Lorentzian-Euclidean black holes: a way to avoid singularities

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OUTLINE

1. SIGNATURE-CHANGING METRICS
2. JUNCTION CONDITIONS AND THIN SHELLS
3. THE LORENTZIAN-EUCLIDEAN BLACK HOLE METRIC
4. THE REGULARIZATION PROCESS
5. AVOIDANCE OF THE SINGULARITY
6. CONCLUSIONS

REFERENCE FOR THIS SEMINAR

- **Main reference:**

"*Avoiding singularities in Lorentzian-Euclidean black holes: the role of atemporality*",
Salvatore Capozziello, Silvia De Bianchi, Emmanuele Battista,
ArXiv: 2404.17267, to appear on *Physical Review D*

SIGNATURE-CHANGING METRICS (1)

- **Metrics whose signature changes from the Lorentzian one to the Euclidean one and vice versa:**

-Studied in classical and quantum General Relativity (GR)

- **Quantum GR:**

-Quantum cosmology

- **Hartle-Hawking no-boundary proposal**
- **Linde proposal**
- **Vilenkin proposal (tunneling from nothing)**

-Loop quantum cosmology

-Supergravity and String theory

SIGNATURE-CHANGING METRICS (2)

- **Classical GR:**

-Not forbidden by Einstein field equations

-Homogeneous and isotropic Friedman-Robertson-Walker geometries

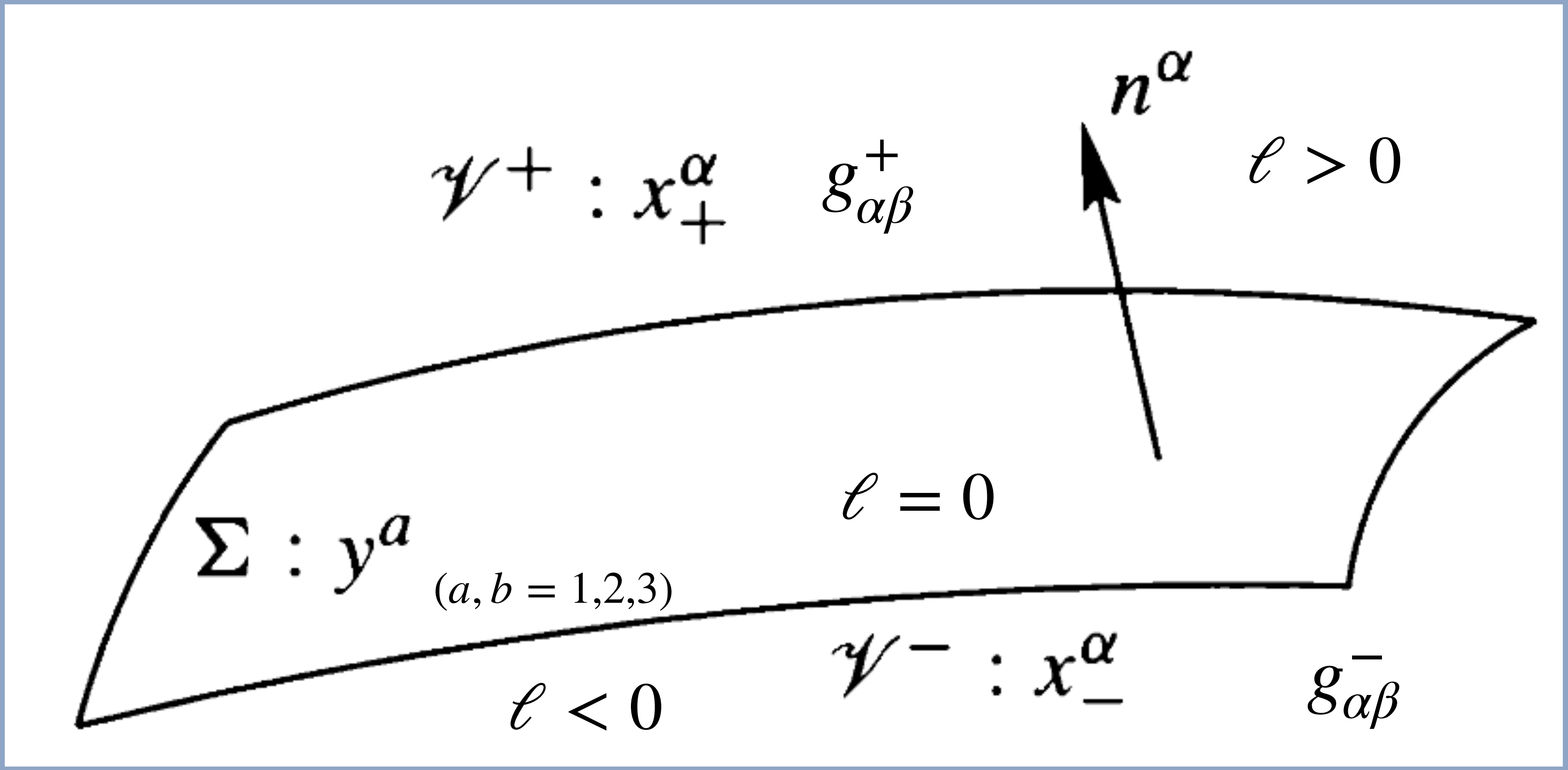
i. **Similar properties with quantum scenarios satisfying the Hartle-Hawking no-boundary condition**

ii. **Related to the real tunneling solutions of Wheeler-DeWitt equation in quantum cosmology**

JUNCTION CONDITIONS AND THIN SHELLS (1)

- **Joining two metrics at a common boundary, which divides the spacetime into two distinct regions**

Israel-Barrabes formalism (metrics with unchanging signature)



$$n_\mu = \alpha \partial_\mu \ell$$

Σ is timelike ($\alpha = 1$) or spacelike ($\alpha = -1$)

The same coordinates y^a installed on both sides of Σ

$$g_{\mu\nu} = \Theta(\ell) g_{\mu\nu}^+ + \Theta(-\ell) g_{\mu\nu}^-$$

metric in coordinates x^μ

JUNCTION CONDITIONS AND THIN SHELLS (2)

What conditions must be imposed on the metric so that $g_{\alpha\beta}$ forms a valid distribution-valued solution of Einstein field equations?

