On the restricted cores and the bounded core of games on distributive lattices

by

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Abstract

We consider TU-games with restricted cooperation, where the set of feasible coalitions is a distributive lattice, hence generated by a partial order on the set of players. In such a situation, the core may be unbounded, and one has to select a bounded part of the core as a solution concept. The restricted core is obtained by imposing equality constraints in the core for sets belonging to so-called normal collections, resulting (if nonempty) in the selection of a bounded face of the core. The bounded core proves to be the union of all bounded faces (restricted cores). The paper aims at investigating in depth the relation between the bounded core and restricted cores, as well as the properties and structures of the restricted cores and normal collections. In particular, it is found that a game is convex if and only if all restricted cores corresponding to the minimal nested normal collections are nonempty. Moreover, in this case the union of these restricted cores already covers the bounded core.

1   Introduction

In cooperative game theory, for a given set of players N, TU-games are functions $v : 2^N \to \mathbb{R}$ which express for each nonempty coalition $S \subseteq N$ of players the best they can achieve by cooperation. In the classical setting, every coalition may form without any restriction, i.e., the domain of v is indeed 2^N. In practice, this assumption is often unrealistic, since some coalitions may not be feasible for various reasons, e.g., players are political parties with divergent opinions, or have restricted communication abilities, or a hierarchy exists among players, and the formation of coalitions must respect the hierarchy, etc.

Many studies have been done on games defined on specific subdomains of 2^N, e.g., antimatroids (Algaba et al., 2004), convex geometries (Bilbao and Edelman, 2000), distributive lattices (Faigle and Kern, 1992), or others (Bilbao et al., 2000). In this paper,

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we focus on the case of distributive lattices. To this end, we assume that there exists some partial order $\preceq$ on N describing some hierarchy or precedence constraint among players, as in Faigle and Kern (1992). We say that a coalition S is feasible if the coalition contains all its subordinates, i.e., $i \in S$ implies that any $j \preceq i$ belongs to S as well. Then by Birkhoff's theorem, feasible coalitions form a distributive lattice.

The main problem in cooperative game theory is to define a rational solution of the game, that is, supposing that the grand coalition N will form, how to share among its members the total worth $v(N)$. The core is the most popular solution concept, since it ensures stability of the game, in the sense that no coalition has an incentive to deviate from the grand coalition. For classical TU-games, the core is either empty or a convex bounded polyhedron. However, for games whose cooperation is restricted, the study of the core is much more complex, since it may be unbounded or even contain no vertices (see a survey in Grabisch (2009)). For the case of games with precedence constraints, it is known that the core is *always* unbounded or empty, but contains no line (i.e., it has vertices). The problem arises then, to select a significant bounded part of the core as a reasonable concept of solution, since unbounded payments make no sense.

As far as we know, two remedies have been proposed in the literature for this problem. The first one is to consider *restricted cores*, i.e., to impose in the core further equality constraints $x(S) = v(S)$ for all S in some collection $\mathcal{N}$, which can be interpreted as binding constraints for some coalitions, in order to exclude any extremal ray in the core (Grabisch and Xie, 2011; Grabisch, 2011). Such a collection is called a normal collection. The second one was proposed by the authors, by considering core elements such that every player takes the maximum of her direct subordinates, in the sense that any transfer from a subordinate to her boss would result in a payoff vector outside the core. The resulting set is called the *bounded core* (Grabisch and Sudhölter, to appear). It happens that the two concepts are related as follows: the bounded core is the union of all restricted cores, taken over all possible normal collections.

The aim of this paper is to investigate more deeply the relation between the two concepts, and to complete the results presented in Grabisch (2011) for restricted cores. In particular, we introduce several remarkable normal collections and study their properties. Nested normal collections, i.e., collections whose sets form a chain, play a fundamental rôle, in particular for convex games. We show that for convex games, only minimal (in the size of the collection) nested normal collections matter, in the sense that the bounded core is the union of restricted cores corresponding to minimal nested collections. Moreover, such restricted cores are never empty. If the game is strictly convex, then this union does not contain any redundant term. Finally, we show that minimal nested collections can be generated by a special class of linear extensions of the partial order $\preceq$ on N. Besides, we show a generalization of the well-known Shapley-Ichiishi theorem for games with precedence constraints.

The paper is organized as follows. Section 2 establishes the basic material for the rest of the paper, and presents the notions of restricted core, normal collection and bounded core. Section 3 studies the set of normal collections, introduces properties and remarkable collections. It shows also how nested collections can be obtained by a closure operator on a certain class of normal collections. Section 4 studies under which conditions the restricted core is nonempty, and provides a balancedness condition. Section 5 investigates in depth

the case of convex games, showing the fundamental rôle played by minimal nested normal collections.

2 Notation, definitions and preliminaries

The characteristic function of a set S is denoted by 1^S. We often omit braces for singletons, writing, e.g., 1^i instead of $1^{\{i\}}$.

Let $(P, \preceq)$ be a finite partially ordered set (*poset* for short), that is, a finite set P endowed with a reflexive, antisymmetric and transitive relation (see, e.g., Davey and Priestley (1990)). We denote by $\prec$ the asymmetric part of $\preceq$. We say that $x \in P$ *covers* $y \in P$, and we denote it by $y \prec\cdot\, x$, if $y \prec x$ and there is no $z \in P$ such that $y \prec z \prec x$.

We denote by $\min(P)$ and $\max(P)$ respectively the set of the minimal and maximal elements of $(P, \preceq)$. The *dual* of the poset $(P, \preceq)$, denoted by $(P, \preceq^\partial)$ (or simply P^∂), is the set P endowed with the reverse order, i.e., $x \preceq y$ if and only if $y \preceq^\partial x$.

Throughout the paper, it is understood that any subset Q of a poset $(P, \preceq)$ is endowed with $\preceq$ restricted to Q (we do not use a special symbol for the restriction).

A *chain* $C \subseteq P$ is a subset of P such that for any two elements $x, y \in C$, we have $x \preceq y$ or $y \preceq x$. The chain is *maximal* if no other chain contains it, or equivalently if $C = \{x_1, \ldots, x_q\}$, with $x_1 \prec\cdot\, x_2 \prec\cdot\, \cdots \prec\cdot\, x_q$ and $x_1 \in \min(P)$, $x_q \in \max(P)$. Its *length* is $q - 1$. The *height* of $i \in P$, denoted by $h(i)$, is the length of a longest chain from a minimal element to i. Elements of same height k form *level* k, denoted by L_k. Hence, $L_0 = \min(P)$ is the set of all minimal elements, $L_1 = \min(P \backslash L_0)$, $L_2 = \min(P \backslash (L_0 \cup L_1))$, etc. The *height* of N, denoted by $h(N)$, is the maximum of $h(i)$ taken over all elements of N. Similarly, we define the *depth $d(i)$* of an element $i \in N$ as its height in the dual poset P^∂. We denote by D_0 the set of all elements of depth 0 and we have $D_0 = \max(P)$, $D_1 = \max(P \backslash D_0)$, $D_2 = \max(P \backslash (D_0 \cup D_1))$, etc.

A *lattice* is a poset $(L, \preceq)$ where for each $x, y \in L$, their supremum $x \vee y$ and infimum $x \wedge y$ exist. The lattice is distributive if $\vee, \wedge$ obey distributivity.

A subset $Q \subseteq P$ is a *downset* of P if $x \in Q$ and $y \preceq x$ implies $y \in Q$. We denote by $\mathcal{O}(P)$ the set of downsets of $(P, \preceq)$. It is a well-known fact that $(\mathcal{O}(P), \subseteq)$ is a distributive lattice and every distributive lattice arises that way (Birkhoff, 1933). We denote by $\downarrow x$ the downset generated by an element $x \in P$, that is, $\downarrow x = \{y \in P \mid y \preceq x\}$. Similarly, for any $Q \subseteq P$, $\downarrow Q = \bigcup_{x \in Q} \downarrow x$.

