

Target group: PhD students and postdocs in mathematics

Course Title: **W algebras – in algebra, geometry and physics**

Background The physicist Alexander Zamolodchikov first discovered W algebras in the 1980's when investigating symmetries in two-dimensional conformal field theory. Having understood them more the past 40 years, we now know they have deep connections with quantisation (the algebraic geometry of nilpotent orbits), automorphic forms, Langlands duality, probability, N=2 SCFTs, and the geometry of surfaces (instantons). The subject is experiencing a period of renewed activity.

Extend: 3 hours per week / 10 weeks / 5 ECTS

Aims The primary aim of this seminar is for students to learn the basics of W algebras. We will follow multiple texts. The course should be a good way to get comfortable with geometric representation theory and vertex algebras, but staying concrete, since we are following a single thread.

Synopsis The W algebras best understood in mathematics are the ones $W(g,e)$ attached to a finite dimensional Lie algebra g and nilpotent element e , its associated graded is functions on the transverse slice. We will begin by talking about the representation theory of $W(g,e)$ and relations to the geometry (of Grassmannians, nilpotent cones, ...). Next we give a mathematical definition of the structures Zamolodchikov was studying: it is a vertex algebra analogue of $W(g,e)$. This opens the road to a huge amount of current mathematics, see the sketch below.

Course schedule sketch

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| 1. Basics: | Basics on D modules and Lie algebras |
| 2. Background: | Whittaker functions, automorphic forms and sheaves |
| 3. Finite W algebras: | Finite W algebras; One dimensional representations and nilpotent orbits; The centre of W and Kostant's theorem; Slodowy slices, Beilinson Drinfeld localisation and microlocalisation [DK, AKM]; Shifted Yangians and the affine Grassmannian [BK] |
| 4. Affine W algebras: | (Poisson) Vertex algebras and conformal field theory; Affine W algebras [FB], and their definition via chiral hamiltonian reduction [AKM]; Centre and opers, Sugenawa construction [FB]; Feigin Frenkel duality [R,FB]; Slodowy slices, Beilinson Drinfeld localisation and microlocalisation [AKM] |
| 5. Exotic W algebras: | Other W algebras: $W(a_1, a_2, \dots)$, quasi primary fields, triplet algebras; Representation theory, minimal models |
| 6. Physics: | Higgs branch [A3, AEM]; Instantons and surfaces; Relation to probability and the Gaussian Free Field [CBV] |

Prerequisites

You do not need to know vertex algebras, but should be comfortable with algebra, and algebraic or complex geometry.

Evaluation method:

The evaluation will be based on homework assignments on the topics covered in the course. The lecturer and an internal censor will evaluate the assignments. The evaluation will be pass/fail.

Teaching method: In person and online lectures

Responsible lecture(s) Alexei Latyntsev

Literature

General overviews and background

- [FB] Frenkel, E. and Ben-Zvi, D., 2004. Vertex algebras and algebraic curves (No. 88). AMS.
- [AMo] Arakawa, T. and Moreau, A., 2016. Lectures on W -algebras. Preprint.
- [L2] Losev, I., 2010. Finite W -algebras. In Proc. ICM 2010, pp. 1281-1307.

Affine W algebras

- [DR] Dhillon, G., Raskin, S., 2020. Localization for affine W -algebras. Preprint arXiv:2010.11434.
- [R] Raskin, S., 2016. W -algebras and Whittaker categories. Preprint arXiv:1611.04937.

Representations of W algebras

- [A1] Arakawa, T., 2017. Introduction to W -algebras and their representation theory. In Perspectives in Lie theory (pp. 179-250). Springer, Cham.
- [A2] Arakawa, T., 2007. Representation theory of W -algebras. Invent. Math., 169(2), pp.219-320.

Quantisation

- [AEM] Arakawa, T., Van Ekeren, J., Moreau, A., 2021. Singularities of nilpotent Slodowy slices and collapsing levels of W -algebras. Preprint arXiv:2102.13462.
- [AKM] Arakawa, T., Kuwabara, T. and Malikov, F., 2015. Localization of affine W -algebras. Comm. Math. Phys., 335(1), pp.143-182.
- [BK] Brundan, J. and Kleshchev, A., 2006. Shifted Yangians and finite W -algebras. Adv. Math., 200(1), pp.136-195.
- [DK] Dodd, C., Kremnizer, K., 2009. A localization theorem for finite W -algebras. ArXiv:0911.2210.
- [GG] Gan, W.L. and Ginzburg, V., 2002. Quantization of Slodowy slices. IMRN, 2002(5), pp.243-255.
- [KWWY] Kamnitzer, J., Webster, B., Weekes, A. and Yacobi, O., 2014. Yangians and quantizations of slices in the affine Grassmannian. Algebra & Number Theory, 8(4), pp.857-893.
- [L1] Losev, I., 2010. Quantizations of nilpotent orbits vs 1-dimensional representations of W -algebras. Preprint arXiv:1004.1669.
- [L3] Losev, I.V., 2007. Quantized symplectic actions and W -algebras. Preprint arXiv:0707.3108.

Whittaker functions and sheaves

- [BBM] Bezrukavnikov, R., Braverman, A. and Mirkovic, I., 2004. Some results about geometric Whittaker model. Advances in Mathematics, 186(1), pp.143-152.

- [FGKV] Frenkel, E., Gaitsgory, D., Kazhdan, D. and Vilonen, K., 1998. Geometric realization of Whittaker functions and the Langlands conjecture. *Journal of the AMS*, 11(2), pp.451-484.
- [FR] Færgeman, J. And Raskin, S., 2022. Non-Vanishing Of Geometric Whittaker Coefficients For Reductive Groups. Preprint arXiv:2207.02955.

Exotic W algebras

- [AMi1] Adamovic, D. and Milas, A., 2008. On the triplet vertex algebra $W(p)$. *Adv. Math.*, 217(6), pp.2664-2699.
- [AMi2] Adamovic, D. and Milas, A., 2007. Logarithmic intertwining operators and $W(2, 2p-1)$ algebras. *Journal of mathematical physics*, 48(7), p.073503.
- [AWM] Honecker, A., Eholzer, W., Flohr, M., Hübel, R. and Varnhagen, R., 1993. W-Algebras in Conformal Field Theory. *HIGH ENERGY PHYSICS AND COSMOLOGY*, p.435.

Higgs branch

- [A3] Arakawa, T., 2017. Representation theory of W-algebras and Higgs branch conjecture. Preprint arXiv:1712.07331.

Instantons

- [BFN] Braverman, A., Finkelberg, M. and Nakajima, H., 2014. Instanton moduli spaces and $\mathscr{W}$ -algebras. Preprint arXiv:1406.2381.
- [BBCCN] Borot, G., Bouchard, V., Chidambaram, N.K., Creutzig, T. and Noshchenko, D., 2018. Higher Airy structures, W algebras and topological recursion. Preprint arXiv:1812.08738.
- [RSYZ] Rapcák, M., Soibelman, Y., Yang, Y. and Zhao, G., 2020. Cohomological Hall algebras, vertex algebras and instantons. *Communications in Mathematical Physics*, 376(3), pp.1803-1873.
- [SV] Schiffmann, O. and Vasserot, E., 2013. Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on A_2 . *Publ. math. l'IHÉS*, 118(1), pp.213-342.

Probability

- [CBV] Cerclé, B., Rhodes, R. and Vargas, V., 2021. Probabilistic construction of simply-laced Toda conformal field theories. Preprint arXiv:2102.11219.

Other seminars

<http://www.math.toronto.edu/dbutson/ls20.html>