Junction conditions that involve three-tensors on Σ

$$[F] := F|_{+} - F|_{-}$$

Jump discontinuity of any tensorial quantity F across Σ

$$[F] = 0$$

F is continuous at Σ

$$[F] \neq 0$$

F is discontinuous across Σ;
$[F]$ is the jump discontinuity of Facross Σ

In our hypotheses $[n^\alpha] = [x^\alpha] = [y^a] = 0$

$$g_{\mu\nu,\gamma} = \Theta(\ell) g^{+}_{\mu\nu,\gamma} + \Theta(-\ell) g^{-}_{\mu\nu,\gamma} + \alpha\delta(\ell)[g_{\mu\nu}] n_\gamma$$

JUNCTION CONDITIONS AND THIN SHELLS (3)

- **First junction condition: the metric is continuous across Σ**

$$[g_{\mu\nu}] = 0$$

In the coordinate system x^α

$$[h_{ab}] = 0$$

Induced metric (coordinate y^a)

coordinate-invariant statement

Metric tangential derivatives are also continuous, but the normal derivatives are not:

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta} n_\gamma$$

JUNCTION CONDITIONS AND THIN SHELLS (4)

- δ-function part of the **Riemann tensor**

$$A^{\alpha}{}_{\beta\gamma\delta} = \frac{\alpha}{2}\left(\kappa^{\alpha}_{\delta} n_{\beta} n_{\gamma} - \kappa^{\alpha}_{\gamma} n_{\beta} n_{\delta} - \kappa_{\beta\delta} n^{\alpha} n_{\gamma} + \kappa_{\beta\gamma} n^{\alpha} n_{\delta}\right)$$

- δ-function part of the **Ricci tensor**

$$A_{\alpha\beta} \equiv A^{\mu}{}_{\alpha\mu\beta} = \frac{\alpha}{2}\left(\kappa_{\mu\alpha} n^{\mu} n_{\beta} + \kappa_{\mu\beta} n^{\mu} n_{\alpha} - \kappa^{\mu}_{\mu} n_{\alpha} n_{\beta} - \alpha\kappa_{\alpha\beta}\right)$$

- δ-function part of the **Ricci scalar**

$$A \equiv A^{\alpha}_{\alpha} = \alpha\left(\kappa_{\mu\nu} n^{\mu} n^{\nu} - \alpha\kappa^{\mu}_{\mu}\right)$$

JUNCTION CONDITIONS AND THIN SHELLS (5)

Einstein field equations give the following expression for the stress-energy tensor:

$$T_{\alpha\beta} = \theta(\ell)T^{+}_{\alpha\beta} + \theta(-\ell)T^{-}_{\alpha\beta} + \delta(\ell)S_{\alpha\beta}$$

with $8\pi S_{\alpha\beta} = A_{\alpha\beta} - \frac{1}{2}Ag_{\alpha\beta}$

The δ-function term of $T_{\alpha\beta}$ is associated with the presence of a thin distribution of matter, which is referred to as surface layer or thin shell

The stress-energy tensor of the thin shell is $S_{\alpha\beta}$

JUNCTION CONDITIONS AND THIN SHELLS (6)

Explicitly, the thin shell stress-energy tensor depends on the jump discontinuity of the extrinsic curvature tensor K_{ab} of Σ:

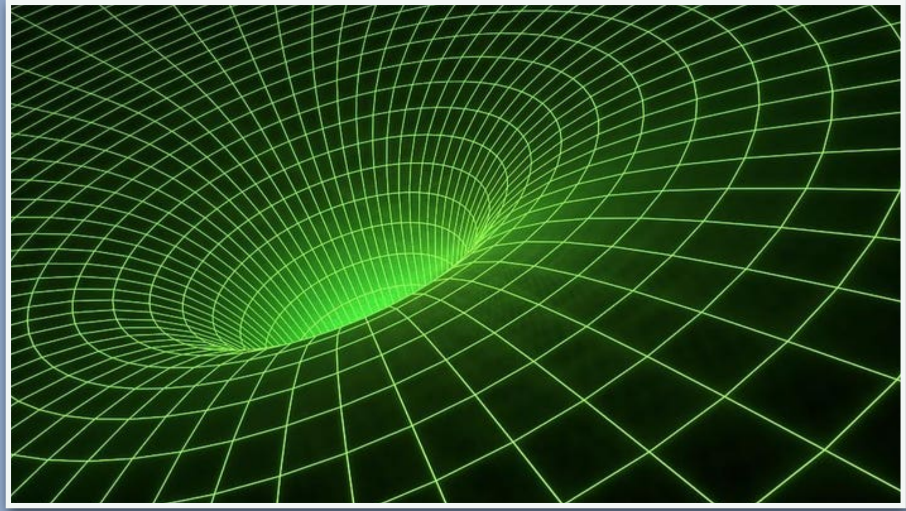
$$S_{ab} = -\frac{\alpha}{8\pi}\left([K_{ab}] - [K]h_{ab}\right)$$

- **Second junction condition: $[K_{ab}] = 0$, which implies $A^{\alpha}{}_{\beta\gamma\delta} = 0$**

When junction conditions are satisfied, then the two metrics $g^{\pm}_{\mu\nu}$ can be joined smoothly through Σ

JUNCTION CONDITIONS AND THIN SHELLS (7)

- When Σ is either spacelike or timelike, then only the Ricci part of the Riemann tensor can show a distributional singularity



- When Σ is null, then both the Ricci and Weyl part of the Riemann tensor can present Dirac-delta singularities

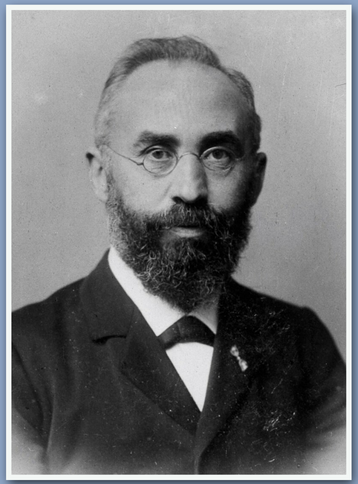
Thin shell

Impulsive gravitational wave

LORENTZIAN-EUCLIDEAN BLACK HOLE (1)

Lorentzian-Euclidean Schwarzschild metric in standard coordinates $\{t, r, \theta, \phi\}$

$$ds^2 = -\varepsilon\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2,$$



where

$$\varepsilon = \operatorname{sign}\left(1 - \frac{2M}{r}\right) = 2H\left(1 - \frac{2M}{r}\right) - 1,$$

Sign function

Step function

H(0)=1/2

LORENTZIAN-EUCLIDEAN BLACK HOLE (2)