Let N be a finite set of n players. A *set system* $\mathcal{F}$ on N is a collection of subsets of N containing N and $\emptyset$. Any nonempty subset in $\mathcal{F}$ is called a *feasible coalition*. We define a *cooperative TU-game with restricted cooperation* (or simply a *game*) on $\mathcal{F}$ as the pair $(\mathcal{F}, v)$, with $v : \mathcal{F} \to \mathbb{R}$, such that $v(\emptyset) = 0$.

In this paper we focus on a particular case of set systems, introduced by Faigle and Kern (1992) (games with precedence constraints). We consider a partial order $\preceq$ on N, which may express precedence constraints among players, or hierarchical relations. A coalition S is *feasible* if whenever $i \in S$, all subordinates of i also belong to S, i.e., S is a downset of $(N, \preceq)$. In other words, $\mathcal{F} = \mathcal{O}(N)$, hence $\mathcal{F}$ is a distributive lattice where supremum and infimum are respectively $\cup, \cap$.

A game $(\mathcal{F}, v)$ with $\mathcal{F} = \mathcal{O}(N)$ is *convex* if

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T) \tag{1}$$

for every $S, T \in \mathcal{F}$. It is *superadditive* if the above inequality is restricted to disjoint sets S, T. It is *strictly convex* if the above inequality is strict for all $S, T \in \mathcal{F}$ such that $S \setminus T \neq \emptyset$ and $T \setminus S \neq \emptyset$.

The following lemma extends a classical result when $\mathcal{F} = 2^N$.

Lemma 1. A game $(\mathcal{F}, v)$ is convex if and only if for all $i \in N$, for all $P \subseteq Q \subseteq N \setminus i$ such that $P \cup i, Q \in \mathcal{F}$,

$$v(P \cup i) - v(P) \leqslant v(Q \cup i) - v(Q). \tag{2}$$

Proof. In order to show that (1) implies (2), put $S = P \cup i$ and $T = Q$, and observe that $S \cap T = P$ and $S \cup T = Q \cup i$. For the other implication we may assume that $S \setminus T \neq \emptyset$ so that there exists $i_1, \ldots, i_p \in N$, where $p = |S \setminus T|$, such that $(S \cap T) \cup \{i_1, \ldots, i_m\} \in \mathcal{F}$ for all $m = 1, \ldots, p-1$ and $S \setminus T = \{i_1, \ldots, i_p\}$ (it suffices that $i_k \not\prec\cdot\, i_\ell$ for $k > \ell$). By (2),

$$v(S) - v(S \cap T) = \sum_{m=1}^{p} (v((S \cap T) \cup \{i_1, \ldots, i_m\}) - v((S \cap T) \cup \{i_1, \ldots, i_m\}))$$

$$\leqslant \sum_{m=1}^{p} (v(T \cup \{i_1, \ldots, i_m\}) - v(T \cup \{i_1, \ldots, i_{m-1}\})) = v(S \cup T) - v(T).$$

$\square$

The *core* of a game $(\mathcal{F}, v)$ is defined as follows:

$$\mathcal{C}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geqslant v(S) \text{ for all } S \in \mathcal{F}, \text{ and } x(N) = v(N)\},$$

where $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. If no ambiguity menaces, we write simply $\mathcal{C}(v)$. By definition, it is a convex closed polyhedron. Unless $\mathcal{F} = 2^N$, the core is unbounded or empty. Derks and Gilles (1995) showed (as well as Tomizawa (1983) in a refined form) that it contains no line, and found its rays (see also Fujishige (2005, Theorem 3.26)). It is well known from the theory of polyhedra that the core can be written as the Minkowski sum of its convex part and its conic part:

$$\mathcal{C}(v) = \operatorname{conv}(\operatorname{ext}(\mathcal{C}(v))) + \mathcal{C}(0),$$

where $\operatorname{ext}(\cdot)$, $\operatorname{conv}(\cdot)$ denote the extreme points of some convex set, and the convex hull of a set. Note that the conic part is obtained by replacing v by the zero function, hence the conic part depends solely on $\mathcal{F}$. Extremal rays of the core are generated by $1^j - 1^i$ for every $i, j \in N$ such that $j \prec\cdot\, i$ in $(N, \preceq)$. Therefore, extremal rays correspond bijectively to edges in the Hasse diagram of $(N, \preceq)$.

To avoid unboundedness, a natural solution is to take a certain subset of the core. In Grabisch (2011), some of the inequalities $x(S) \geqslant v(S)$ are turned into equalities so that no extremal ray exists any more. These equalities can be considered as additional binding constraints on certain coalitions. We call *normal collection* any collection $\mathcal{N} \subseteq \mathcal{F} \setminus \{\emptyset, N\}$ such that

$$\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geqslant v(S)\ \forall S \in \mathcal{F}, \quad x(S) = v(S)\ \forall S \in \mathcal{N}, \text{ and } x(N) = v(N)\}$$

is bounded for all games $(\mathcal{F}, v)$ on $\mathcal{F}$. Note that $\mathcal{N}$ is normal if and only if $\mathcal{C}_{\mathcal{N}}(0) = \{0\}$. It is remarked that the empty collection is normal if and only if $\mathcal{F} = 2^N$. We call $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ the *restricted core* w.r.t. $\mathcal{N}$, and we write simply $\mathcal{C}_{\mathcal{N}}(v)$ if no ambiguity menaces.

We denote by $\mathcal{NC}(\mathcal{F})$ the set of normal collections on $\mathcal{F}$. In Grabisch (2011), several normal collections are proposed (see Section 3.1). It is proved that a normal collection contains at least $h(N)$ sets, where $h(N)$ is the height of $(N, \preceq)$.

We say that an extremal ray r of $\mathcal{C}(0)$ is *deleted* by equality $x(S) = 0$ if $\mathcal{C}_{\{S\}}(0) = \{x \in \mathcal{C}(0) \mid x(S) = 0\}$ does not contain r any more. The following result from (Grabisch, 2011) is fundamental.

Lemma 2. Let $\mathcal{F} = \mathcal{O}(N)$. For $i, j \in N$ such that $j \prec\!\cdot\, i$, the extremal ray generated by $1^j - 1^i$ is deleted by equality $x(S) = 0$ if and only if $S \ni j$ and $S \not\ni i$.

Geometrically, a restricted core $\mathcal{C}_{\mathcal{N}}(v)$, whenever nonempty, is a bounded face of the core $\mathcal{C}(v)$ because it is bounded and defined by just turning some constraints that determine the core into binding constraints.

On the other hand, the authors have proposed the notion of bounded core (Grabisch and Sudhölter, to appear). The *bounded core* $\mathcal{C}^b(v)$ of $(\mathcal{F}, v)$ is the set of elements of $\mathcal{C}(v)$ that satisfy the following condition for any $i, j \in N$ such that $j \prec\!\cdot\, i$: There is no $\epsilon > 0$ such that $x + \epsilon 1^i - 1^j \in \mathcal{C}(v)$. Hence the bounded core is the set of core elements such that every player takes the maximum of her direct subordinates, in the sense that any transfer from a subordinate to her boss would result in a payoff vector outside the core. Also, the bounded core is the union of all bounded faces of $\mathcal{C}(v)$. If nonempty, it coincides with the core if and only if $\mathcal{F} = 2^N$.

The following result shows the relation between the two concepts (Grabisch and Sudhölter, to appear).

Proposition 1. Let $\mathcal{F} = \mathcal{O}(N)$, and consider any game $(\mathcal{F}, v)$. Then

$$\mathcal{C}^b(v) = \bigcup_{\mathcal{N} \in \mathcal{NC}(\mathcal{F})} \mathcal{C}_{\mathcal{N}}(v).$$

3 The set of normal collections

Let $(N, \preceq)$ be a poset and consider $\mathcal{F} = \mathcal{O}(N)$. In order to avoid pathologic cases we assume throughout this section that $\preceq\, \neq \emptyset$, i.e., $h(N) > 0$. We now define some possible properties of a normal collection.

