad \hat{\mathscr{H}}\,|0> = 0$$

$$\Rightarrow \quad \text{if} \quad Q\,|0> \neq 0, \quad \text{then} \quad \mathscr{H}\,|0> \neq 0$$

there is then a SSB mechanism at work: $\underbrace{Q,\ \mathscr{H}}_{\text{broken}} \quad \rightarrow \quad \underbrace{\hat{\mathscr{H}}}_{\text{conserved}}$

... on the cylinder: $D,\ Q \quad \rightarrow \quad \hat{D}$

Some background, part 2: Moduli effective theory

- What we are working with: CFT w/ SSB of conformal symmetry + internal symmetry G

$$G \xrightarrow{\text{SSB}} H$$

$$\Rightarrow \quad \text{moduli space:} \quad \Phi \quad \text{(dilaton)} \quad + \quad \dim(G/H) \quad \text{NGBs}$$

- For $E \ll m$ (gap of massive states) the dynamics of the theory is described by the EFT of the NGBs
- EFT of these modes constrained by non-linear realisation of conformal symmetry and of the spontaneously broken internal symmetry

… in practice:

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- We write the EFT imposing Weyl invariance
- Introduce: $\Phi = f^{-1} e^{f\sigma(x)}$

transforms as a scalar field in d-dimensions

$$\sigma \xrightarrow{x \to e^{\alpha} x} \sigma - \frac{\alpha}{f}$$

the actual dilation

- Other NGBs: $\pi^a, \quad a = 1,\ldots,\dim(G/H)$ (non-SUSY case)
- In flat space:

all other m=0 states e.g. strongly coupled sector

$$\mathcal{S}_{EFT} = \int d^d x \left[\frac{1}{2} G_{ab}(\Phi, \pi)\partial_\mu \pi^a \partial^\mu \pi^b + \frac{1}{2} G_{\Phi\Phi}(\pi)(\partial\Phi)^2 + G_{a\Phi}\partial_\mu \pi^a \partial^\mu \Phi\right] + \mathcal{S}_{CFT_{IR}}$$

An important note:

- The absence of relevant couplings in $\mathcal{S}_{EFT}$ is important!
- Conformal invariance always admits a cosmological constant term:

$$\int d^d x \sqrt{g}\, |\Phi|^d f(\phi)$$

constrained by the internal symmetry
(might just be a constant)

- Such a term creates a potential for the dilaton
- For the CFT to admit a moduli space all relevant couplings of the dilaton must be zero
- We will use $\mathcal{S}_{EFT}$ self-consistently w/o worrying about the mechanism that sets these terms to zero (be it SUSY or fine-tuning)

Derivation of leading order result

Setup: d – dim. CFT $\qquad \mathscr{M} = \mathbb{R} \times S^{d-1}$ (Lorentzian)

Spontaneously broken conf. inv. $\Rightarrow \quad \Phi$

$$G \xrightarrow{SSB} H \qquad\qquad Q^a \to \pi^a$$

broken charges $\qquad$ NGBs

$$\mathcal{S}_{EFT} = \int d^d x \sqrt{g} \left[\frac{\Phi^2}{2} \hat{G}_{ab} \partial_\mu \pi^a \partial^\mu \pi^b + \frac{\hat{G}_{\Phi\Phi}}{2} (-\Phi \partial^2 \Phi - m_d^2 \Phi^2) \right] + \ldots$$

$$m_d^2 = \frac{(d-2)^2}{4R^2}$$

conformal mass of the dilaton

- We look for a solution to the EOMs with the ansatz

$$\Phi = v^{\frac{d-2}{2}} \qquad \pi^a = \underbrace{m^a}_{\text{chemical potential}} t/R$$