Therefore, the spacetime manifold is divided as $V = V_+ \cup V_-$ **and**

- $\varepsilon = 1$ **if** $r > 2M$**: Lorentzian signature** $(-+++)$
- $\varepsilon = 0$ **if** $r = 2M$**: metric is degenerate** $\det g_{\mu\nu} = 0$
- $\varepsilon = -1$ **if** $r < 2M$**: metric has a Euclidean structure and attains ultrahyperbolic signature** $(--++)$
- $\Sigma \; : \quad r = 2M$ **change surface (null hypersurface)**
- **Metric and its derivatives are discontinuous across the change surface**

$$[g_{\alpha\beta}] \neq 0 \qquad\qquad [g_{\alpha\beta,\mu}] \neq 0$$

LORENTZIAN-EUCLIDEAN BLACK HOLE (3)

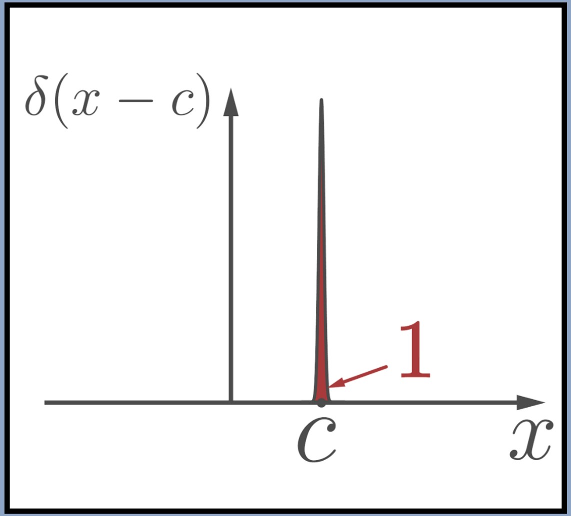
Metric in Gullstrand-Painlevé coordinates $(\mathscr{T}, r, \theta, \phi)$

$$ds^2 = -\varepsilon d\mathscr{T}^2 + \left(dr + \sqrt{\varepsilon}\sqrt{\frac{2M}{r}}d\mathscr{T}\right)^2 + r^2 d\Omega^2.$$

The only pathology is related to the fact that the metric becomes degenerate on the change surface Σ**, i.e., when** $r = 2M$ **and** $\varepsilon = 0$

THE REGULARIZATION PROCESS (1)

Recall that $[g_{\alpha\beta}] \neq 0$ and $[g_{\alpha\beta,\mu}] \neq 0$ → first junction condition cannot be satisfied



- **Dirac-delta-like contributions arising in the Riemann tensor**
- **Terms proportional to $\varepsilon', (\varepsilon')^2, \varepsilon'' \Rightarrow$ Linear and quadratic terms in the Dirac-delta function $\delta(r-2M)$ in the Riemann tensor**

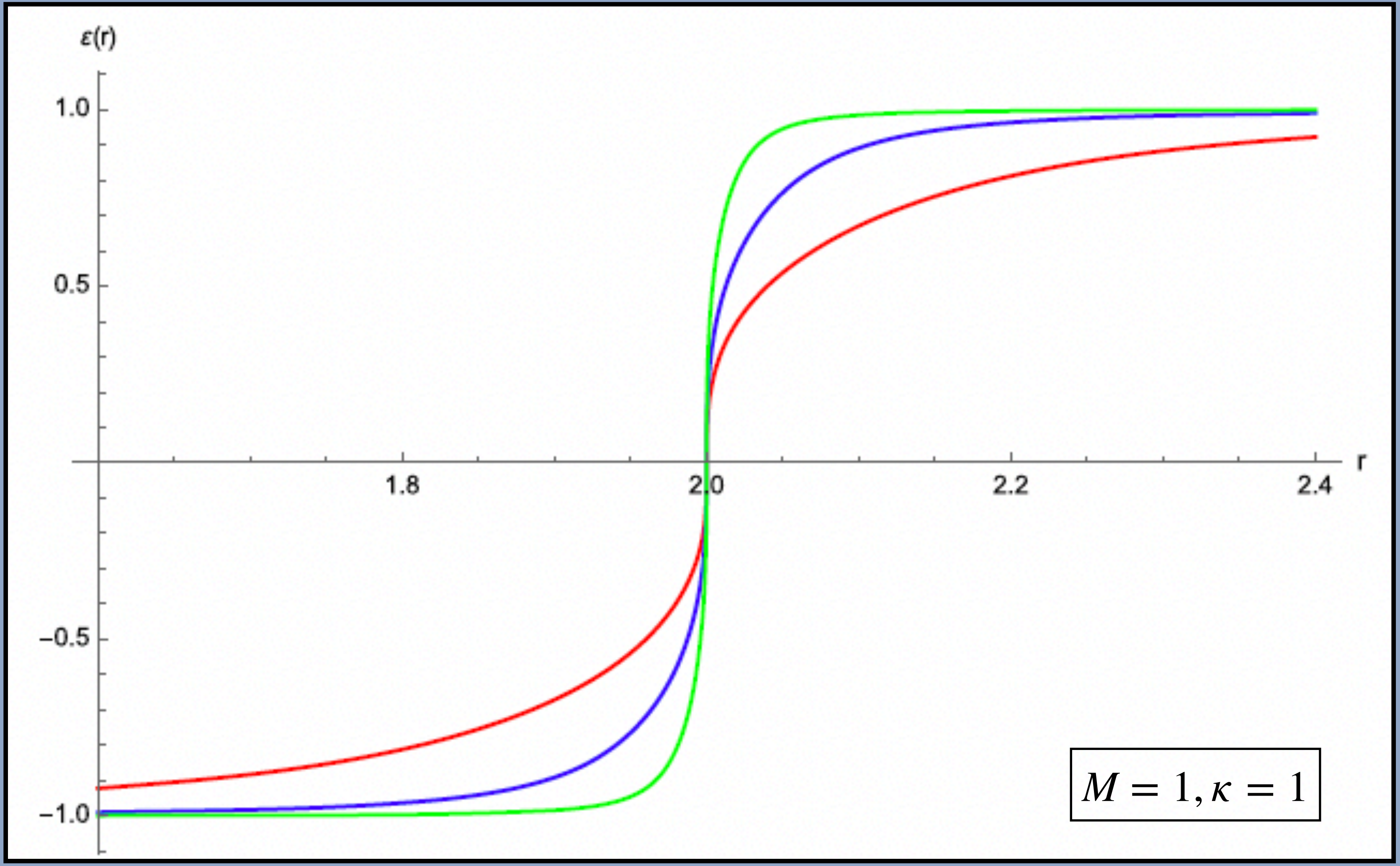
Proper regularization scheme

THE REGULARIZATION PROCESS (2)