Definition 1. Let $\mathcal{N} = \{N_1, \ldots, N_q\}$ be a normal collection.

(i) $\mathcal{N}$ is a *minimal* collection if no proper subcollection is normal.

(ii) $\mathcal{N}$ is a *thin* collection if no $S \in \mathcal{N}$ may be replaced by a proper subset of S without losing normality.

(iii) $\mathcal{N}$ is a *short* collection if it contains exactly $h(N)$ subsets.

(iv) $\mathcal{N}$ is a *nested* collection if it is a chain in $\mathcal{F}$.

Note that any short normal collection is minimal, but the converse is not true (see Example 4).

We give some elementary properties of normal collections.

Lemma 3. Let $\mathcal{N} = \{N_1, \ldots, N_q\}$ be a normal nested collection. Then $N_k \setminus N_{k-1}$ is an antichain in $(N, \preceq)$ for $k = 1, \ldots, q$, with $N_0 = \emptyset$.

Proof. If $N_k \setminus N_{k-1}$ is not an antichain, then there exist $i, j \in N_k \setminus N_{k-1}$ such that $i \prec\!\!\cdot\ j$. Since $\mathcal{N}$ is nested, no set in $\mathcal{N}$ will contain i and not j. Then by Lemma 2, the ray $1^i - 1^j$ is not deleted by $\mathcal{N}$. $\qquad\square$

Lemma 4. Suppose that $\mathcal{N}$ is a collection containing a set S such that $N \setminus S \in \mathcal{F}$. If $\mathcal{N}$ is normal, then $\mathcal{N} \setminus \{S\}$ is normal.

Proof. If there were an element $S \in \mathcal{N}$ such that $N \setminus S \in \mathcal{F}$, then the condition $x(S) = 0$ would not eliminate any extremal ray because $j \in S$ and $j \prec\!\!\cdot\ i$ would imply $i \in S$. Therefore, S can be discarded from $\mathcal{N}$. $\qquad\square$

Lemma 5. Let $\mathcal{N}$ be a normal collection that contains two disjoint sets P and Q with $P \cup Q \neq N$. If $\mathcal{N}' = (\mathcal{N} \setminus \{P, Q\}) \cup \{P \cup Q\}$, then $\mathcal{N}'$ is a normal collection, and $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \subseteq \mathcal{C}_{\mathcal{N}'}(\mathcal{F}, v)$ for every superadditive game $(\mathcal{F}, v)$.

Proof. As $\mathcal{F} = \mathcal{O}(N)$, $P \cup Q \in \mathcal{F}$. Let $i, j \in N$ such that $j \prec\!\!\cdot\ i$. If $j \in P \not\ni i$, then $i \notin Q$ because $Q \in \mathcal{O}(N)$ and $P \cap Q = \emptyset$. Similarly, if $j \in Q \not\ni i$, then $i \notin P$. Hence $\mathcal{N}'$ is still normal by Lemma 2. Now, let $x \in \mathcal{C}_{\mathcal{N}}(v)$. Then $x(T) = v(T)$ for all $T \in \mathcal{N}' \setminus \{P \cup Q\}$, $x(S) \geqslant v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{N}'$, $x(N) = v(N)$, $x(P) = v(P)$, and $x(Q) = v(Q)$, $x(P \cup Q) \geqslant v(P \cup Q) \geqslant v(P) + v(Q)$, where the last inequality is valid by superadditivity. Hence $x(P \cup Q) = v(P \cup Q)$, so that $x \in \mathcal{C}_{\mathcal{N}'}(v)$. $\qquad\square$

3.1 Some remarkable normal collections

We introduce some notation. We say that an element x of a poset $(P, \preceq)$ is *disconnected* if it is incomparable with any other element of P (i.e., it has no link in the Hasse diagram of $(P, \preceq)$), equivalently, if it is both minimal and maximal. We denote by P° the set of elements of P which are not disconnected:

$$P^\circ = P \setminus (\min(P) \cap \max(P)).$$

For any subset $Q \subseteq P$, we define $\underline{Q} = \{y \in P \mid y \prec\!\!\cdot\ x \text{ for some } x \in Q\}$.

We introduce the *upwards* normal collection $\mathcal{N}^{\mathrm{u}} = \{N_1^{\mathrm{u}}, \ldots, N_q^{\mathrm{u}}\}$, defined as follows (Grabisch, 2011).

$$
\begin{aligned}
N_1^{\mathrm{u}} &= \min(N^\circ) \\
N_2^{\mathrm{u}} &= \downarrow\!\min((N \setminus N_1^{\mathrm{u}})^\circ) = \downarrow\!\min((N \setminus L_0)^\circ) \\
&\vdots \quad = \vdots \\
N_q^{\mathrm{u}} &= \downarrow\!\min((N \setminus (N_1^{\mathrm{u}} \cup \cdots \cup N_{q-1}^{\mathrm{u}}))^\circ) = \downarrow\!\min((N \setminus (L_0 \cup \cdots \cup L_{q-2}))^\circ)
\end{aligned}
$$

where q is minimal such that $N \setminus (N_1^{\mathrm{u}} \cup \cdots \cup N_q^{\mathrm{u}})^\circ = \emptyset$ (for the definitions of $h(P)$, L_k, D_k, etc., see Section 2). Note that since $(N \setminus (N_1^{\mathrm{u}} \cup \cdots \cup N_{k-1}^{\mathrm{u}}))^\circ = (N \setminus (L_0 \cup \cdots \cup L_{k-2}))^\circ$

for $k > 1$, we deduce that $q = h(N)$, hence the upwards collection is short, therefore minimal. Grabisch (2011) proved that the upwards collection is also thin.

The upwards collection is built from the successive removal of minimal elements, hence its name. One can proceed as well from maximal elements. This gives the *downwards* collection $\mathcal{N}^d = \{N_1^d, \ldots, N_q^d\}$, defined as follows.

$$N_1^d = \downarrow\overline{\max(N)}$$
$$N_2^d = \downarrow\overline{\max(N \setminus \max(N))} = \downarrow\overline{\max(N \setminus D_0)}$$
$$\vdots = \vdots$$
$$N_q^d = \downarrow\overline{\max(N \setminus (D_0 \cup \cdots \cup D_{q-2}))}$$

with $q = h(N)$ again, since $\overline{\max(N \setminus (D_0 \cup \cdots \cup D_{h(N)-1}))} = \emptyset$. Therefore the downwards collection is short, hence minimal. Observe that the operation $()^\circ$ is not necessary here, since any disconnected element is eliminated by the $\overline{()}$ operation.

The downwards collection can be expressed in a simpler way. Observe that $\underline{N} = D_1 \cup \cdots \cup D_{h(N)}$, $\underline{N \setminus D_0} = D_2 \cup \cdots \cup D_{h(N)}$, etc. Moreover, $\underline{N} = \downarrow\overline{\max(N)}$, $\underline{N \setminus D_0} = \downarrow\overline{\max(N \setminus D_0)}$, etc., hence we obtain

$$N_1^d = D_1 \cup D_2 \cup \cdots \cup D_{h(N)}$$
$$N_2^d = D_2 \cup \cdots \cup D_{h(N)}$$
$$\vdots = \vdots$$
$$N_{h(N)}^d = D_{h(N)},$$

which shows that the downwards collection is nested. In addition, we have the following result.

Proposition 2. The downwards collection is thin.