- Describes a solution where both time translations D/R and Q^a are spontaneously broken, leaving $D - m^a Q^a$ unbroken... like in large charge w/o moduli space!
- ... but this time from the EOMs we find:

$$j_a^0 = \frac{Q_a}{R^{d-1}\Omega_{d-1}} = \hat{G}_{ab} m^b v^{d-1}/R \xrightarrow{Q^a = \delta_1^a Q} v = \frac{1}{R}\left(\frac{Q}{\Omega_{d-1}\hat{G}_{1b}m^b}\right)^{\frac{1}{d-2}}$$

$$\xrightarrow[\text{corresp.}]{\text{state-op.}} \Delta_{min}(Q) = R\int d^{d-1}x\sqrt{g}\,T_{00} =$$

$$= R\int d^{d-1}x\sqrt{g}\left(\frac{j_0^a m^a}{R} - \underbrace{\cancel{\mathscr{L}}}_{\text{0 on the EOMs}}\right) =$$

$$= \alpha_0 Q$$

$\alpha_0 = \mathcal{O}(1)$ coeff. determined by the W.C. of the EFT

- This is a large charge result: analysis is reliable as long as the dilaton vev is much larger than the derivatives of the Goldstone field

$$v \sim Q^{\frac{1}{d-1}}/R \gg |\partial\pi^a| \sim m^a/R \qquad \Rightarrow \qquad Q \gg 1$$

- **Example:** free th. of a charged scalar ϕ

$$\mathscr{L}_{EFT} = |\partial\phi|^2 - m_d^2|\phi|^2, \qquad \phi = \frac{\Phi e^{-i\pi}}{\sqrt{2}}$$

$$\Rightarrow \qquad \phi_0 = e^{-im_d t} v^{\frac{d-2}{2}}$$

$$j^0 = 2m_d v^{d-2}, \quad T_{00} = m_d j^0 \quad \Rightarrow \quad \Delta_{min}(Q) = \frac{d-2}{2}Q$$

Quantum corrections

- Expand around the classical solution

$$\Phi = v^{\frac{d-2}{2}} + \delta\Phi, \qquad \pi^a = \frac{m^a t}{R} + \frac{\delta\pi^a}{v^{\frac{d-2}{2}}}$$

- From the quadratic Lagrangian for the perturbations:

$$\omega_k(\ell) = \sqrt{\frac{n_k^2}{R^2} + J_\ell^2}, \qquad J_\ell^2 = \frac{\ell(\ell + d - 2)}{R^2}, \qquad n_k^2 \sim \mathcal{O}(1)$$

$$\Rightarrow \quad \delta\Delta^{(1-loop)} = \frac{R}{2}\sum_{k=1}^{n}\sum_{\ell} \deg(\ell)\omega_k(\ell)$$

$$\delta\Delta^{(1-loop)} = \begin{cases} \alpha_0 & d \text{ odd} \\ \alpha_0 \log Q & d = 4 \end{cases}$$

no C.T.s $\Rightarrow$ the sum must be finite

Correlation functions

- The EFT can also be used to compute correlation functions
- A neutral operator w/ scaling dimension $\Delta \sim \mathcal{O}(1)$ is represented in the EFT in terms of the dilaton:

$$\mathcal{O} = c_{\mathcal{O}} \Phi^{\frac{2\Delta}{d-2}} + \dots$$

$$\Rightarrow \quad \lambda_{Q\mathcal{O}Q} = < Q | \mathcal{O} | Q > \overset{<\Phi> = v^{\frac{d-2}{2}}}{\simeq} c_{\mathcal{O}} v^{\Delta}$$

$$\Rightarrow \quad \lambda_{Q\mathcal{O}Q} \propto Q^{\frac{\Delta}{d-2}}$$

Approximate moduli in non-SUSY theories

- Exact moduli spaces are hard to construct in interacting theories w/o SUSY
- BUT: several models admit a situation where the dilaton potential is <u>parametrically suppressed:</u>

$$\delta^{\frac{4}{d-2}} \int d^d x \sqrt{g}\, |\Phi|^d f(\phi), \qquad \delta \to 0$$

- It is found:

$$\Delta_{min}(Q) \propto \begin{cases} Q, & 1 \ll g^2 Q \ll \frac{1}{\delta^2} \\ \frac{(\delta^2 g^2 Q)^{\frac{d}{d-1}}}{\delta^2 g^2}, & g^2 Q \gg \frac{1}{\delta^2} \end{cases}$$