- **Smooth approximation of $\varepsilon(r) = 2H(1 - 2M/r) - 1$:**

$$\varepsilon(r) = \frac{(r-2M)^{1/(2\kappa+1)}}{\left[(r-2M)^2+\rho\right]^{1/2(2\kappa+1)}},$$

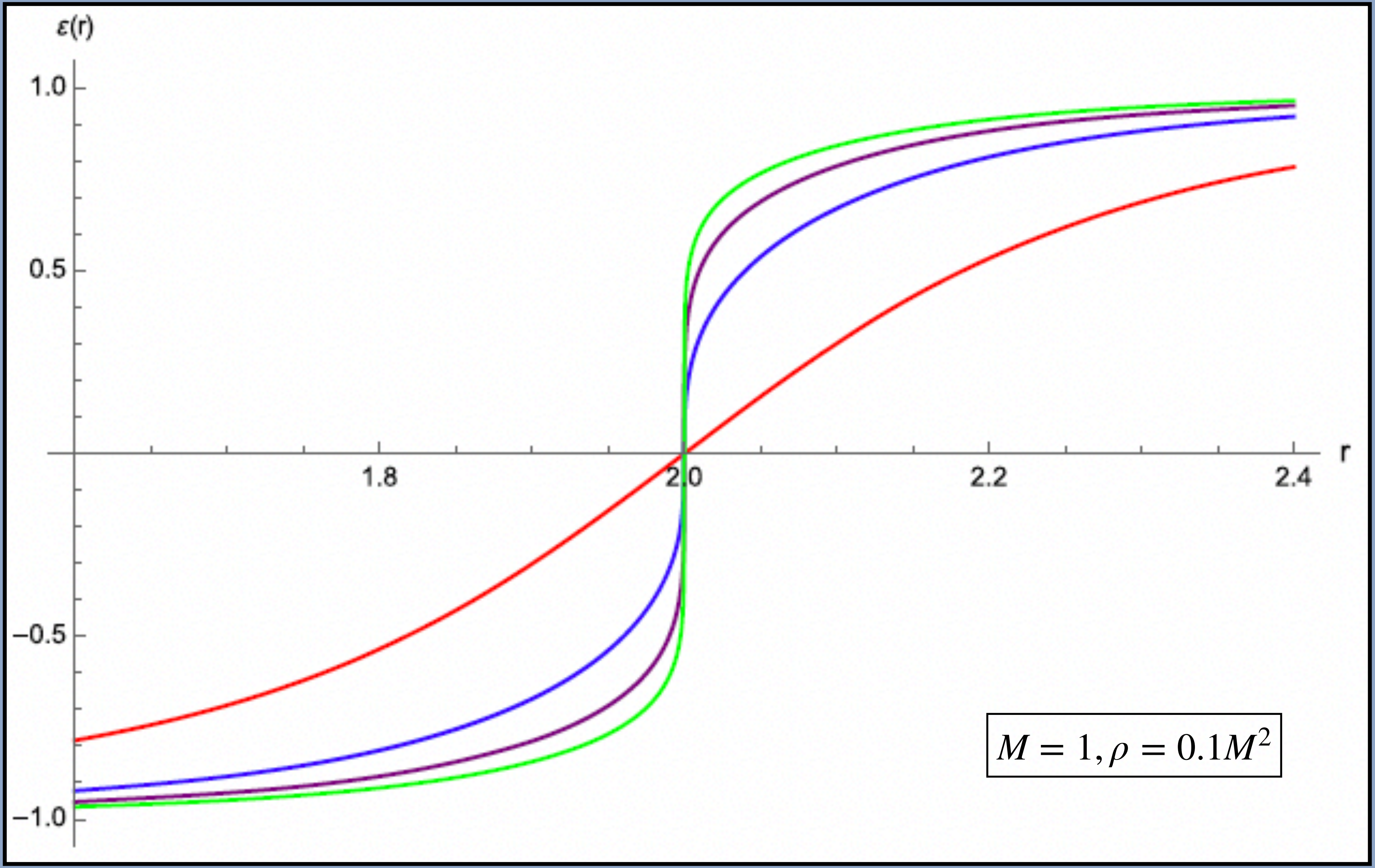
ρ/M^2 : **small positive quantity**
κ : **positive integer**



The smaller ρ/M^2, the sharper $\varepsilon(r)$

$\rho = 0.1M^2$: **red curve**
$\rho = 0.01M^2$: **blue curve**
$\rho = 0.001M^2$: **green curve**

THE REGULARIZATION PROCESS (3)



The larger κ, the steeper $\varepsilon(r)$

$\kappa = 0$: **red curve**
$\kappa = 1$: **blue curve**
$\kappa = 2$: **purple curve**
$\kappa = 3$: **green curve**

We will see that our regularization scheme requires $\kappa \geq 1$

THE REGULARIZATION PROCESS (4)

- **The Riemann tensor contains linear-in-delta ill-defined terms of the type**

$$\int \mathrm{d}r \, \frac{\delta(r-2M)}{\varepsilon(r)},$$

Hadamard *partie finie* regularization method & approximation of $\varepsilon(r)$:

$$\frac{\delta(x)}{|x|^n} \equiv 0,$$

n: **positive integer**

$x := r - 2M$

THE REGULARIZATION PROCESS (5)

- **Let $F(\xi; a)$ be a function of ξ which diverges as ξ approaches a. We assume that near $\xi = a$**

$$F(\xi; a) = \sum_{n=0}^{n_{\max}} s^{-n} f_n(s; a) + \mathrm{O}(s),$$

$$s = |\xi - a|$$

- **The function diverges as $s^{-n_{\max}}$ when $\xi \to a$ and does not have a well-defined value at $\xi = a$**

↓

- **We can regularize it by extracting its *partie finie* at the singular point $\xi = a$, which is defined by**

$$\langle F \rangle(a) := \frac{1}{2\pi} \int_0^{2\pi} f_0(s; a)\, \mathrm{d}\theta$$

Angular average of the zeroth term $f_0(s; a)$ of the Laurent series

THE REGULARIZATION PROCESS (6)