Proof. Consider $N_\ell^d = D_\ell \cup \cdots \cup D_{h(N)}$ for some $1 \leqslant \ell \leqslant h(N)$, and remove an element k from it which is maximal in this set (if not, one cannot remove it since $N_\ell^d \setminus \{k\}$ would not be a downset). Note that $k \in D_\ell$. Then there exists an element j such that $k \prec\cdot j$. Therefore the ray $1^k - 1^j$ is not deleted by $N_\ell^d \setminus \{k\}$. Since $k \in D_\ell$, it follows that $k \notin N_{\ell+1}^d$, so that ray $1^k - 1^j$ remains. $\square$

A third remarkable collection is the one proposed by Grabisch and Xie (2011), and is defined as follows: $\mathcal{N}^{GX} = \{L_0, L_0 \cup L_1, \ldots, L_0 \cup \cdots \cup L_{h(N)-1}\}$. It is a short nested collection, and as observed by Grabisch (2011), there is no short nested collection such that each set includes a set of $\mathcal{N}^{GX}$. Observe that the downwards collection can be seen as a "dual" of $\mathcal{N}^{GX}$, since $\mathcal{N}^d = \{D_1 \cup \cdots \cup D_{h(N)}, \ldots, D_{h(N)-1} \cup D_{h(N)}, D_{h(N)}\}$.

Example 1. Consider the poset $(N, \leqslant)$ of 9 elements depicted below.

We have $L_0 = \{1, 2, 3\}$, $L_1 = \{4, 5, 6, 9\}$, $L_2 = \{7, 8\}$, and $D_0 = \{7, 8, 9\}$, $D_1 = \{2, 4, 5, 6\}$, $D_2 = \{1, 3\}$. The upwards collection is $\{123, 13456\}$, the downwards collection is $\{13, 123456\}$, and the Grabisch-Xie collection is $\{123, 1234569\}$ (where it is understood that 123 is a shorthand for $\{1, 2, 3\}$, etc.).

3.2 Nested closure of a normal collection

Let $\mathcal{N}$ be a normal collection and let q denote its cardinality. An *ordering* of $\mathcal{N}$ is a bijection $\sigma : \mathcal{N} \to \{1, \ldots, q\}$. For any ordering σ define its corresponding *nested closure* by $\overline{\mathcal{N}}_\sigma = \left\{ \bigcup_{i=1}^{k} \sigma^{-1}(i) \mid k = 1, \ldots, q \right\}$. We say that σ is a *feasible ordering* of $\mathcal{N}$ if

$$\sigma^{-1}(k) \setminus \left(\sigma^{-1}(k-1) \cup \cdots \cup \sigma^{-1}(1) \right) \text{ is an antichain of } (N, \preceq) \text{ for all } k = 1, \ldots, q, \quad (3)$$

where $\sigma^{-1}(0) = \emptyset$.

Example 2. (Example 1 cont.) Consider the normal collection $\mathcal{N} = \{136, 134, 125\}$. Then $\mathcal{N}$ has no feasible ordering because none of its elements is an antichain.

Theorem 1. Let $\mathcal{N}$ be a normal collection and σ be an ordering of $\mathcal{N}$. Then the nested closure $\overline{\mathcal{N}}_\sigma$ is normal if and only if σ is a feasible ordering of $\mathcal{N}$.

Proof. Consider a normal collection $\mathcal{N}$, an ordering σ of $\mathcal{N}$, and denote by $\overline{\mathcal{N}}_\sigma = \{\overline{N}_1, \ldots, \overline{N}_q\}$ its nested closure, i.e., $\overline{N}_k = \bigcup_{i=1}^{k} \sigma^{-1}(i)$.

Suppose that $\overline{\mathcal{N}}_\sigma$ is not normal. Then there exists a ray $1^i - 1^j$ with $i \prec\cdot j$ which is not deleted by $\overline{\mathcal{N}}_\sigma$, i.e., any set in $\overline{\mathcal{N}}_\sigma$ either contains both i and j or none of them. Since $\overline{\mathcal{N}}_\sigma$ is nested, there exists some $k \in \{1, \ldots, q\}$ such that $\overline{N}_1, \ldots, \overline{N}_k$ contain neither i nor j, and $\overline{N}_{k+1}, \ldots, \overline{N}_q$ contain both i and j. Therefore, in $\mathcal{N}$, $\sigma^{-1}(1), \ldots, \sigma^{-1}(k)$ contain neither i nor j, while $\sigma^{-1}(k+1)$ contains them both. Since $i \prec\cdot j$, this contradicts (3), hence σ is not a feasible ordering.

Conversely, suppose that σ is not feasible for $\mathcal{N}$. Then there exist $i, j \in N$ with $i \prec j$, and $k \in \{1, \ldots, q\}$ such that $\sigma^{-1}(k)$ contains both i, j, while the sets $\sigma^{-1}(k-1), \ldots, \sigma^{-1}(1)$ contain none of them. Then, $\overline{N}_k \setminus \overline{N}_{k-1}$ is not an antichain, and by Lemma 3, we conclude that $\overline{\mathcal{N}}_\sigma$ is not normal. $\qquad \square$

Proposition 3. For the upwards collection the ordering σ defined by $\sigma(N_k^{\mathrm{u}}) = k$, for $k = 1, \ldots, h(N)$, is feasible.

Proof. Noting that $\min(N \setminus (L_0 \cup \cdots \cup L_{k-2})) = L_{k-1}$, we see that $N_k^{\mathrm{u}} \subseteq L_0 \cup \cdots \cup L_{k-1}$. Therefore, elements in $N_k^{\mathrm{u}} \cap L_{k-1}$ do not belong to $N_1^{\mathrm{u}} \cup \cdots \cup N_{k-1}^{\mathrm{u}}$. We claim that if $x \in N_k^{\mathrm{u}} \cap (L_0 \cup \cdots \cup L_{k-2})$, then it belongs to $N_1^{\mathrm{u}} \cup \cdots \cup N_{k-1}^{\mathrm{u}}$. Therefore, $N_k^{\mathrm{u}} \setminus (N_1^{\mathrm{u}} \cup \cdots \cup N_{k-1}^{\mathrm{u}})$ is an antichain, since it is a subset of L_{k-1}. This being true for $k = 1, \ldots h(N)$, it follows that σ is a feasible ordering.

Proof of the claim: Suppose w.l.o.g. that $x \in L_j$ for some $j \in \{0, \ldots, k-2\}$. Clearly, $x \notin N_1^{\mathrm{u}} \cup \cdots \cup N_j^{\mathrm{u}}$. Suppose that $x \notin N_{j+1}^{\mathrm{u}}$. Since $N_{j+1}^{\mathrm{u}} = \downarrow\min((L_j \cup \cdots \cup L_{h(N)})^\circ)$, it follows that x is disconnected in $L_j \cup \cdots \cup L_{h(N)}$, i.e., there is no $y \in L_j \cup \cdots \cup L_{h(N)}$ such that $x \prec y$. But $x \notin L_{k-1}$ and $x \in N_k^{\mathrm{u}}$ (which is a downset) imply that it must exist some $y \in L_{k-1}$ such that $x \prec y$, a contradiction. Therefore $x \in N_{j+1}^{\mathrm{u}}$. $\qquad \square$

It follows by Theorem 1 that the nested closure of $\mathcal{N}^{\mathrm{u}}$ is a normal collection, called by Grabisch (2011) the *Weber collection* $\mathcal{N}^{\mathrm{W}} = \{N_1^{\mathrm{W}}, \ldots, N_{h(N)}^{\mathrm{W}}\}$, and we have

$$N_1^{\mathrm{W}} = N_1^{\mathrm{u}}$$
$$N_2^{\mathrm{W}} = N_1^{\mathrm{u}} \cup N_2^{\mathrm{u}}$$
$$\vdots = \vdots$$
$$N_{h(N)}^{\mathrm{W}} = N_1^{\mathrm{u}} \cup \cdots \cup N_{h(N)}^{\mathrm{u}}.$$

The Weber collection is short, therefore minimal, but not thin in general, as shown by the next result and Example 3.