...an example:

- A model displaying this behaviour is the scalar $O(N) \times O(N)$ CFT in $2 < d < 4$ with potential

$$V(\vec{\phi}_1^{\,2}, \vec{\phi}_2^{\,2}) = \frac{\lambda_{11}}{8N}(\vec{\phi}_1^{\,2})^2 + \frac{\lambda_{22}}{8N}(\vec{\phi}_2^{\,2})^2 + \frac{\lambda_{12}}{4N}(\vec{\phi}_1^{\,2})(\vec{\phi}_2^{\,2})$$

- Fixed point: $\lambda_{11} = \lambda_{22} = -\lambda_{12} = \lambda > 0, \quad \lambda = 16\pi^2\epsilon, \quad d = 4 - \epsilon$

$$V|_{F.P.} = \frac{\lambda}{8N}(\vec{\phi}_1^{\,2} - \vec{\phi}_2^{\,2})^2 \quad \Rightarrow \quad \text{flat direction:} \quad <\vec{\phi}_1^{\,2}> = <\vec{\phi}_2^{\,2}>$$

$\Rightarrow$ There will be operators whose scaling dimension obey

$$\Delta(Q) = \alpha Q + \ldots, \quad \text{for} \quad 1 \ll \frac{\epsilon Q}{N} \ll N$$

Comments on the case w/o broken charges

- The presence of broken global charges is an important hypothesis
- What about the case with no broken charges?
- Spoiler: we don't find a tower of states s.t. $\Delta \propto Q$, but there might be resonant states
- Basic idea: consider the EFT of a real dilaton on the cylinder

$$\mathcal{S} = \int d^d x \left[\frac{1}{2}(\partial\Phi)^2 - \frac{m_d^2}{2}\Phi^2 \right] + \dots$$

$$\xrightarrow{EOM} \quad \Phi = \frac{v^{\frac{d-2}{2}}}{\sqrt{2}} \cos(m_d t + t_0) \qquad\qquad v^{d-2} = \frac{n}{(d-2)R^{d-2}\Omega_{d-2}}$$

describes a state s.t. $\Delta_n = n$ for $n \gg 1$

- This naively leads us to think that CFTs w/ moduli spaces always admit states s.t. $\Delta_n \simeq n,\ n \gg 1$
- <u>However</u>: even for $n \gg 1$ our solution Φ is NOT valid for the full period $T = 2\pi/m_d$, because the CFT is only valid for

$$(\partial\Phi)^2 \ll |\Phi|^{\frac{2d}{d-2}}$$

- The solution exits the EFT validity regime during its oscillation period for a time δt

$$\frac{\delta t}{T} \sim n^{-1/d}$$

- The result above is not valid because the EFT itself loses validity
- ...so, what does the Φ state above describe? Speculation: **resonant states** on the cylinder

- Remember: you can integrate out light states at the cost of losing unitarity
- Let us apply this idea to our resonant state
- For $\delta t/T \ll 1$ we can treat the dilaton interactions with other states with

$$\delta \mathcal{S} = c \int d^d x \delta(\Phi)[(\partial\Phi)^4 + \ldots]^{\frac{1}{4}+\frac{d-2}{4d}}$$

can be complex!

- $\delta\mathcal{S}$ brings a correction to the stress-energy tensor of the form

$$T_{00} \propto cn^{\frac{d-1}{2}}$$

- If c is complex this leads to a decay width for the state!

Summary and Outlook

- Take-home message

$$\text{CFT with SSB of conformal inv.} + \text{global charges} \quad \Rightarrow \quad \exists\ \mathcal{O}_\Delta \text{ s.t. } \ \Delta(Q) = \alpha Q + \ldots$$

- In non-SUSY theories the spontaneous breaking of conformal symmetry might require fine-tuning, but the above property still holds approximately
- In absence of broken global charges the above property does not hold, but a signature of the spontaneous breaking of conformal symmetry might be the presence of resonant states on the cylinder

Thanks!

Questions? Comments? Complaints?