- **The *partie finie* can be used to make sense of the product of F with the delta function $\delta(\xi - a)$, since we declare that**

$$F(\xi;a)\delta(\xi - a) \equiv \langle F \rangle(a)\ \delta(\xi - a)$$

$$\Downarrow$$

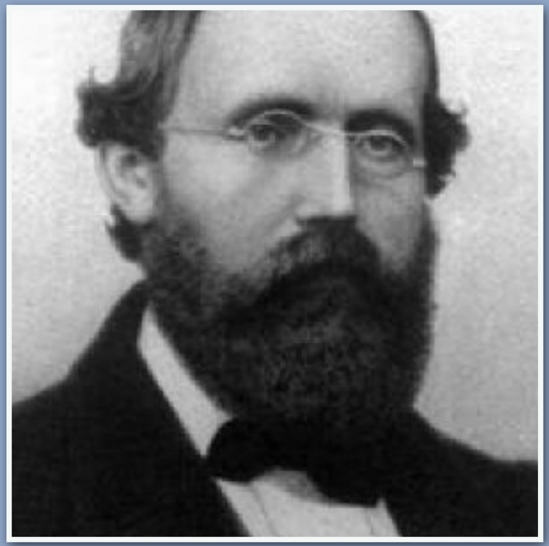
$$\int F(\xi;a)\delta(\xi - a)\ \mathrm{d}\xi = \langle F \rangle(a)$$

- **In our case** $\dfrac{\delta(x)}{|x|^n} \equiv 0,$ → $F = |x|^{-n} := |r - 2M|^{-n}$; $\langle F \rangle = 0$

THE REGULARIZATION PROCESS (7)

- Quadratic-in-delta ill-defined terms occurring in the Riemann tensor

Regularized within our model since their coefficient vanish when $r = 2M$



Terms of the type $\delta^2(r - 2M)$ give vanishing contribution in the distributional sense to the Riemann tensor

THE REGULARIZATION PROCESS (8)

- **An example: regularization of $R^r{}_{r\mathscr{T}r}$**

$$R^r{}_{r\mathscr{T}r} = \sqrt{\frac{M}{r}}\,\frac{r^2(2M-r)\varepsilon'^2 + 2r\varepsilon\left[r(r-2M)\varepsilon'' + 3M\varepsilon'\right] - 8M\varepsilon^2}{2\sqrt{2}\,r^3\varepsilon^{3/2}}$$

-Terms linear in $\varepsilon'(x)$ yield an integral proportional to (recall $x := r - 2M$)

$$\int \mathrm{d}x \frac{\delta(x)}{\varepsilon^{1/2}} = \int \mathrm{d}x\, \delta(x) \frac{\left(x^2+\rho\right)^{1/4(2\kappa+1)}}{x^{1/2(2\kappa+1)}} = \int \mathrm{d}x \left(\frac{\delta(x)}{x^p x^{1/2(2\kappa+1)}}\right)\left[x^p\left(x^2+\rho\right)^{1/4(2\kappa+1)}\right]$$

Approximation for $\varepsilon(r)$

$\delta(x)/|x|^n \equiv 0$
(Hadamard prescription)

vanishing in $x = 0$

THE REGULARIZATION PROCESS (9)

-**Terms depending on $(\varepsilon')^2$ lead to an integral proportional to**

$$\int \mathrm{d}x\ \frac{x\delta^2(x)}{\varepsilon^{3/2}} = \int \mathrm{d}x\ \delta^2(x)\left(x^2+\rho\right)^{3/4(2\kappa+1)} x^{(4\kappa-1)/2(2\kappa+1)},$$

Vanishing contribution in the distributional sense as the coefficient of $\delta^2(x)$ is zero in $x=0$ if we suppose $\kappa \geq 1$

-**Terms depending on ε'' give an integral proportional to**

$$\int \mathrm{d}x\ \frac{x\varepsilon''(x)}{\varepsilon^{1/2}} = 2\int \mathrm{d}x\ \frac{x\delta'(x)}{\varepsilon^{1/2}} == -2\int \mathrm{d}x\ \delta(x)\frac{\left(x^2+\rho\right)^{1/4(2\kappa+1)}}{x^{1/2(2\kappa+1)}}$$

$$+2\int \mathrm{d}x\ \delta^2(x)x\frac{\left(x^2+\rho\right)^{3/4(2\kappa+1)}}{x^{3/2(2\kappa+1)}},$$

Vanishing contribution in the distributional sense

$\delta(x)/|x|^n \equiv 0$
(Hadamard prescription)

THE REGULARIZATION PROCESS (10)

The regularized $R^r{}_{r\mathscr{T}r}$ assumes this form

$$R^r{}_{r\mathscr{T}r} = -2\sqrt{2}\left(\frac{M}{r}\right)^{3/2}\frac{\sqrt{\varepsilon}}{r^2}$$

Remaining regularized Riemann tensor components read as

$$\begin{aligned}
R^r{}_{\theta\theta r} &= \frac{M}{r}, \\
R^r{}_{\phi\phi r} &= \sin^2\theta\, R^r{}_{\theta\theta r}, \\
R^r{}_{\mathscr{T}\mathscr{T}r} &= \frac{2M\varepsilon(r-2M)}{r^4}, \\
R^\theta{}_{r\theta r} &= -\frac{1}{r^2}R^r{}_{\theta\theta r}, \\
R^\theta{}_{r\mathscr{T}\theta} &= -\frac{1}{2}R^r{}_{r\mathscr{T}r}, \\
R^\theta{}_{\phi\phi\theta} &= -2\sin^2\theta\, R^r{}_{\theta\theta r},
\end{aligned}$$

$$\begin{aligned}
R^\theta{}_{\mathscr{T}\theta r} &= \frac{1}{2}R^r{}_{r\mathscr{T}r}, \\
R^\theta{}_{\mathscr{T}\mathscr{T}\theta} &= -\frac{1}{2}R^r{}_{\mathscr{T}\mathscr{T}r} \\
R^\phi{}_{r\phi r} &= -\frac{1}{r^2}R^r{}_{\theta\theta r}, \\
R^\phi{}_{r\mathscr{T}\phi} &= -\frac{1}{2}R^r{}_{r\mathscr{T}r}, \\
R^\phi{}_{\theta\phi\theta} &= 2R^r{}_{\theta\theta r}, \\
R^\phi{}_{\mathscr{T}\phi r} &= \frac{1}{2}R^r{}_{r\mathscr{T}r},
\end{aligned}$$

$$\begin{aligned}
R^\phi{}_{\mathscr{T}\mathscr{T}\phi} &= -\frac{1}{2}R^r{}_{\mathscr{T}\mathscr{T}r}, \\
R^\mathscr{T}{}_{r\mathscr{T}r} &= \frac{2}{r^2}R^r{}_{\theta\theta r}, \\
R^\mathscr{T}{}_{\theta\mathscr{T}\theta} &= -R^r{}_{\theta\theta r}, \\
R^\mathscr{T}{}_{\phi\mathscr{T}\phi} &= -\sin^2\theta\, R^r{}_{\theta\theta r}, \\
R^\mathscr{T}{}_{\mathscr{T}\mathscr{T}r} &= -R^r{}_{r\mathscr{T}r}.
\end{aligned}$$