Proposition 4. Consider the downwards and the Weber collections. It holds

$$N_{h(N)}^{\mathrm{d}} \subseteq N_1^{\mathrm{W}}, \quad N_{h(N)-1}^{\mathrm{d}} \subseteq N_2^{\mathrm{W}}, \ldots, \quad N_1^{\mathrm{d}} \subseteq N_{h(N)}^{\mathrm{W}}. \tag{4}$$

Proof. Recall that the Weber collection is $\{N_1^{\mathrm{u}}, N_1^{\mathrm{u}} \cup N_2^{\mathrm{u}}, \ldots, N_1^{\mathrm{u}} \cup N_2^{\mathrm{u}} \cup \cdots \cup N_{h(N)}^{\mathrm{u}}\}$. We will prove (4) by induction. We first prove that $N_{h(N)}^{\mathrm{d}} \subseteq N_1^{\mathrm{u}}$. Take $i \in N_{h(N)}^{\mathrm{d}}$. Then i is covered by some j, and does not cover any element in N by construction. Therefore i is minimal in N and is not disconnected, hence it belongs to N_1^{u}.

Induction hypothesis: $N_{h(N)-l+1}^{\mathrm{d}} \subseteq N_1^{\mathrm{u}} \cup \cdots \cup N_l^{\mathrm{u}}$ for $l = 1, \ldots, k < h(N)$. Let us prove that the property still holds for $l = k + 1$. Take any $i \in N_{h(N)-k}^{\mathrm{d}} \setminus N_{h(N)-k+1}^{\mathrm{d}} = D_{h(N)-k}$ and note that it is not a disconnected element in $(N, \preceq)$. Then either i covers some $j \in N_{h(N)-k+1}^{\mathrm{d}}$, or i is incomparable with these elements. If i covers some j in $N_{h(N)-k+1}^{\mathrm{d}}$, since by induction hypothesis, all these j's are members of $N_1^{\mathrm{u}} \cup \cdots \cup N_k^{\mathrm{u}}$, either i belongs to $N_1^{\mathrm{u}} \cup \cdots \cup N_k^{\mathrm{u}}$, or to $\min((N \setminus (N_1^{\mathrm{u}} \cup \cdots \cup N_k^{\mathrm{u}}))^{\circ})$, in which case $i \in N_{k+1}^{\mathrm{u}}$. On the other hand, if i is incomparable with elements of $N_{h(N)-k+1}^{\mathrm{d}}$, it does not cover any element in N (otherwise this element would belong to $N_{h(N)-k+1}^{\mathrm{d}}$, and i would be comparable with an element of $N_{h(N)-k+1}^{\mathrm{d}}$, a contradiction), which implies that it is minimal in N, and therefore belongs to N_1^{u}. $\qquad \square$

Example 3. (Example 1 continued) The Weber collection is $123, 123456$, hence in the above proposition there are cases where strict inclusion occurs for some set.

Take any nested normal collection $\mathcal{N} = \{N_1, \ldots, N_q\}$, $N_1 \subset \cdots \subset N_q$, and define its *(unnested) opening* by

$$\mathcal{N}^{\circ} = \{N_1, \downarrow(N_2 \setminus N_1), \ldots, \downarrow(N_q \setminus (N_1 \cup \cdots \cup N_{q-1}))\}.$$

Proposition 5. If $\mathcal{N}$ is a nested normal collection, then its opening $\mathcal{N}^{\circ}$ is normal and there exists a feasible ordering of $\mathcal{N}^{\circ}$ whose corresponding nested closure is $\mathcal{N}$.

Proof. Suppose $\mathcal{N}^{\circ}$ is not normal. Then there exists a ray $1^i - 1^j$, with $i \prec\!\cdot\, j$, which is not deleted by $\mathcal{N}^{\circ}$, i.e., all sets in $\mathcal{N}^{\circ}$ either contain both i, j or none of them. Hence, the same property holds for the sets in $\mathcal{N}$, which implies that $\mathcal{N}$ is not normal, a contradiction.

Put $\mathcal{N}^{\circ} = \{N_1^{\circ}, \ldots, N_q^{\circ}\}$. Since $\mathcal{N}^{\circ}$ and $\mathcal{N}$ are normal, and $\overline{\mathcal{N}^{\circ}}_{\sigma} = \mathcal{N}$ with $\sigma(N_k^{\circ}) = k$ for $k = 1, \ldots, q$, by Theorem 1, it follows that σ is feasible. $\qquad \square$

Corollary 1. Every nested normal collection arises as the closure of a normal collection for which there is a feasible ordering, or is itself an open collection (i.e., invariant under the opening).

4 Balancedness conditions

Let $\mathcal{N}$ be a normal collection and $(\mathcal{F}, v)$ a game. Whether $\mathcal{C}_{\mathcal{N}}(v)$ is empty or not depends on both the normal collection and the game. It may happen that $\mathcal{C}_{\mathcal{N}}(v)$ is empty while $\mathcal{C}(v)$ is not. If $\mathcal{C}_{\mathcal{N}}(v)$ is nonempty, then it is a bounded face of $\mathcal{C}(v)$.

The following result has been shown by Grabisch (2011).

Proposition 6. Suppose that $(\mathcal{F}, v)$ is a convex game. Then for any normal nested collection $\mathcal{N}$, the restricted core $\mathcal{C}_{\mathcal{N}}(v)$ is nonempty.

A direct application of the Farkas Lemma leads to the following.

Proposition 7. Let $\mathcal{N}$ be a normal collection. $\mathcal{C}_{\mathcal{N}}(v) \neq \emptyset$ if and only if for all vectors $y \in \mathbb{R}^{\mathcal{F} \setminus \{\emptyset\}}$ satisfying

$$\sum_{S \ni i, S \in \mathcal{F}} y_S = 0, \quad i \in N$$

$$y_S \geqslant 0, \quad S \in \mathcal{F} \setminus \{\mathcal{N} \cup N\},$$

it holds

$$\sum_{S \in \mathcal{F} \setminus \{\emptyset\}} y_S v(S) \leqslant 0.$$

Another form can be obtained proceeding as in the classical Bondareva-Shapley theorem (Bondareva, 1963; Shapley, 1971). Considering a normal collection $\mathcal{N}$, we say that a collection $\mathcal{B} \subseteq \mathcal{F}$ is $\mathcal{N}$-*balanced* if there exists $y_S > 0$, $S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} y_S 1^S = \sum_{S \in \mathcal{N} \cup N} 1^S$. We call $\{y_S\}_{S \in \mathcal{B}}$ a *system of $\mathcal{N}$-balancing weights*.

Theorem 2. Let $\mathcal{N}$ be a normal collection. $\mathcal{C}_{\mathcal{N}}(v) \neq \emptyset$ if and only if for every $\mathcal{N}$-balanced collection $\mathcal{B}$ with $\mathcal{N}$-balancing weights $\{y_S\}_{S \in \mathcal{B}}$, it holds

$$\sum_{S \in \mathcal{B}} y_S v(S) \leqslant \sum_{S \in \mathcal{N} \cup N} v(S). \tag{5}$$

Proof. We consider the following linear program with $x \in \mathbb{R}^n$:

$$
\begin{array}{rrcl}
\min & z & = & \sum_{S \in \mathcal{N} \cup N} x(S) \\
\text{s.t.} & x(S) & \geqslant & v(S), \quad S \in \mathcal{F}
\end{array}
$$

The optimal value z^* of z is $\sum_{S \in \mathcal{N} \cup N} v(S)$ if and only if $\mathcal{C}_{\mathcal{N}}(v) \neq \emptyset$. The dual problem reads

$$
\begin{array}{rrcl}
\max & w & = & \sum_{S \in \mathcal{F}} y_S v(S) \\
\text{s.t} & \sum_{S \ni i, S \in \mathcal{F}} y_S & = & \sum_{S \ni i, S \in \mathcal{N} \cup N} y_S, \quad i \in N \\
& y_S & \geqslant & 0, \quad S \in \mathcal{F}.
\end{array}
$$

By the duality theorem, $w^* = z^*$, which implies that any feasible solution satisfies $\sum_{S \in \mathcal{F}} y_S v(S) \leqslant \sum_{S \in \mathcal{N} \cup N} v(S)$. $\qquad \square$

Note that this generalizes a result by Grabisch and Xie (2011). As it can be observed, the proof follows the classical argument, where in the right members of (in)equalities, N is formally replaced by $\mathcal{N} \cup N$. Therefore, a strong form of the theorem can be obtained as well, which we give without proof.