THE REGULARIZATION PROCESS (11)

-The regularized Riemann tensor does not depend on the Dirac-delta function and is discontinuous across Σ, as $[R^{\alpha}_{\ \beta\gamma\delta}] \neq 0$

-The ensuing Ricci tensor, Ricci scalar, and consequently Einstein tensor vanish

Σ does not represent a thin shell

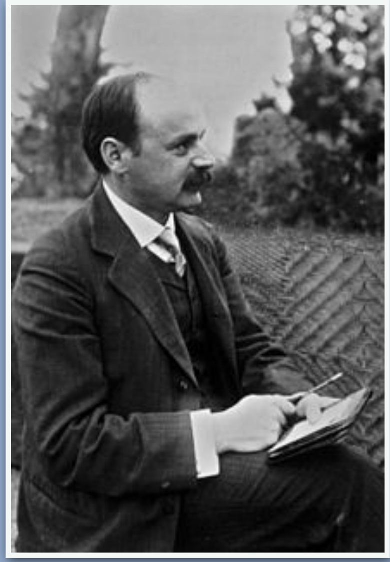
-Regularized Kretschmann invariant

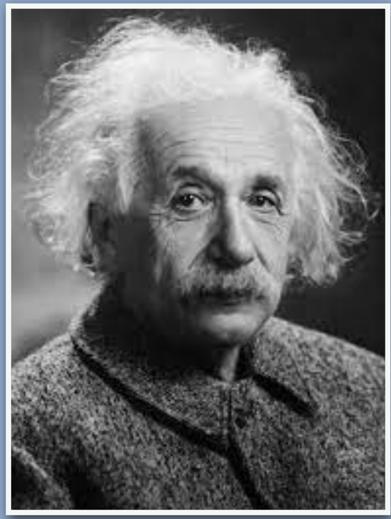
$$R_{\alpha\beta\gamma\mu}R^{\alpha\beta\gamma\mu} = \frac{48M^2}{r^6}$$

Σ does not give rise to a new curvature singularity

THE REGULARIZATION PROCESS (12)

-The Weyl tensor stemming from the regularized Riemann tensor is discontinuous across Σ, but it does not depend on Dirac-delta function





No impulsive gravitational wave on Σ

The Lorentzian-Euclidean Schwarzschild metric
is a valid signature-changing solution
of vacuum Einstein field equations

AVOIDANCE OF THE SINGULARITY (1)

Henceforth, we will use Schwarzschild coordinates $\{t, r, \theta, \phi\}$

$$ds^2 = -\varepsilon\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2,$$

with

$\varepsilon = 1$ **if** $r > 2M$**,** $\varepsilon = 0$ **if** $r = 2M$**, and** $\varepsilon = -1$ **if** $r < 2M$**.**

Let us study the motion of bodies radially approaching the Lorentzian-Euclidean black hole

AVOIDANCE OF THE SINGULARITY (2)

- **Geodesic motion**

-Observer starting at rest at some finite distance $r_i > 2M$

-Describe the radial variable via the relation

$$r(\eta) = r_i \cos^2(\eta/2), \quad \eta \in [0, \eta_H]$$

-Equations governing infalling radial geodesics are

$$\dot{r} = -\sqrt{\frac{\varepsilon^4 \sin^2\left(\eta/2\right) + E^2\left[\cos^2\left(\eta/2\right) - \varepsilon^4\right]}{\varepsilon^3 \cos^2\left(\eta/2\right)}}$$

$$\dot{t} = \frac{E}{\varepsilon^2} \frac{\cos^2\left(\eta/2\right)}{\cos^2\left(\eta/2\right) - \left(1 - E^2\right)}$$

AVOIDANCE OF THE SINGULARITY (3)

along with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta} = (\dot{r})^{-1}\frac{\mathrm{d}r}{\mathrm{d}\eta} = r_i \sin\left(\eta/2\right)\cos^2\left(\eta/2\right)\sqrt{\frac{\varepsilon^3}{\varepsilon^4 \sin^2\left(\eta/2\right) + E^2\left[\cos^2\left(\eta/2\right) - \varepsilon^4\right]}}$$

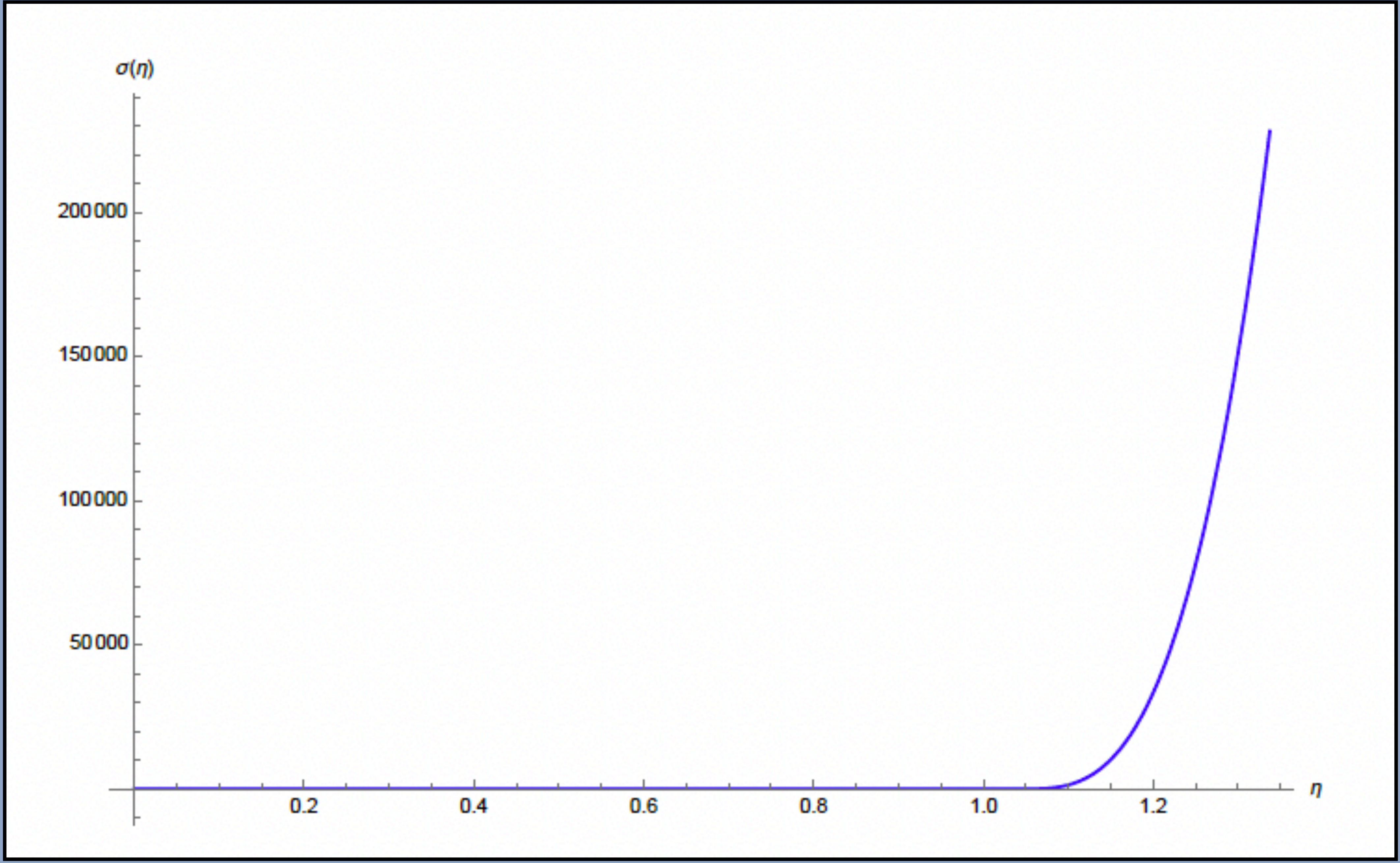
$$\frac{\mathrm{d}t}{\mathrm{d}\eta} = \dot{t}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\eta} = \frac{E}{\varepsilon^2}\frac{r_i \cos^4\left(\eta/2\right)\sin\left(\eta/2\right)}{\cos^2\left(\eta/2\right) - \left(1 - E^2\right)}\sqrt{\frac{\varepsilon^3}{\varepsilon^4 \sin^2\left(\eta/2\right) + E^2\left[\cos^2\left(\eta/2\right) - \varepsilon^4\right]}}$$

-The radial velocity $\dot{r}$, and the derivatives $\mathrm{d}\sigma/\mathrm{d}\eta$, $\mathrm{d}t/\mathrm{d}\eta$ assume imaginary values as $r < 2M$

- The radial velocity $\dot{r}$ vanishes at $r = 2M$

AVOIDANCE OF THE SINGULARITY (4)

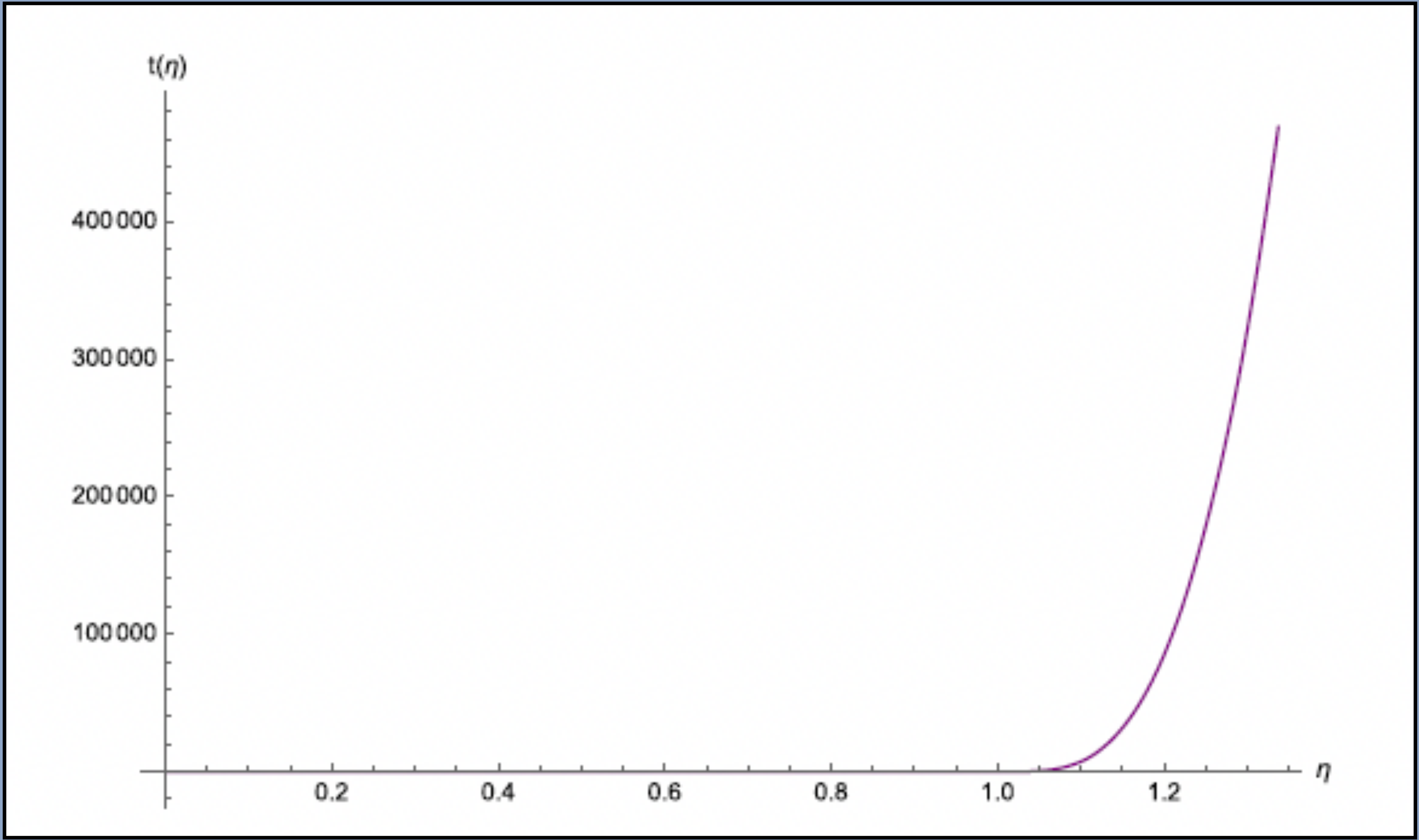
-The observer in radial free fall takes **an infinite amount of proper time σ** to stop at the event horizon



event horizon:
$\eta \approx 1.3$

AVOIDANCE OF THE SINGULARITY (5)