We say that a collection $\mathcal{B} \subseteq \mathcal{F}$ is *minimal $\mathcal{N}$-balanced* if $\mathcal{B}$ is $\mathcal{N}$-balanced and no proper subcollection is. Similarly to the classical case, we obtain that a minimal $\mathcal{N}$-balanced collection has a unique system of $\mathcal{N}$-balancing weights, and we get the following result.

Theorem 3. Let $\mathcal{N}$ be a normal collection. $\mathcal{C}_{\mathcal{N}}(v) \neq \emptyset$ if and only if (5) holds for any minimal $\mathcal{N}$-balanced collection $\mathcal{B}$, where $\{y_S\}_{S \in \mathcal{B}}$ is the unique system of $\mathcal{N}$-balancing weights for $\mathcal{B}$.

Convexity is not a necessary condition for balancedness. Indeed, Grabisch and Sudhölter (to appear, Lemma 3.2) have shown that if $(N, \preceq)$ is connected (that is, for any $i, j \in N$, there is a sequence of elements $i = i_1, i_2, \ldots, i_k = j$ such that i_ℓ and $i_{\ell+1}$ are comparable, for $\ell = 1, \ldots, k-1$), then the core $\mathcal{C}(\mathcal{F}, v)$ is nonempty for any game v, in particular for nonconvex games. Since $\mathcal{C}(\mathcal{F}, v)$ contains no line, the bounded core too is nonempty, implying that there exists a nonempty restricted core for any game v.

5   The case of convex games

By Proposition 6, we know that for convex games, nested normal collections play a central rôle. The next result proves that moreover, one only needs to consider nested normal collections, and the main result of this section (Theorem 5) will show the converse of Proposition 6 for strictly convex games.

Proposition 8. For any normal collection $\mathcal{N}$, there exists a nested normal collection $\mathcal{N}'$ such that, for any convex game v,

$$\mathcal{C}_{\mathcal{N}}(v) \subseteq \mathcal{C}_{\mathcal{N}'}(v).$$

The proof is based on the following technical lemma. For any collection $\emptyset \neq \mathcal{G} \subseteq \mathcal{F}$ we define

$$F(\mathcal{G}) = \{(\mathcal{G} \setminus \{T, T'\}) \cup \{T \cap T', T \cup T'\} \mid T, T' \in \mathcal{G}\}.$$

Note that any element of $F(\mathcal{G})$ is nonempty and does not possess a larger cardinality than $\mathcal{G}$ and that $\mathcal{G} \in F(\mathcal{G})$.

Lemma 6. With $g = |\mathcal{G}|$, the $\frac{g(g-1)}{2}$-fold composition of F applied to $\mathcal{G}$, i.e., $F^{\frac{g(g-1)}{2}}(\mathcal{G})$, contains a nested collection.

Proof. We proceed by induction on g. If $g = 1$, then $\mathcal{G}$ is already nested. Assume that the lemma is valid for any $g < k$ for some $k > 1$. Now, if $g = k$, then let $\mathcal{G} = \{T_1, \ldots, T_g\}$, define $T_1' = \bigcup_{j=1}^{g} T_j$, $T_k' = T_k \cap \bigcup_{j=1}^{k-1} T_k$ for $k = 2, \ldots, g$ and let $\mathcal{G}' = \{T_1', \ldots, T_g'\}$. Note that $\mathcal{G}' \in F^{g-1}(\mathcal{G})$ and that $2 \leqslant g' = |\mathcal{G}'| \leqslant g$. By the inductive hypothesis, $F^{\frac{(g'-1)(g'-2)}{2}}(\mathcal{G}' \setminus \{T_1'\})$ contains a nested collection $\mathcal{G}''$. By construction all elements of $\mathcal{G}''$ are contained in T_1'. Hence, $\mathcal{G}'' \cup \{T_1'\}$ is a nested collection in $F^{g-1+\frac{(g'-1)(g'-2)}{2}}(\mathcal{G}) \subseteq F^{\frac{g(g-1)}{2}}(\mathcal{G})$. $\qquad\square$

Proof. (of Proposition 8) We may assume that $\mathcal{N} \neq \emptyset$. Let $T, T' \in \mathcal{N}$ and v be a convex game. In view of Lemma 6 it suffices to show that $\mathcal{N}' = (\mathcal{N} \setminus \{T, T'\}) \cup \{T \cup T', T \cap T'\}$ is (a) normal and (b) $\mathcal{C}_{\mathcal{N}}(v) \subseteq \mathcal{C}_{\mathcal{N}'}(v)$. In view of Lemma 2 and by interchanging the roles of T and T' if necessary, in order to show (a) it suffices to prove that, for any $i, j \in N$ such that $i \in T \not\ni j$ and $i \prec\cdot j$ either $j \notin T \cup T'$ or $i \in T \cap T'$. Now, if $j \in T \cup T'$, i.e., $j \in T'$, then $i \in T'$ because T' is a downset. Hence, $i \in T \cap T'$ in this case. In order to show (b) let $x \in \mathcal{C}_{\mathcal{N}}(v)$. In order to show that $x \in \mathcal{C}_{\mathcal{N}'}(v)$ it suffices to show that $x(T \cup T') = v(T \cup T')$ and $x(T \cap T') = v(T \cap T')$. As the game is convex,

$$v(T \cup T') + v(T \cap T') \leqslant x(T \cup T') + x(T \cap T')$$
$$= x(T) + x(T') = v(T) + v(T') \leqslant v(T \cup T') + v(T \cap T')$$

so that the desired equalities follow immediately. $\qquad\square$

We recall the notion of marginal vector and restricted marginal vector introduced by Grabisch (2011). We consider the set of maximal chains in $\mathcal{F}$. This set is in a one-to-one correspondence with the set $\mathcal{L}(\preceq)$ of linear extensions of $(N, \preceq)$, i.e., to any maximal chain $C = \{\emptyset, S_1, S_2, \dots, S_n\}$, $\emptyset \subset S_1 \subset \cdots \subset S_n = N$, corresponds a unique permutation π on N with $S_i := \{\pi(1), \dots, \pi(i)\}$, $i = 1, \dots, n$, and vice versa. The linear extension is given by the sequence $\pi(1), \pi(2), \dots, \pi(n)$. Considering a game $(\mathcal{F}, v)$, the *marginal vector* $a^\pi(v) \in \mathbb{R}^N$ associated to the linear extension π (equivalently, $a^C(v)$ associated to the maximal chain C) is the payoff vector defined by

$$a^\pi_{\pi(i)}(v) := v(S_i) - v(S_{i-1}) = v(\{\pi(1), \dots, \pi(i)\}) - v(\{\pi(1), \dots, \pi(i-1)\}), \quad i \in N.$$

Consider a nested collection $\mathcal{G}$ (not necessarily normal). A *restricted maximal chain* w.r.t. $\mathcal{G}$ is a maximal chain (from $\emptyset$ to N) in $\mathcal{F}$ containing $\mathcal{G}$. Associated linear extensions are called *restricted linear extensions*, and the set of restricted linear extensions w.r.t. $\mathcal{G}$ is denoted by $\mathcal{L}_{\mathcal{G}}(\preceq)$. A *restricted marginal vector* is a marginal vector whose underlying maximal chain is restricted.

The following result is noteworthy and extends the classical result of Shapley (1971) and Ichiishi (1981)[1].

Theorem 4. A game $(\mathcal{F}, v)$ is convex if and only if $a^\pi(v) \in \mathcal{C}(v)$ for every $\pi \in \mathcal{L}(\preceq)$.