-The observer in radial free fall takes an infinite amount of time to stop at the event horizon also from the point of view of an observer stationed at infinity



event horizon:
$\eta \approx 1.3$

AVOIDANCE OF THE SINGULARITY (6)

- **Accelerated motion**

-Radially accelerated observer whose trajectory begins at rest from a large distance from the black hole

$$a^\lambda = \frac{\mathrm{d}U^\lambda}{\mathrm{d}\sigma} + \Gamma^\lambda_{\mu\nu} U^\mu U^\nu$$

$$U^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}$$

-Radial-directed orbit (θ, ϕ constant)

$$a^t = \frac{\mathrm{d}U^t}{\mathrm{d}\sigma} + 2\Gamma^t_{tr} U^t U^r$$

$$a^r = \frac{\mathrm{d}U^r}{\mathrm{d}\sigma} + \Gamma^r_{tt} U^t U^t + \Gamma^r_{rr} U^r U^r$$

Christoffel symbols regularized via our technique

AVOIDANCE OF THE SINGULARITY (7)

- **Radial velocity**

$$U^r = -\sqrt{\varepsilon}\sqrt{\mathscr{F}^2 - (1 - 2M/r)}$$

$$\mathscr{F} = f(\sigma)\sqrt{1 - 2M/r},$$

$$f(\sigma) > 1$$

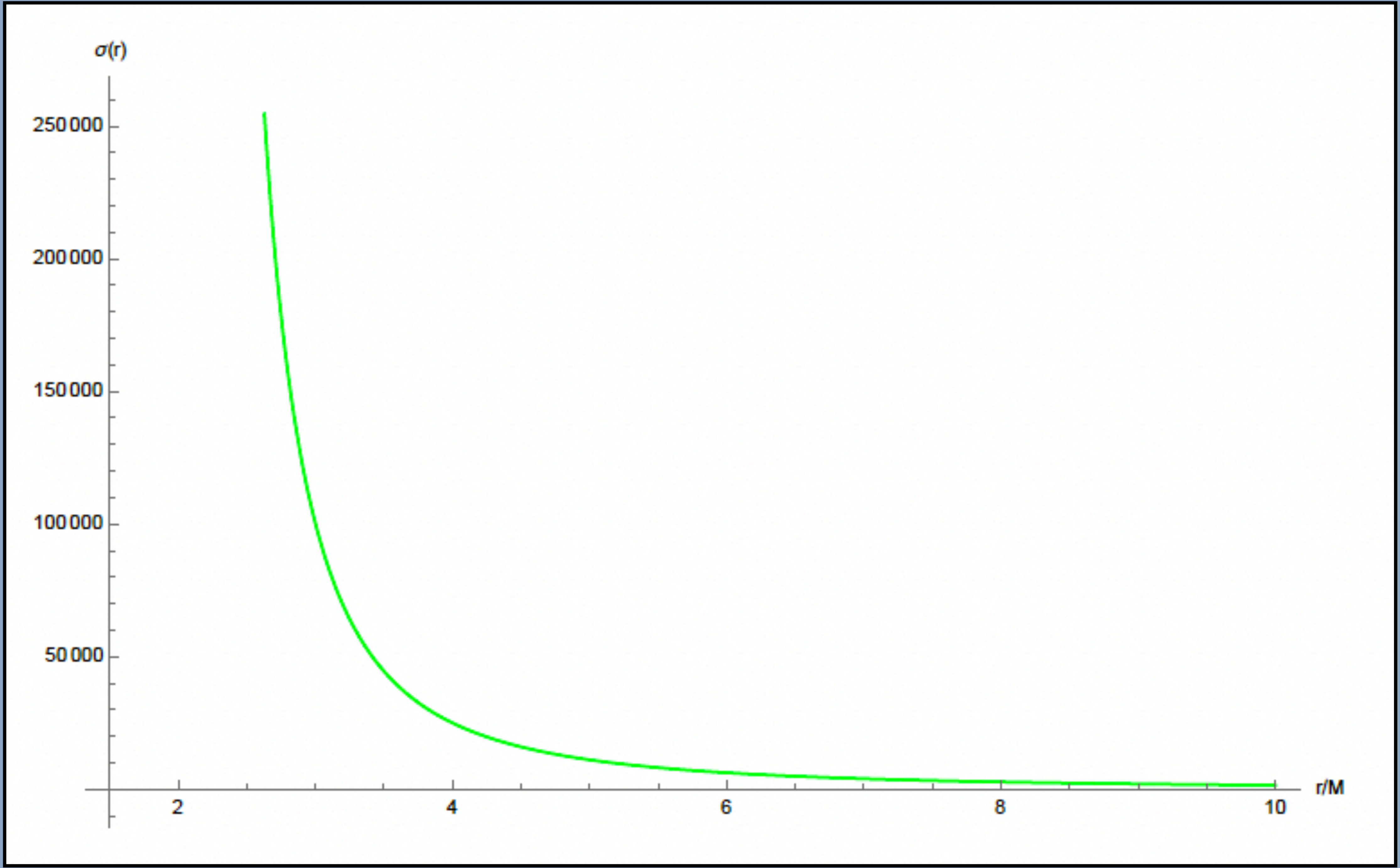
U^r **vanishes on the event horizon and becomes imaginary inside it**

-Differential equation for σ

$$\frac{\mathrm{d}\sigma}{\mathrm{d}r} = -\frac{1}{\sqrt{\varepsilon\left[\mathscr{F}^2 - (1 - 2M/r)\right]}}$$

AVOIDANCE OF THE SINGULARITY (8)

The accelerated observer takes an infinite amount of proper time σ to stop at the event horizon



CONCLUSIONS

- **The signature change of the Lorentzian-Euclidean metric can be ascribed to the emergence of an imaginary time variable t when $r < 2M$. We propose to relate this feature to the concept of "*atemporality*"**

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Atemporality **configures in our model as the dynamical mechanism which permits one to avoid the black-hole singularity**

- **Bunch of particles accumulating on the event horizon: observational feature of the model?**