Proof. Necessity: Assume that v is convex, let $S \in \mathcal{F} \setminus \{\emptyset\}$, and $\pi \in \mathcal{L}(\preceq)$. We have to show that

$$\sum_{i \in S} a^\pi_i(v) \geqslant v(S). \tag{6}$$

Let $i_1, \dots, i_s \in S$, $s = |S|$, be chosen so that $\pi^{-1}(i_1) < \cdots < \pi^{-1}(i_s)$. Then $T_k = \{i_1, \dots, i_k\} = S_{\pi^{-1}(i_k)} \cap S \in \mathcal{F}$ for any $k = 1, \dots, s$, using the above notation. By (2),

$$v(S_{\pi^{-1}(i_k)}) - v(S_{\pi^{-1}(i_k)} \setminus i_k) \geqslant v(T_k) - v(T_{k-1}) \text{ for all } k = 1, \dots, s,$$

---

[1] This result is in fact already known. It has been proved for acyclic permission structures by Derks and Gilles (1995), while it is known from Algaba et al. (2004) that these set systems are equivalent to set systems of the form $\mathcal{O}(N)$. Also, Grabisch and Xie (2008) proved it in an unpublished paper. The "only if" part is known from Fujishige and Tomizawa (1983) (also cited in (Fujishige, 2005, Theorem 3.22)). We provide a simpler proof of this result, thereby also making the current paper more self-contained.

where $T_0 = \emptyset$. Summing up all these inequalities yields (6).

Sufficiency: Let v be a game and assume that $a^\pi(v) \in \mathcal{C}(v)$ for all $\pi \in \mathcal{L}(\preceq)$. Let $S, T \in \mathcal{F}$ so that $S \setminus T \neq \emptyset \neq T \setminus S$. Let $S \cap T = \{i_1, \ldots, i_r\}$, $T \setminus S = \{i_{r+1}, \ldots, i_t\}$, $S \setminus T = \{i_{t+1}, \ldots, i_q\}$, and $N \setminus (S \cup T) = \{i_{q+1}, \ldots, i_n\}$ such that, for any $j \in N$, $\{i_1, \ldots, i_j\} \in \mathcal{F}$. Then the permutation π defined by $\pi(j) = i_j$ for any $j \in N$ is a linear extension. Hence

$$
\begin{aligned}
v(S) &\leqslant& \sum_{i \in S} a_i^\pi(v) = \sum_{i \in S} (v(S_{\pi^{-1}(i)}) - v(S_{\pi^{-1}(i)} \setminus i)) \\
&=& \sum_{j=1}^{r} (v(\{i_1, \ldots, i_j\}) - v(\{i_1, \ldots, i_{j-1}\})) + \sum_{j=t+1}^{q} (v(T \cup \{i_{t+1}, \ldots, i_j\}) \\
&& - v(T \cup \{i_{t+1}, \ldots, i_{j-1}\})) \\
&=& v(S \cap T) + v(S \cup T) - v(T),
\end{aligned}
$$

so that the proof is complete. $\qquad\square$

In Grabisch (2011, Theorems 4 and 5), it is proved that for any nested normal collection $\mathcal{N}$, the set of restricted marginal vectors is the set of extreme points of $\mathcal{C}_\mathcal{N}(v)$ if v is convex.

Proposition 9. For any nested normal collection $\mathcal{N}$ of $\mathcal{F}$ and any convex game v, $\{a^\pi(v) \mid \pi \in \mathcal{L}_\mathcal{N}(\preceq)\}$ is the set of extreme points of $\mathcal{C}_\mathcal{N}(v)$.

An inspection of the proof shows that the result extends to any nested collection, not necessarily normal.

We are now in position to show the main result of this section. Let $\mathcal{MNNC}(\mathcal{F})$ denote the set of minimal nested normal collections of $\mathcal{F}$.

Theorem 5. (i) For any convex game v and any nested normal collection $\mathcal{N}$ of $\mathcal{F}$, $\mathcal{C}_\mathcal{N}(v) \neq \emptyset$. Moreover, if v is strictly convex, then $\dim \mathcal{C}_\mathcal{N}(v) = n - |\mathcal{N}| - 1$.

(ii) For any convex game v,

$$
\mathcal{C}^b(v) = \bigcup_{\mathcal{N} \in \mathcal{MNNC}(\mathcal{F})} \mathcal{C}_\mathcal{N}(v).
$$

Moreover, no term in the union is redundant if v is strictly convex.

(iii) Let $\mathcal{N}$ be a normal collection of $\mathcal{F}$. If v is strictly convex, then $\mathcal{C}_\mathcal{N}(v) \neq \emptyset$ if and only if $\mathcal{N}$ is nested.

Proof. (i) The first assertion is Proposition 6. By Proposition 9, for any $\pi \in \mathcal{L}_\mathcal{N}(\preceq)$, $a^\pi(v) \in \mathcal{C}_\mathcal{N}(v)$. Let $x = \frac{1}{|\mathcal{L}_\mathcal{N}(\preceq)|} \sum_{\pi \in \mathcal{L}_\mathcal{N}(\preceq)} a^\pi(v)$. If v is strictly convex, then in order to show the equation it suffices to prove that $x(S) > v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{N} \cup \{\emptyset, N\}$. Let $\mathcal{N} \cup \{\emptyset, N\} = \{T_0, \ldots, T_r\}$, where $\emptyset = T_0 \subsetneq T_1 \subsetneq \cdots \subsetneq T_r = N$. Suppose there exists $j \in \{1, \ldots, r-1\}$ such $T_j \setminus S \neq \emptyset \neq S \setminus T_j$, then

$$
v(S) + v(T_j) < v(S \cap T_j) + v(S \cup T_j) \leqslant x(S \cap T_j) + x(S \cup T_j) = x(S) + x(T_j) = x(S) + v(T_j)
$$

by strict convexity and because $x \in \mathcal{C}(v)$. Otherwise there exists $\ell \in \{0, \ldots, r-1\}$ such that $T_\ell \subsetneq S \subsetneq T_{\ell+1}$. Let $S' = T_\ell \cup (T_{\ell+1} \setminus S)$. Note that since $T_{\ell+1} \setminus T_\ell$ is an antichain by Lemma 3, $S' \in \mathcal{F}$. Then there exists $\widetilde{\pi} \in \mathcal{L}_{\mathcal{N}}(\preceq)$ such that $S' = \{\widetilde{\pi}(1), \ldots, \widetilde{\pi}(|S'|)\}$, i.e., $\sum_{i \in S'} a_i^{\widetilde{\pi}}(v) = v(S')$. By strict convexity we conclude that $\sum_{i \in S} a_i^{\widetilde{\pi}} > v(S)$. For any $\pi \in \mathcal{L}_{\mathcal{N}}(\preceq)$, $\sum_{i \in S} a_i^{\pi}(v) \geq v(S)$, hence $x(S) > v(S)$.

(ii) The equation follows from Propositions 8 and 1, and the fact that minimal normal collections give largest restricted cores. In order to show the final statement, let v be strictly convex and let x be defined as in (i). We have seen that $x(S) = v(S)$ if and only if $S \in \mathcal{N} \cup \{\emptyset, N\}$ so that there is no other minimal normal collection $\mathcal{N}'$ with $x \in \mathcal{C}_{\mathcal{N}'}(v)$.

(iii) One direction follows from (i). For the other direction let $\mathcal{N}$ be a normal collection that is not nested. Hence, there are $S, T \in \mathcal{N}$ such that $S \setminus T \neq \emptyset \neq T \setminus S$. By strict convexity, $v(S) + v(T) < v(S \cup T) + v(S \cap T)$ so that any $y \in \mathbb{R}^N$ with $y(S) = v(S)$ and $y(T) = v(T)$ either satisfies $y(S \cap T) < v(S \cap T)$ or $y(S \cup T) < v(S \cup T)$. We conclude that $\mathcal{C}_{\mathcal{N}}(v) = \emptyset$.

$\square$

It remains to find all nested minimal normal collections. For this, the following lemma is useful.

Lemma 7. The nested normal collection $\mathcal{N} = \{N_1, \ldots, N_q\}$, $\emptyset \neq N_1 \subsetneq \cdots \subsetneq N_q$, is minimal if and only if

(i) $N_q \setminus N_{q-1}$ contains an element that is not maximal (w.r.t. $\preceq$), and

(ii) $N_k \setminus N_{k-1}$ contains an element i that covers some element j (i.e., $j \prec\!\!\cdot\, i$) of $N_{k-1} \setminus N_{k-2}$ for $k = 2, \ldots, r$, where $N_0 = \emptyset$.

Proof. If (i) is not satisfied, then $\mathcal{N} \setminus \{N_q\}$ is still normal, and if (ii) is not satisfied for some k, then $\mathcal{N} \setminus \{N_{k-1}\}$ is still normal. In order to verify the opposite implication assume that $\mathcal{N}$ is not minimal and let $p \in \{1, \ldots, q\}$ such that $\mathcal{N} \setminus \{N_p\}$ is still normal. If $p = q$, then $\mathcal{N}$ violates (i), and if $p < q$, then $\mathcal{N}$ violates (ii) for $k = p + 1$. $\square$

Every $\pi \in \mathcal{L}(\preceq)$ *generates* a collection $\mathcal{N}^\pi \subseteq \mathcal{F}$ defined as follows: Let $0 = t_0 < t_1 < \cdots < t_q < t_{q+1} = n$ be defined by the requirements that

- $\{\pi(t_j + 1), \ldots, \pi(t_{j+1})\}$ is an antichain for all $j = 0, \ldots, q$;

- $\{\pi(t_j + 1), \ldots, \pi(t_{j+1} + 1)\}$ is not an antichain for all $j = 0, \ldots, q-1$.

Then define $N_j = \{\pi(1), \ldots, \pi(t_j)\}$ for all $j = 1, \ldots, q$ and put $\mathcal{N}^\pi = \{N_1, \ldots, N_q\}$. As $N_j \setminus N_{j-1}$ for $j = 1, \ldots, q$ is an antichain (where $N_0 = \emptyset$), $\mathcal{N}^\pi$ is a nested normal collection. Moreover, $\mathcal{N}^\pi$ is a minimal normal collection by Lemma 7. Conversely, let $\mathcal{N} = \{N_1, \ldots, N_q\}, N_1 \subsetneq \cdots \subsetneq N_q$, be a minimal nested normal collection. Choose $\pi \in \mathcal{L}(\preceq)$ such that

- for all $j = 1, \ldots, q$, $\pi^{-1}(N_j) = \{1, \ldots, |N_j|\}$;

- $\pi(|N_{k-1}| + 1)$ covers some element of $N_{k-1} \setminus N_{k-2}$ for all $k = 2, \ldots, q$.

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By Lemma 7 such π exists. By construction $\mathcal{N} = \mathcal{N}^\pi$.

We summarize that every linear extension[2] of $(N, \preceq)$ (i.e., every $\mathcal{F}$-admissible ordering of N) generates a unique minimal nested normal collection of $\mathcal{F}$ and that minimal nested normal collection is generated by some (not necessarily unique) linear extension of $(N, \preceq)$.

Example 4. Let $N = \{1, 2, 3, 4\}$ and $\preceq$ be determined by $1 \prec 3$ and $2 \prec 4$. Then minimal nested normal collections are

$$\mathcal{N}_1 = \{12\}, \mathcal{N}_2 = \{1, 123\}, \text{ and } \mathcal{N}_3 = \{2, 124\}.$$

Note that $\mathcal{N}_2$ and $\mathcal{N}_3$, although minimal, are not short. The $\mathcal{F}$-admissible permutations are

$$\pi_1 = (1, 2, 3, 4), \pi_2 = (1, 2, 4, 3), \pi_3 = (2, 1, 3, 4), \pi_4 = (2, 1, 4, 3), \pi_5 = (1, 3, 2, 4),$$
$$\text{and } \pi_6 = (2, 4, 1, 3).$$

The permutations $\pi_1, \ldots, \pi_4$ generate $\mathcal{N}_1$, π_5 generates $\mathcal{N}_2$, and π_6 generates $\mathcal{N}_3$. However, for any convex game v,

$$\mathcal{C}_{\mathcal{N}_1} = \text{conv}(\{a^{\pi_1}(v), \ldots, a^{\pi_4}(v)\}), \quad \mathcal{C}_{\mathcal{N}_2}(v) = \text{conv}(\{a^{\pi_1}(v), a^{\pi_5}(v)\}),$$
$$\text{and } \mathcal{C}_{\mathcal{N}_3}(v) = \text{conv}(\{a^{\pi_4}(v), a^{\pi_6}(v)\}).$$

Finally, if $v(S) = |S|^2$ for $S \in \mathcal{F}$, then v is strictly convex and $\frac{a^{\pi_5}(v) + a^{\pi_6}(v)}{2} = (3, 3, 5, 5) \notin \mathcal{C}^b(v)$ so that the bounded core may be non-convex even for convex games.

The remarkable nested collections $\mathcal{N}^d, \mathcal{N}^{GX}, \mathcal{N}^W$ presented in Section 3.1 are generated by particular linear extensions among those which satisfy the two above requirements. The $\mathcal{N}^{GX}$ collection is generated by any such linear extension which in addition satisfies

$$\pi(1), \ldots, \pi(|L_0|) \text{ is a permutation on } L_0$$
$$\pi(|L_0| + 1), \ldots, \pi(|L_0 \cup L_1|) \text{ is a permutation on } L_1$$
$$\vdots$$
$$\pi(|L_0 \cup \cdots \cup L_{h(N)-2}| + 1), \ldots, \pi(|L_0 \cup \cdots \cup L_{h(N)-1}|) \text{ is a permutation on } L_{h(N)-1}.$$

The downwards collection $\mathcal{N}^d$ is in turn generated by linear extensions which in addition satisfy

$$\pi(1), \ldots, \pi(|D_{h(N)}|) \text{ is a permutation on } D_{h(N)}$$
$$\pi(|D_{h(N)}| + 1), \ldots, \pi(|D_{h(N)-1} \cup D_{h(N)}|) \text{ is a permutation on } D_{h(N)-1}$$
$$\vdots$$
$$\pi(|D_2 \cup \cdots \cup D_{h(N)}| + 1), \ldots, \pi(|D_1 \cup \cdots \cup D_{h(N)}|) \text{ is a permutation on } D_1.$$

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[2]Linear extensions, also known as topological sorting, can be generated in linear time in the number of linear extensions; see, e.g., Pruesse and Ruskey (1994). However, the problem of counting all linear extensions of a finite partial order is #P-complete; see Brightwell and Winkler (1991).

Finally, the Weber collection arises from any linear extension satisfying in addition

$$\pi(1), \ldots, \pi(|\min(N^\circ)|) \text{ is a permutation on } \min(N^\circ)$$
$$\pi(|\min(N^\circ)| + 1), \ldots, \pi(|\min(N^\circ) \cup \min((N \setminus L_0)^\circ)|) \text{ is a permutation on}$$
$$\min((N \setminus L_0)^\circ)$$
$$\vdots$$
$$\pi(|\min(N^\circ) \cup \cdots \cup \min((N \setminus (L_0 \cup \cdots \cup L_{q-3}))^\circ)| + 1), \ldots$$
$$\ldots, \pi(|\min(N^\circ) \cup \cdots \cup \min((N \setminus (L_0 \cup \cdots \cup L_{q-2}))^\circ)|) \text{ is a permutation on}$$
$$\min((N \setminus (L_0 \cup \cdots \cup L_{q-2}))^\circ).$$

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